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Fig. 1

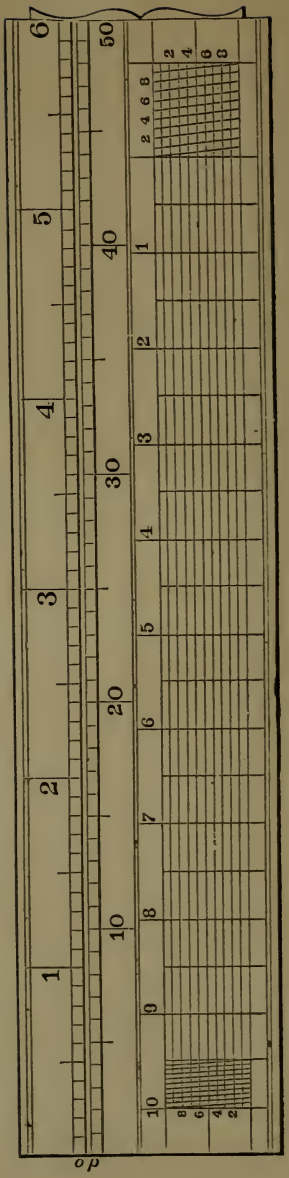

Fig. 2.

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PREFACE.

THIS treatise on the *Special* or *Elementary Geometry* consists of four parts.

PART I. is designed as an introduction. In it the student is made familiar with the geometrical concepts, and with the fundamental definitions and facts of the science. The definitions here given, are given once for all. It is thought that the pupil can obtain his first conception of a geometrical fact, as well, at least, from a correct, scientific statement of it, as from some crude, colloquial form, the language of which he will be obliged to replace by better, after the former shall have become so firmly fixed in his mind, as not to be easily eradicated. No attempt at demonstration is made in this part, although most of the fundamental facts of Elementary Plane Geometry are here presented, and amply and familiarly illustrated. This course has been taken in obedience to the canon of the teacher's art, which prescribes "facts before theories." Moreover, such has been the historic order of development of this, and most other sciences; viz., the *facts* have been known, or conjectured, long before men have been able to give any logical account of them. And does not this indicate what *may* be the natural order in which the individual mind will receive science? When the student has become familiar with the things (concepts) about which his mind is to be occupied, and knows some of the more important of their properties and relations, he is better prepared to reason upon them.

PART II. contains all the essential propositions in Plane, Solid, and Spherical Geometry, which are found in our common text-books, with their demonstrations. The subject of triedrals and the doctrine of the sphere are treated with more than the ordinary fullness. The earlier sections of this part are made short, each treating of a single subject, and the propositions are made to stand out prominently. At the close of each section are *Exercises* designed to illustrate and apply the principles contained in the section, rather than to extend the pupil's knowledge of geometrical facts. These features, together with the synopses at the close of the sections, practical teachers cannot fail to appreciate.

PART III., which is contained only in the *University Edition*, has

been written with special reference to the needs of students in the *University of Michigan*. Our admirable system of public High-Schools, of which schools there is now one in almost every considerable village, promises ere long to become to us something near what the German Gymnasia are to their Universities. In order to promote the legitimate development of these schools, it is necessary that the University resign to them the work of instruction in the elements of the various branches, as fast and as far as they are prepared in sufficient numbers to undertake it. It is thought that these schools should now give the instruction in Elementary Geometry, which has hitherto been given in our ordinary college course. The first two parts of this volume furnish this amount of instruction, and students are expected to pass examination upon it on their entrance into the University. This amount of preparation enables students to extend their knowledge of Geometry, during the Freshman year in the University, considerably beyond what has hitherto been practicable. As a text-book for such students, *Part* III. has been written. At this stage of his progress, the student is prepared to learn to investigate for himself. Hence he is here furnished with a large collection of well classified theorems and problems, which afford a review of all that has gone before, extend his knowledge of geometrical truth, and give him the needed discipline in original demonstration. To develop the power of independent thought, is the most difficult, while it is the most important part of the teacher's work. Great pains have therefore been taken, in this part of the work, to render such aid, and *only such*, as a student ought to require in advancing from the stage in which he has been following the processes of others, to that of independent reasoning. In the second place, this part contains what is usually styled *Applications of Algebra to Geometry*, with an extended and carefully selected range of examples in this important subject. A third purpose has been to present in this part an introduction to what is often spoken of as the *Modern Geometry*, by which is meant the results of modern thought in developing geometrical truth upon the direct method. While, as a system of geometrical reasoning, this Geometry is not philosophically different from that with which the student of Euclid is familiar, and which is properly distinguished as the *special* or *direct* method, the character of the facts developed is quite novel. So much so, indeed, that the student who has no knowledge of Geometry but that which our common text-books furnish, knows absolutely nothing of the domain into which most of the brilliant advances of

the present century have been made. He knows not even the terms in which the ideas of such writers as PONCELET, CHASLES, and SALMON, are expressed, and he is quite as much a stranger to the thought. In this part are presented the fundamental ideas concerning *Loci*, *Symmetry*, *Maxima* and *Minima*, *Isoperimetry*, the theory of *Transversals*, *Anharmonic Ratio*, *Polars*, *Radical Axes*, and other modern views concerning the circle.

PART IV. is *Plane* and *Spherical Trigonometry*, with the requisite Tables. While this Part, as a whole, is much more complete than the treatises in common use in our schools, it is so arranged that a shorter course can be taken by such as desire it. Thus, for a shorter course in Plane Trigonometry, see NOTE on page 55. In Spherical Trigonometry, the first three sections, either with or without the *Introduction* on Projection, will afford a very satisfactory elementary course.

A few words as to the manner in which this plan has been executed, may be important. In general, the *Definitions* are those usually given, with such slight alterations as have been suggested by reflection and experience. There are, however, a few exceptions. Among these is the definition of an *Angle*. I can but regard the attempt to define an angle as *The difference in direction between two lines*, or *The amount of divergence*, as needlessly vague, abstract, and perplexing to a student, as well as questionable on philosophical grounds. The definition given in the text will be seen to be, at bottom, the old one, the conception being slightly altered to bring it into more close connection with common thought, and also with the idea of an angle as generated by the revolution of a line. As to *Parallels*, and the definition of *similarity*, my experience as a teacher is decidedly in favor of retaining the old notions. So also in adopting a definition of a *Trigonometrical Function*, I am compelled to adhere to the geometrical conception. A ratio is a complex concept, and consequently not so easy of application as a simple one. For this reason, among others, I prefer the *differential* to the *differential coefficient*, in the calculus, and a *line* to a *ratio*, in Trigonometry. Moreover, I have found that students invariably rely upon the geometrical conception, even when first taught the other; hence I am not surprised that all our writers who define a trigonometrical function as a ratio, hasten to tell the pupil what it means, by giving him the geometrical illustrations. Nor are the superior facility which the geometrical conception affords for a full elucidation of the doctrine of the signs of the functions, and its admirable adaptation to fix these laws in the mind, considerations to be lost sight of in selecting the definition.

Surely no apology is needed, at the present day, for introducing the idea of *motion* into Elementary Geometry, notwithstanding the rigorous and disdainful manner with which its entrance was long resisted by the old Geometers. And, having admitted this idea, the conception of loci as generated by motion would seem to follow as a logical necessity. In like manner, I take it, the *Infinitesimal* method must come in. Its directness, simplicity, and necessity in applied mathematics, demand its recognition in the elements. In two or three instances, I have presented the *reductio ad absurdum*, where the methods are equivalents, and have always in presenting the infinitesimal method woven in the idea of *limits*, which I conceive to be fundamentally the same as the infinitesimal. Thus we bring the lower and higher mathematics into closer connection.

The *order of arrangement* in *Plane Geometry* (Chap. I.), is thought to be simple, philosophical, and practical. A glance at the table of contents will show what it is. This arrangement secures the very important result, that each section presents some particular *method of proof*, and holds the student to it, until it is familiar. True, it requires that a larger number of propositions be demonstrated from fundamental truths; but who will consider this an objection?

To such as consider it the sole province of geometrical demonstration, to convince the mind of the truth of a proposition, not a few theorems in these and ordinary pages must seem quite superfluous. To them, *Prop.* I., page 121, may afford some merriment. But those who, with myself, consider Geometry as a branch of practical logic, the aim of which is to detect and state the steps which actually lie between premise and conclusion, will see the propriety of such demonstrations; and for each individual of the other class, a separate treatise will be needed, since no two minds will intuitively grant exactly the same propositions.

To Ex-President HILL, of Harvard, I am indebted for the confirmation of an opinion which had been previously forming in my mind, that the study of Geometry as a branch of logic, should be preceded by a presentation of its leading facts. The works of COMPAGNON, TAPPAN, and our lamented countryman, CHAUVENET, have been within reach during the entire work of preparation, and this volume would have been different, in some respects, if any one of these able treatises had not appeared before it.

In the preparation of PART III. the works of ROUCHÉ et COMBEROUSSE and MULCAHY have been freely used. For the very concise and elegant form in which the principle of Delambre, for the pre-

cise calculations of Trigonometrical Functions near their limits, is embodied in TABLE III., I am indebted to the recent work of President ELI T. TAPPAN, of Kenyon College, Ohio.

My long and intimate intercourse with Professor G. B. MERRIMAN, now of the department of Physics in the University, has been a source of great profit to me in the preparation of the entire work. His sound, practical judgment as a teacher of Geometry, and cultivated taste and skill as a Mathematician, have been ever at my service, and have done more than I can tell, in giving form to the work, both as respects its matter and its spirit.

EDWARD OLNEY.

UNIVERSITY OF MICHIGAN,
ANN ARBOR, *January*, 1872.

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SPECIAL OR ELEMENTARY

GEOMETRY.

INTRODUCTION.

SECTION I.

LOGICO-MATHEMATICAL TERMS.*

1. ***A Proposition*** is a statement of something to be considered or done.

ILL.—Thus, the common statement, "Life is short," is a proposition; so, also, we make, or state a proposition, when we say, "Let us seek earnestly after truth."—"The product of the divisor and quotient, plus the remainder, equals the dividend," and the requirement, "To reduce a fraction to its lowest terms," are examples of Arithmetical propositions.

2. Propositions are distinguished as *Axioms, Theorems, Lemmas, Corollaries, Postulates,* and *Problems.*

3. ***An Axiom*** is a proposition which states a principle that is so simple, elementary, and evident as to require no proof.

ILL.—Thus, "A part of a thing is less than the whole of it," "Equimultiples of equals are equal," are examples of axioms. If any one does not admit the truth of axioms, when he understands the terms used, we say that his mind is not sound, and that we cannot reason with him.

4. ***A Theorem*** is a proposition which states a real or supposed fact, whose truth or falsity we are to determine by reasoning.

ILL.—"If the same quantity be added to both numerator and denominator of a proper fraction, the value of the fraction will be increased," is a *theorem.* It is a statement the truth or falsity of which we are to determine by a course of reasoning.

---

* That is, terms used in the science in consequence of its logical character. The science of the Pure Mathematics may be considered as a department of practical logic.

5. A Demonstration is the course of reasoning by means of which the truth or falsity of a theorem is made to appear. The term is also applied to a logical statement of the reasons for the processes of a rule. A solution tells *how* a thing is done: a demonstration tells *why* it is so done. A demonstration is often called *proof.*

6. A Lemma is a theorem demonstrated for the purpose of using it in the demonstration of another theorem.

ILL.—Thus, in order to demonstrate the rule for finding the greatest common divisor of two or more numbers, it may be best first to prove that "A divisor of two numbers is a divisor of their sum, and also of their difference." This theorem, when proved for such a purpose, is called a *Lemma.*

The term *Lemma* is not much used, and is not very important, since most theorems, once proved, become in turn auxiliary to the proof of others, and hence might be called lemmas.

7. A Corollary is a subordinate theorem which is suggested, or the truth of which is made evident, in the course of the demonstration of a more general theorem, or which is a direct inference from a proposition, or a definition.

ILL.—Thus, by the discussion of the ordinary process of performing subtraction in Arithmetic, the following *Corollary* might be suggested: "Subtraction may also be performed by addition, as we can readily observe what number must be added to the subtrahend to produce the minuend."

8. A Postulate is a proposition which states that something can be done, and which is so evidently true as to require no process of reasoning to show that it is possible to be done. We may or may not know how to perform the operation.

ILL.—Quantities of the same kind can be added together.

9. A Problem is a proposition to do some specified thing, and is stated with reference to developing the method of doing it.

ILL.—A problem is often stated as an incomplete sentence, as, "To reduce fractions to a common denominator."—This incomplete statement means that "We propose to show how to reduce fractions to a common denominator." Again, the problem "To construct a square," means that "We propose to draw a figure which is called a square, and to tell how it is done."

10. A Rule is a formal statement of the method of solving a general problem, and is designed for practical application in solving special examples of the same class. Of course a rule requires a demonstration.

11. A Solution is the process of performing a problem or an example. It should usually be accompanied by a demonstration of the process.

12. A Scholium is a remark made at the close of a discussion, and designed to call attention to some particular feature or features of it.

ILL.—Thus, after having discussed the subject of multiplication and division in Arithmetic, the remark that "Division is the converse of multiplication," is a scholium.

SYNOPSIS.

Subject of the section.
Proposition. *Ill.*
Varieties of propositions.
Axiom. *Ill.*
One who will not admit the truth of axioms.
Theorem. *Ill.*
Demonstration. Difference between a solution and a demonstration.
Lemma. *Ill.* Why the term is unimportant.
Corollary. *Ill.*
Postulate. *Ill.*
Problem. How stated. *Ill.*
Rule.
Solution.
Scholium. *Ill.*

SECTION II.

THE GEOMETRICAL CONCEPTS.*

POINTS.

13. A Point is a place without size. Points are designated by letters.

ILL.—If we wish to designate any particular point (place) on the paper, we put a letter by it, and sometimes a dot on it. Thus, in *Fig.* 3, the ends of the line, which are points, are designated as "point A," "point D;" or, simply, as A and D. The points marked on the line are designated as "point B," "point C," or as B and C. F and E are two points above the line.

FIG. 3.

---

* A concept is a thing thought about;—a thought-object. Thus, in Arithmetic, number is the concept; in Botany, plants; in Geometry, as will appear in this section, points, lines, and solids. These may also be said to constitute the *subject-matter* of the science.

LINES.

14. A Line is the path of a point in motion. Lines are represented upon paper by marks made with a pen or pencil, the point of the pen or pencil representing the moving point. A line is designated by naming the letters written at its extremities, or somewhere upon it.

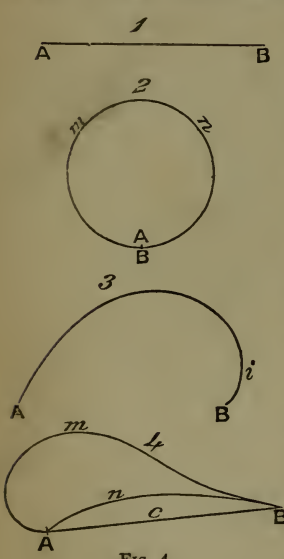

Fig. 4.

Ill.—In each case in *Fig.* 4, conceive a point to start from A and move along the path indicated by the mark to B. The path thus traced is a line. *Since a true point has no size, a line has no breadth*, though the marks by which we represent lines have some breadth. The first and third lines in the figure are each designated as "the line AB." The second line is considered as traced by a point starting from A and coming around to A again, so that B and A coincide. This line may be designated as the line A*mn*A, or A*mn*B. In the fourth case, there are three lines represented, which are designated, respectively, as A*m*B, A*n*B, and A*c*B; or, the last, as AB.

15. Lines are of ***Two Kinds,*** *Straight* and *Curved.* A straight line is also called a *Right Line.* A curved line is often called simply a *Curve.*

16. A Straight Line* is a line traced by a point which moves constantly in the same direction.

17. A Curved Line is a line traced by a point which constantly changes its direction of motion.

Ill's.—Thus in 1, *Fig.* 4, if the line AB is conceived as traced by a point moving from A to B, it is evident that this point moves in the same direction throughout its course; hence AB is a straight line. If a body, as a stone, be let fall, it moves constantly toward the centre of the earth; hence its path represents a straight line. If a weight be suspended by a string, the string represents a straight line. Considering the line represented by A*i*B, *Fig.* 4, as the path of a point moving from A to B, we see that the direction of motion is constantly changing. For example, if this were a line traced on a map, we

* The word "line" used alone signifies "straight line."

would say, that, starting from A, the point begins to move nearly north, but keeps changing its direction more and more toward the east, until at 3 it moves directly east; and from 3 it continues to change its course and moves more and more toward the south, till at *i* it is moving directly south. The same general truth is illustrated in 2 and 4, *Fig.* 4. The path of a ball thrown into the air, in any direction except directly up, represents a curved line. Most of the lines seen in nature are curved, as the edges of leaves, the shore of a river or lake, etc. Sometimes a path like that represented in *Fig.* 5 is called, though improperly, a *Broken Line.* It is not *a* line at all; that is, not *one* line: it is a series of straight lines.

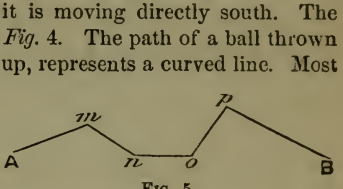

FIG. 5.

SURFACES.

18. A Surface is the path of a line in motion.*

19. Surfaces are of ***Two Kinds,*** *Plane* and *Curved.*

20. A Plane Surface, or simply a *Plane,* is a surface with which a straight line may be made to coincide in any direction. Such a surface may always be conceived as the path of a straight line in motion.

21. A Curved Surface is a surface in which, if lines are conceived to be drawn in all directions, some or all of them will be curved lines.

ILL'S.—Let AB, *Fig.* 6, be supposed to move to the right, so that its extremities A and B move at the same rate and in the same direction, A tracing the line AD, and B, the line BC. The path of the line, the figure ABCD, is a surface. This page is a surface, and may be conceived as the path of a line sliding like a ruler from top to bottom of it, or from one side to the other. Such a path will have length and breadth, being in the latter respect unlike a line, which has only length.

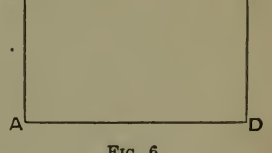

FIG. 6.

As a second illustration, suppose a fine wire bent into the form of the curve A*m*B, *Fig.* 7, and its ends A and B stuck into a rod, XY. Now, taking the rod XY in the fingers and rolling it, it is evident that the path of the line represented by the wire A*m*B, will be the surface of a ball (sphere).

* Should it be said that irregular surfaces are not included in this definition, the sufficient reply is, that such surfaces are not subjects of Geometrical investigation, except approximately, by means of regular surfaces.

Again, suppose the rod XY be placed on the surface of this paper so that the wire A*m*B shall stand straight up from the paper, just as it would be if we could take hold of the curve at *m* and raise it right up, letting XY lie as it does in the figure. Now slide the rod straight up or down the page, making both ends move at the same rate. The path of A*m*B will be like the surface of a half-round rod (a semi-cylinder).

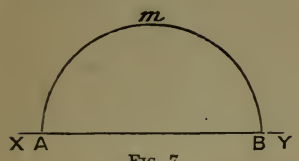

Fig. 7.

Thus we see how surfaces plane and curved may be conceived as the paths of lines in motion.

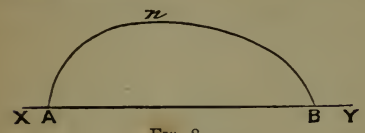

Fig. 8.

Ex. 1. If the curve A*n*B, *Fig.* 8, be conceived as revolved about the line XY, the surface of what object will its path be like?

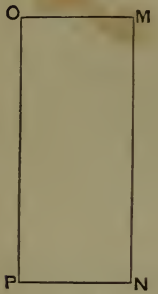

Fig. 9.

Ex. 2. If the figure OMNP, *Fig.* 9, be conceived as revolved about OP, what kind of a path will MN trace? What kind of paths will PN and OM trace?

Ans. One path will be like the surface of a joint of stove-pipe, *i. e.*, a cylindrical surface; and one will be like a flat wheel, *i. e.*, a circle.

Ex. 3. If you fasten one end of a cord at a point in the ceiling and hang a ball on the other end, and then make the ball swing around in a circle, what kind of a surface will the string describe?

[Note.—The student is not necessarily expected to give the geometrical name of the surface, but rather to tell in his own way what it is like, so as to make it clear that he conceives the thing itself.]

Ex. 4. If you were to draw lines in all directions on the surface of the stove-pipe, might any of them be straight? Could *all* of them be straight? What kind of a surface is this, therefore?

Ex. 5. Can you draw a straight line on the surface of a ball? On the surface of an egg? What kind of surfaces are these?

Ex. 6. When the carpenter wishes to make the surface of a board perfectly flat, he takes a ruler whose edge is a straight line, and lays this straight edge on the surface in all directions, watching closely

to see if it always touches. Which of our definitions is he illustrating by his practice?

Ex. 7. When the miller wishes to make flat the surface of one of the large stones with which wheat is ground into flour, he sometimes takes a ruler with a straight edge, and smearing the edge with paint, applies it in all directions to the surface, and then chips off the stone where the paint is left on it. What principles is he illustrating?

Ex. 8. How can you conceive a straight line to move so that it shall not generate a surface?

ANGLES.

22. A Plane Angle, or simply an *Angle,* is the opening between two lines which meet each other. The point in which the lines meet is called the ***vertex,*** and the lines are called the ***sides.*** An angle is designated by placing a letter at its vertex, and one at each of its sides. In reading, we name the letter at the vertex when there is but one vertex at the point, and the three letters when there are two or more vertices at the same point. In the latter case, the letter at the vertex is put between the other two.

Ill.—In common language an angle is called a *corner.* The opening between the two lines AB and AC, in which the figure 1 stands, is called the angle A; or, if we choose, we may call it the angle BAC. At L there are two vertices, so that were we to say the angle L, one would not know whether we meant the angle (corner) in which 4 stands, or that in which 5 stands. To avoid this ambiguity, we say the angle HLR for the former, and RLT for the latter. The angle ZAY is the corner in which 11 stands; that is, the opening between the two lines AY and AZ. In designating an angle by three letters, it is immaterial which letter stands first so that the one at the vertex is put between the other two. Thus, PQS and SQP are both designations of the angle in which 6

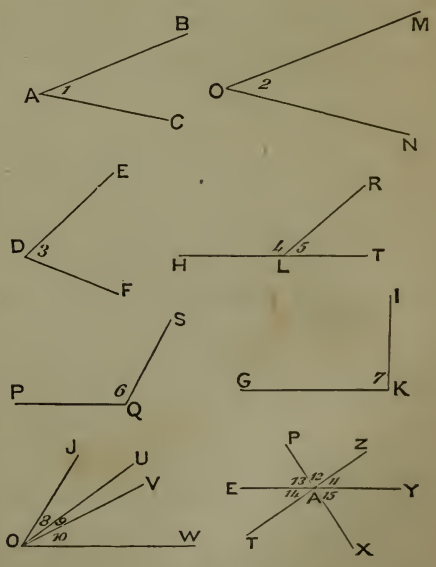

Fig. 10.

stands. An angle is also frequently designated by putting a letter or figure in it and near the vertex.

23. The Size of an Angle depends upon the rapidity with which its sides separate, and not upon their length.

ILL.—The angles BAC and MON, *Fig.* 10, are equal, since the sides separate at the same rate, although the sides of the latter are more prolonged than those of the former. The sides DF and DE separate faster than AB and AC, hence the angle EDF is greater than the angle BAC.

24. Adjacent Angles are angles so situated as to have a common vertex and one common side lying between them.

ILL.—In *Fig.* 10, angles 4 and 5 are adjacent, since they have the common vertex L, and the common side LR. Angles 9 and 10 are also adjacent, as are also 8 and 9.

25. Angles are distinguished as *Right Angles* and *Oblique Angles.* Oblique angles are either *Acute* or *Obtuse.*

26. A Right Angle is an angle included between two straight lines which meet each other in such a manner as to make the adjacent angles equal. ***An Acute Angle*** is an angle which is less than a right angle, *i. e.*, one whose sides separate less rapidly. ***An Obtuse Angle*** is an angle which is greater than a right angle, *i. e.*, one whose sides separate more rapidly.

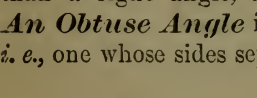

FIG. 11.

ILL.—As in common language an angle is called a *corner*, so a right angle is called a *square corner*; an acute, a *sharp corner;* and an obtuse angle might be called a blunt corner. In *Fig.* 11, BAC and DAB are right angles. In *Fig.* 10, 1, 2, 3, 5, 8, 9, and 10 are acute angles, 4 and 6 are obtuse, and 7 is a right angle.

A SOLID.

27. A Solid is a limited portion of space. It may also be conceived as the path of a surface in motion.

ILL.—Suppose you have a block of wood like that represented in *Fig.* 12, with all its corners (angles) square corners (right angles). Hold it still in your

fingers a moment, and fix your mind upon it. Now take the block away and think of the space (place) where it was. This space will be of just the same form as the block of wood, and by a little effort you can think of it just as well as of the wood. This *space* is an example of what we call a *Solid* in Geometry. In fact, the solids of Geometry are not solids at all in the common sense of solids; they are only just *places of certain shapes.*

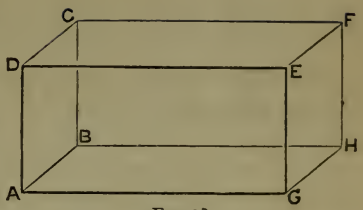

FIG. 12.

Again, hold your ball still a moment in your fingers and then let it drop, and think of the place it filled when you had it in your fingers. It is this *place*, shaped just like your ball, that we think about, and talk about as a *solid*, in Geometry.

In order to see how a solid may be conceived as the path of a surface, suppose you cut out a piece of paper of just the same size as the end of the block represented in Fig. 12. Let ABCD represent this piece of paper. Now, holding the paper in a perpendicular position, as ABCD is represented in the figure, move it along to the right, so that its angles shall trace the lines AC, BH, DE, and CF. When the paper has moved to the position GHFE, its path will be just the same space as the block of wood occupied. This path, or the space through which the surface represented by the piece of paper moved, is the solid.

Ex. 1. If a semicircle is conceived as revolved around its diameter, what is the path through which it moves? See *Fig.* 7.

Ex. 2. If the surface OMNP, *Fig.* 9, is conceived as revolved around OP, what is the path through which it moves?

CAUTION.—The student needs to be careful and distinguish between the *surface* traced by the *line* MN, and the *solid* traced by the *surface* OMNP.

Ex. 3. If the surface represented by ABC be conceived as revolved about its side CA, what kind of a solid is its path?

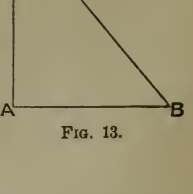

FIG. 13.

[NOTE.—As has been said before, the student is not necessarily expected to *name* these solids, but rather to show, in his own language, that he has the conception.]

Ex. 4. As you fill a vessel with water, what is the solid traced by the surface of the water?

Ans. The same as the space within the vessel.

Ex. 5. If a circle is conceived as lying horizontally, and then moved directly up, what will be the solid described, *i. e.*, its path? Do not confound the surface described with the solid. What describes the surface? What the solid?

EXTENSION AND FORM.

28. ***Extension*** means a stretching or reaching out. Hence, a *Point* has no extension. It has only position (place). A *Line* stretches or reaches out, but only in length, as it has no width. Hence, a line is said to have *One Dimension*, viz., length. A *Surface* extends not only in length, but also in breadth; and hence has *Two Dimensions*, viz., length and breadth. A *Solid* has *Three Dimensions*, viz., length, breadth, and thickness.

ILL.—Suppose we think of a point as capable of stretching out (extending) in one direction. It would become a line. Now suppose the line to stretch out (extend) in another direction—to widen. It would become a surface. Finally, suppose the surface capable of thickening, that is, extending in another direction. It would become a solid.

29. The *Limits* (extremities) of a line are points.
The *Limits* (boundaries) of a surface are lines.
The *Limits* (boundaries) of a solid are surfaces.

30. ***Magnitude*** (size) is the result of extension. Lines, surfaces, and solids are the geometrical magnitudes. A point is not a magnitude, since it has no size. The magnitude of a line is its length; of a surface, its area; of a solid, its volume.

31. ***Figure or Form*** (shape) is the result of position of points. The form of a line (as straight or curved) depends upon the relative position of the points in the line. The form of a surface (as plane or curved) depends upon the relative position of the points in it. The form of a solid depends upon the relative position of the points in its surface. Lines, surfaces, and solids are the geometrical figures.*

ILL.—In *Fig.* 14, it is easy to conceive the form of the lines by knowing the position of points in the lines. By taking a quantity of common pins of different lengths, sticking them upright in a board, and conceiving the heads to represent points in a surface, we can readily see how the position of the points in a surface determine its form.

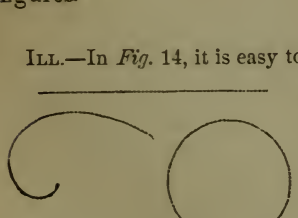

FIG. 14.

EX. 1. Suppose a line to begin to con-

* Lines, surfaces, and solids are called magnitudes when reference is had to their *extent*, and figures when reference is had to their *form*.

tract in length, and continue the operation till it can contract no longer, what does it become? That is, what is the minor limit of a line?

Ex. 2. If a surface contracts in one dimension, as width, till it reaches its limit, what does it become? If it contracts to its limit in both dimensions, what does it become?

Ex. 3. If a solid contracts to its limit in one dimension, what does it pass into? If in two dimensions? If in three dimensions?

Ex. 4. What kind of a surface is that, every point in which is equally distant from a given point?

32. Geometry treats of *magnitude* and *form* as the result of extension and position.

The *Geometrical Concepts* are points, lines, surfaces (including plane and spherical angles), and solids (including solid angles).

The *Object* of the science is the measurement and comparison of these concepts.

Plane Geometry treats of figures all of whose parts are confined to one plane. *Solid Geometry*, called also *Geometry of Space*, and *Geometry of Three Dimensions*, treats of figures whose parts lie in different planes. The division of Part II. into two chapters is founded upon this distinction. In the *Higher* or *General* Geometry these divisions are marked by the terms "*Of Loci in a Plane*," and "*Of Loci in Space*."

SYNOPSIS.

- GEOMETRICAL CONCEPTS.
  - POINT ....
    - What.—How designated.—*Ill.*
    - Dimensions of.
    - Limit of Line.—Surface.—Solid.
  - LINE .....
    - What.
    - How designated.
    - Dimensions of.
    - Limit of Surface.
    - Kinds
      - Straight.—What.—*Ill.*
      - Curved.—What.—*Ill.*
      - Broken (?).
  - SURFACE..
    - What.
    - Dimensions of.
    - Limit of Solid.
    - Kinds
      - Plane.—What.—*Ill.*
      - Curved.—What.—*Ill.*
    - Angle
      - What.—Size depends on what.—Adjacent.
      - Kinds
        - Right.—What.—*Ill.*
        - Oblique
          - Acute.—What.—*Ill.*
          - Obtuse.—What.—*Ill.*
  - SOLID.....What.—*Ill.*—Examples.
- GEOMETRY..
  - Treats of
    - Magnitude.—What.—Result of what.
    - Figure or form.—What.—Result of what.
  - Concepts.—What.
  - Object.—What.

PART I.

A FEW OF THE MORE IMPORTANT FACTS OF THE SCIENCE.

SECTION I.

ABOUT STRAIGHT LINES.

***33. Prob.**—To measure a straight line with the dividers and scale.*

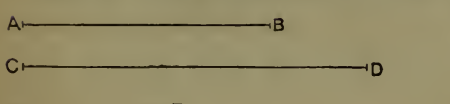

Fig. 15.

Solution.—Let AB, *Fig.* 15, be the line to be measured. Take the dividers, *Fig.* 2 (frontispiece), and placing the sharp point A firmly upon the end A of the line AB, open the dividers till the other point B (the pencil point) just reaches the other end of the line B. Then letting the dividers remain open just this amount, place the point A on the lower end of the left hand scale, as at *o*, *Fig.* 1, and notice where the point B reaches. In this case it reaches 3 spaces beyond the figure 1. Now, as this scale is inches and *tenths* of inches,* the line AB is 1.3 inches long.

Ex. 1. What is the length of CD? *Ans.* .15 of a foot.

Ex. 2. What is the length of EF? *Ans.* .75 of an inch.

Ex. 3. What is the length of GH? *Ans.* 1½ inches.

Ex. 4. What is the length of IK? *Ans.* .18 of a foot.

Ex. 5. Draw a line 3 inches long.

Ex. 6. Draw a line 2.15 inches long.

Ex. 7. Draw a line 1.25 inches long.

Ex. 8. Draw a line .85 of an inch long.

* The next scale to the right is divided into 10ths and 100ths of a foot. Thus from *p* to 1 is 1 tenth of a foot, and the smaller divisions are hundredths.

[NOTE.—Suppose a fine elastic cord were attached by each of its ends to the points A and B of the dividers; when they were opened so as to reach from C to D, *Fig.* 15, the cord would represent the line CD. Now applying the dividers to the scale is the same as laying this cord on the scale. Without the cord, we can imagine the distance between the points of the dividers to be a line of the same length as CD.]

Ex. 9. Find in the same way as above the length and width of this page. Also the distance from one corner (angle) to the opposite one (the diagonal).

---

34. Prob.—To find the sum of two lines.

SOLUTION.—To find the sum of AB and CD, I * first draw the indefinite line E*x*. With the dividers I obtain the length of AB, by placing one point on A and extending the other to B. This length I now lay off on the indefinite line E*x*, by putting one point of the dividers at E and with the other marking the point F. EF is thus made equal to AB.

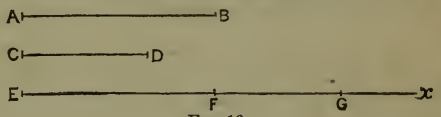

FIG. 16.

In the same manner taking the length of CD with the dividers, I lay it off from F on the line F*x*. Thus I obtain EG=EF+FG=AB+CD. Hence, the sum of AB and CD is EG.

[NOTE.—The student may measure EG by (*33*) and find the sum of AB and CD in inches or feet; but it is most important that he be able to look upon EG as the sum itself.]

Ex. 1. Find the sum of AB and EF, *Fig.* 15.

Ex. 2. Find the sum of EF, CD, and GH, *Fig.* 15.

Ex. 3. Make a line twice as long as CD, *Fig.* 16. Three times as long.

---

35. Prob.—To find the difference of two lines.

SOLUTION.—To find the difference of AB and CD, I take the length of the less line AB with the dividers; and placing one point of the dividers at one extremity of CD, as C, make C*e* = AB. Then is *e*D the difference of AB and CD, since *e*D = CD − C*e* = CD − AB.

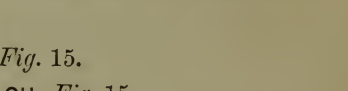

FIG. 17.

Ex. 1. Find the difference of IK and EF, *Fig.* 15.

Ex. 2. Find the difference of GH and CD, *Fig.* 15.

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* These elementary solutions are sometimes put in the singular, as the more simple style.

Ex. 3. Find how much longer IK, *Fig.* 15, is than the sum of EF, *Fig.* 15, and CD, *Fig.* 16.

Ex. 4. Find the difference of the sum of AB and GH, and the sum of CD and EF, *Fig.* 15.

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36. Prob.—*To compare the lengths of two lines; that is, to find their ratio (approximately*).*

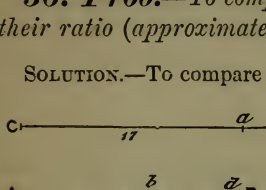

FIG. 18.

SOLUTION.—To compare the lengths of AB and CD, I lay off AB, the shorter, upon CD, as C*a*. (If AB could be applied two or more times to CD, I should apply it as many times as CD would contain it.) Now I apply the remainder of CD, viz., *a*D, to AB, as many times as AB will contain it, which is once with the remainder *b*B. This remainder I now apply to *a*D, and find it contained once with a remainder *c*D. Again, I apply this last remainder to *b*B, and find it contained twice with a remainder *d*B. This last remainder I now apply to *c*D, and find it contained 3 times, without any remainder. This last measure, *d*B, is a common measure of the two lines. Calling *d*B 1, I now observe that

$$
\begin{aligned}
d\mathrm{B} &= 1;\\
c\mathrm{D} &= 3d\mathrm{B} = 3;\\
bd &= 2c\mathrm{D} = 6;\\
ac &= b\mathrm{B} = bd + d\mathrm{B} = 7;\\
a\mathrm{D} &= ac + c\mathrm{D} = 10;\\
\mathrm{AB} &= \mathrm{A}b + b\mathrm{B} = a\mathrm{D} + ac = 17;\\
\mathrm{CD} &= \mathrm{C}a + a\mathrm{D} = \mathrm{AB} + a\mathrm{D} = 27.
\end{aligned}
$$

Hence the lines AB and CD are to each other as the numbers 17 and 27; AB is $\frac{17}{27}$ of CD; or, expressed in the form of a proportion, AB : CD :: 17 : 27.

[NOTE.—This process will be seen to be the same as that developed in Arithmetic and Algebra for finding the greatest or highest Common Measure of two numbers, and should be studied in connection with a review of those processes. See COMPLETE ARITHMETIC (*115*), and COMPLETE SCHOOL ALGEBRA (*137*).]

Ex. 1. Find, as above, the approximate ratio of AB to CD, *Fig.* 15.
Ratio, 13 : 18.

Ex. 2. Find, as above, the approximate ratio of CD and IK, *Fig.* 15.
Ratio, 5 : 6.

---

* This method does not get the *exact* ratio, because of the imperfection of measurement, and also because lines are sometimes incommensurable, as will appear hereafter.

Ex. 3. Find, as above, the approximate ratio of EF to GH, *Fig.* 15.
Ratio, 1 : 2.

Ex. 4. Find, as above, the approximate ratio of EF to CD, *Fig.* 15.
Ratio, 5 : 12.

---

37. To Intersect is to cross; and a crossing is called an *Intersection.*

38. To Bisect anything is to divide it into two *equal* parts.

39. Prob.—*To bisect a given line.*

SOLUTION.—To bisect the line AB, I take the dividers; and opening them so that the line between their points is more than half as long as AB, I place the sharp point A on the point A, and holding it firmly there, make a little mark with the pencil point B, as nearly as I can guess, opposite the middle of the line. Then, being careful to keep the dividers open just the same, I place the sharp point on B, and make a mark intersecting the first one, as at *m*. Now, doing just the same on the other side of the line, I make two marks intersecting each other, as at *n*. Finally, I draw a line from *m* to *n*, and where this line crosses AB is its middle point; that is, AO is equal to OB. [Why this is so we do not propose to tell now. The student needs only to learn how to do it. He should *measure* AO and OB, and thus test the accuracy of his work.]

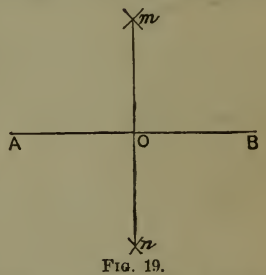

FIG. 19.

Ex. 1. Is it necessary that the dividers be opened just as wide when the marks are made through *n*, as when they are made through *m*? Try it.

Ex. 2. Suppose you make the marks through *m* as directed, but, in making those through *n*, you have the dividers *wider* open when you put the point on A than when you put it on B; will the line joining *m* and *n* then cross AB in the middle? If not, on which side of the middle will O be? Try it.

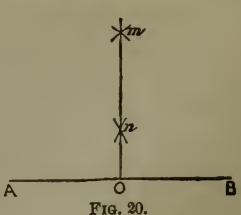

FIG. 20.

Ex. 3. Can you bisect a line by making the marks all on one side of it? If so, do it.

40. Axiom.—A straight line is the shortest path between two points.*

ILL.—If a cord is *stretched* across the table, it marks a straight line. In this way the carpenter marks a straight line. Having rubbed a cord, called a chalk-line, with chalk, he *stretches it tightly* from one point to another on the surfaces upon which he wishes to mark the line, and then raising the middle of the cord, lets it snap upon the surface. So the gardener makes the edges of his paths straight by *stretching* a cord along them. These operations depend upon the principle that when the line between the points is the shortest possible, it is straight.

***41. Axiom.**—Two points in a straight line determine its position.*

ILL.—If the farmer wants a straight fence built, he sets two stakes to mark its ends. From these its entire course becomes known. This is the principle upon which aligning (or sighting) depends. Having given two points in the required line, by looking in the direction of one from the other, we look along a straight line, and are thus able to locate other points in the line. If the points A and B are marked, by putting the eye at A and looking steadily towards B, we can tell whether D and E are in the same straight line with A and B, or not. So we can observe that C′ and C″ are *not* in the line; but that C is. This process of discovering other points in a line with two given points is called aligning, or sighting. In this way a row of trees is made straight, or a line of stakes set. It is the principle upon which the surveyor runs his lines, and the hunter aims his gun. In the latter case, the two sights are the given points, and the mark, or game, is a third point, which the marksman wishes to have in the same straight line as the sights.

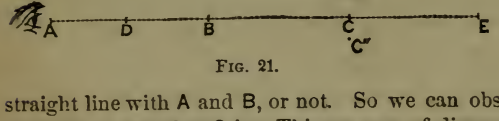

FIG. 21.

***42. Axiom.**—Between the same two points there is one straight line, and only one.*

ILL.—Let any two letters on this page represent the situation of two points; we readily see that there is one, and only one, straight path between them. Again, let a corner of the desk represent one point and a corner of the ceiling of the room represent another point; we perceive at once that, if a point is conceived to pass in a straight line from one to the other, it will always trace

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* An axiom may be *illustrated*, but it needs no *demonstration*. We may explain the terms used and elaborate the condensed statement; but if, when its meaning is clearly understood, any one does not grant the *truth* of its statement, he has not a sound mind, and we cannot reason with him.

the same path. In short, as soon as two points are mentioned, we think of the distance between them as a single straight line,—for example, the centre of the earth and the centre of the sun.

Once more, conceive A and B, *Fig.* 21, to be two points in the path of a point moving from A in the direction of B. Now *all* the points in the same direction from A as B is, are in this path; and any point out of this line, as C′ or C″, is in a different direction from A.

In this manner we draw a straight line on paper by laying the straight edge of a ruler on two points through which we wish the line to pass, and passing a pen or pencil along this edge.

Cor.—*Two straight lines can intersect in but one point; for, if they had two points common, they would coincide and not intersect.*

Ex. 1. A railroad is to be run from the town A to town B. If it is made straight, through what points will it pass? Can it pass through any points not in the same direction from A as B is?

Ex. 2. If I live on the south side of a straight railroad, and my friend on the north side, but five miles farther east, and two miles farther north, and the road from my house to his is straight, how many times does it cross the railroad?

Ex. 3. Can you always draw a straight line which shall cut a curve (whatever curve it may be) in two points? Try it.

Ex. 4. Detroit is directly east of where I live. How could I drive my horse there and never turn his head to the east? Would he have to travel in straight lines or in a curve? If I drive him on a curve, how can I manage it so that his head will be east for but an instant? If his head is all the time east, what is the line in which I drive him?

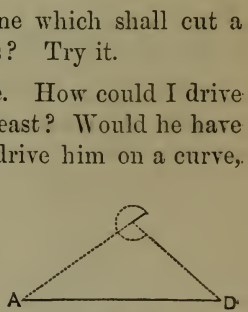

Fig. 22.

Sug.—The figure will suggest how the first may be accomplished.

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43. A Perpendicular to a given line is a line which makes a right angle (***26***) with the given line. The latter is also perpendicular to the former. *Oblique Lines* are such as are not perpendicular to each other, and which meet if sufficiently extended.

Ill.—In *Fig.* 11, BA is perpendicular to DC; so also AC is perpendicular to BA. In *Fig.* 10, KG and KI are perpendicular to each other. The other lines in *Fig.* 10 are oblique to each other.

44. Prob.—*To erect a perpendicular to a given line at a given point in the line.*

SOLUTION.—Suppose I want to erect a perpendicular to the line XY, at the point A. With the dividers I measure off a distance AB on one side of the point A, and *an equal distance* AC on the other side. Then opening the dividers a little wider, I put the sharp point on B and make a mark with the pencil point, as at O, about where I think the perpendicular will go. Then, keeping the dividers *open just the same*, I put the sharp point on C, and make a mark intersecting the former one at O. Now, drawing a line through O and A, it is the perpendicular sought.

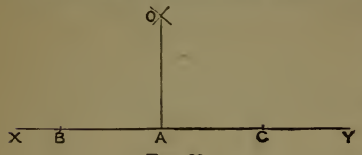

FIG. 23.

Ex. 1. Suppose I make a mistake and close up the dividers a little after making the first mark through O, and then make the second mark; which way will the line lean? Will it be a perpendicular or an oblique line in this case? What kind of an angle would OAY be? What OAX? What kind of angles are these when OA is a perpendicular?

Ex. 2. Suppose I should mistake a point nearer to A than B was taken, and use it as I did C, having the dividers open just alike when I made the two marks through O; which way would the line lean (incline)? (Same questions as in the last.)

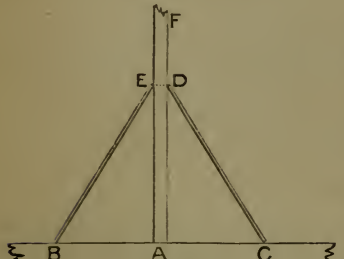

FIG. 24.

ILL.—A carpenter wishes to get the piece of timber AF at right angles to MN, into which it is mortised at A. So he measures off AB and AC, equal distances from A; and taking two poles of equal length (say 10 feet long), has the end of one held steadily at B and the end of the other at C, and moves (racks, as he calls it) the end F to the right or left until the ends E and D of the poles are exactly opposite, as in the figure. AF is then perpendicular to MN.

---

45. Prob.—*From a point without a given line, to draw a perpendicular to the line.*

SOLUTION.—I wish to draw a perpendicular from O to the line XY. I first open the dividers wide enough, so that when I place the sharp point on O the pencil will mark the line XY in two points, as B and C, when it swings around. Marking these two points, I put the sharp point first on B and afterward on C, *keeping them open just* alike in both cases, and make the two marks intersecting at D. Placing the straight edge of the ruler on the points O and D, I draw the line OA along its edge. OA is the perpendicular required.

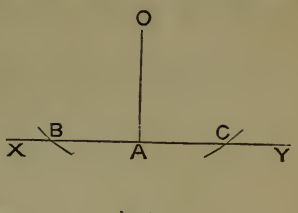

FIG. 25.

Ex. 1. Let fall a perpendicular from a point, as O, upon a straight line, as XY, without making any marks on the opposite side of XY from O.

Ex. 2. A mason wishes to build a wall from O, in the wall AB, "straight across" (perpendicular) to the wall CD, which is 8 feet from AB. He has only his 10-foot pole, which is subdivided into feet and inches, with which to find the point in the opposite wall at which the cross wall must join. How shall he find it?

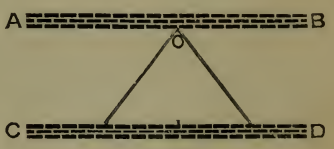

FIG. 26.

SECTION II.

ABOUT CIRCLES.

46. A Circle is a plane surface bounded by a curved line every point in which is equally distant from a point within.

47. The Circumference of a Circle is the curved line every point in which is equally distant from a point within.

48. The Centre of a Circle is the point within, which is equally distant from every point in the circumference.

49. An Arc is a part of a circumference.

50. A Radius is a line drawn from the centre to any point in the circumference of a Circle.

51. A Diameter of a Circle is a line passing through the centre and terminating in the circumference.

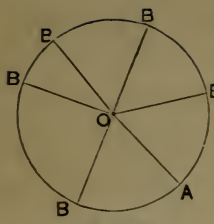

FIG. 27.

ILL.—A circle may be conceived as the path of a line, like OB, *Fig.* 27, one end of which, O, remains at the same point, while the other end, B, moves around it in the plane (say of the paper). OB is the *Radius*, and the path described by the point B is the *Circumference*. AB is a diameter. In *Fig.* 28, the curved line ABCDA (going clear around) is the *Circumference*, O is the *Centre*, and the space within the circumference is the *Circle*. Any part of a circumference as AB, or any of the curved lines BB, *Fig.* 27, is an arc. So also AM and EF, *Fig.* 29, are arcs. EF is an arc drawn from O′ as a centre, with the radius O′B.

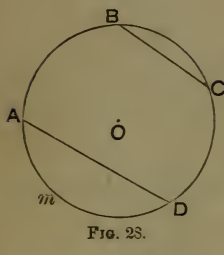

FIG. 28.

52. A Chord is a straight line joining any two points in a circumference, but not passing through the centre, as BC or AD, *Fig.* 28. The portion of the circle included between the chord and its arc, as A*m*D, is a SEGMENT.

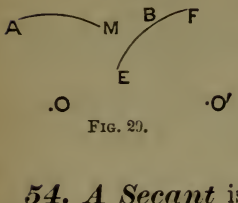

FIG. 29.

53. A Tangent to a circle is a straight line which touches the circumference, but does not intersect it, how far soever the line be produced.

54. A Secant is a straight line which intersects the circumference in two points.

Ex. 1. Suppose DC, *Fig.* 11, to represent a small wooden rod, and BA a wire stuck into it at right angles. Now if you take the end C of the rod in your fingers and place the end D on the table so that the rod shall stand upright, and then revolve the rod once around like a shaft, what will the wire describe? What the end B? What any point in BA? If you only revolve the rod a little way, what will the point B describe? What does BA represent?

Ex. 2. If you take a string, OP, and hold one end at a particular point, O, on your slate or blackboard, while with the other hand you hold the other end, P, of the string upon the end of a pencil or crayon, and then move the end P around O, making a mark as it goes, what will the mark made represent when the pencil or crayon has gone clear

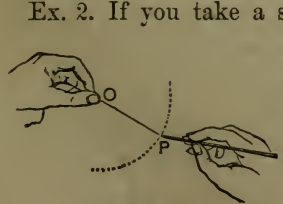

FIG. 30.

around? What will the string represent? What is the surface passed over by the string?

Ex. 3. If you take the dividers, *Fig.* 1, and open them (say 2 inches), and then place the sharp point, A, firmly on the paper while you turn them around, making the pencil point, B, mark the paper as it goes, what kind of a line will be described? What is the line joining the points of the dividers?* What line describes the circle? If the dividers only turn a little way, what is the line described?

Ex. 4. If a boy skating on the ice makes a curve which bends everywhere just alike, what kind of a path will he make? Does the boy describe a circle? How might you conceive the circle inclosed by his path, as described? Is a circle described by a point or by a line?

[NOTE.—The word "circle" is used in common language as equivalent to "circumference." It is also thus used in General Geometry. But, however the words may be used, the pupil should be taught to mark the distinction between the plane surface inclosed and the bounding line.]

Ex. 5. In how many points can a straight line intersect a circumference? In how many points can one circumference intersect another?

Ex. 6. There is a piece of ground in the form of a circle, the radius of which is 100 rods, by which run two roads; one road runs within 80 rods of the centre, and the other within 100 rods. How do the roads lie with reference to the ground?

Ex. 7. When you unwind a thread by drawing it off a spool in the ordinary way, what geometrical line does the unwound thread represent?

Ex. 8. In a circle whose diameter is 50 feet, there are drawn two chords, one is 20 feet long, and the other 30 feet. Which is nearer the centre?

Ex. 9. There are two circles whose radii are respectively 12 and 18 feet. The distance from the centre of one to the centre of the other is 25 feet. Do the circumferences intersect? Would they intersect if the centres were 3 feet apart? How would they lie in reference to each other in the latter case? How if their centres were 30 feet apart? How if they were 35 feet apart?

* The imagination may be aided by supposing a fine elastic cord stretched between the points of the dividers and carried by them.

Ex. 10. What kind of a line is represented by water flying from a swiftly-revolving grindstone?

Ex. 11. If you draw two chords in the same circle, one of which is twice as long as the other, will the arc cut off by the longer chord be twice as long as the arc cut off by the shorter? Will it be more than twice as long, or less?

---

55. Theorem.—*The chord of a sixth part of the circumference of a circle is just equal to the radius of the same circle.*

ILL.—If I draw a circle, and then, being careful not to open or close the dividers, place the sharp point on the circumference at some point, as A, and mark the circumference at another point, as B, with the pencil point, and then move the sharp point to B and mark again, as C, I find that when I have measured off six such chords, each equal to the radius, I return exactly to A, the point of starting.

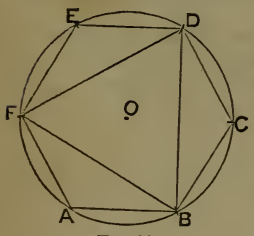

FIG. 31.

Moreover, if I draw the chords AB, BC, etc., I have a regular figure with six equal sides. A figure with six sides is called a hexagon. This hexagon is called *regular*, because its sides are equal each to each, and its angles are also mutually equal.

Again, if I unite the alternate angles of the regular hexagon, as FB, BD, and DF, I have a regular triangle, called an equilateral triangle.

56. Inscribed Figures are figures drawn in a circle, and having the vertices of all their angles in the circumference, as the hexagon and triangle in the last illustration. When the figure is without, and all its sides touch but do not cut the circumference, it is *circumscribed* about the circle.

Ex. 1. Draw a regular hexagon whose side is two inches.

Ex. 2. Inscribe an equilateral triangle in a circle whose radius is one inch.

---

57. Prob.—*To find the centre of a circle when the circumference is drawn* (or, as we usually say, *known*).

SOLUTION.—The circumference of my circle is drawn, but the centre is not

marked. So I want to find the centre. I draw any two chords, as AB and CD (the nearer they are at right angles to each other the better for accuracy). I then bisect each chord with a perpendicular, as AB with the perpendicular MN, and CD with RS (*39*). The intersection of these two perpendiculars, as O, is the centre of the circle. [The pupil must *do* everything with his pencil, ruler, and dividers, just as he says. He must not be of those who "*say* and *do* not." He must do the things told, "over and over," till he can do them neatly and easily.]

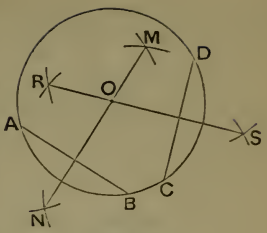

FIG. 32.

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58. Prob.—*To pass a circumference through three given points.*

SOLUTION.—I wish to pass a circumference through the three given points A, B, and C. [The pupil should first designate three points by dots on his paper, slate, or board, and then proceed according to the solution.] In order to do this, I join A and B with a line, and also B and C. I now bisect these lines with the perpendiculars MN and RS, as in the last problem. The intersection of these perpendiculars, O, is the centre of the required circle. Now setting the sharp point of the dividers upon O and opening them till the pencil point just reaches A (B or C will answer as well), I draw the circumference with O as its centre and the radius OA, and find that it passes through the three given points A, B, and C.

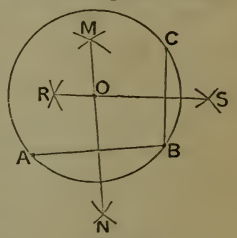
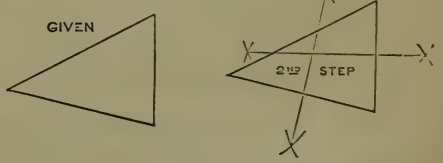


FIG. 33.

Ex. 1. To pass a circumference through the three vertices of a triangle, *i. e.*, to circumscribe a circumference about a triangle, as this operation is technically called.

SUG.—This is just like the last, A, B, and C being the vertices of the triangle. The four figures in the margin represent the successive steps in the solution. First draw the given triangle. Then take the first step in the solution, then the second, etc.



Ex. 2. Given the centre of a circle and a point in the circumference, to draw the circle.

SUG.—Make a dot on the board to indicate the centre, and another dot to indicate the point in the circumference to be found. This is what is given. You are

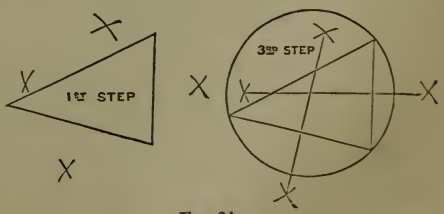


FIG. 34.

then to draw the circumference, which shall pass through the latter point, and have the former for its centre.

Ex. 3. Draw an arc of a circle, and rub out the mark, if you make any, at the centre, so that you cannot see where the centre is. Then find the centre, and complete the circumference according to these problems.

Sug.—Mark three points in the given arc, and then the example is just like the last. [Do not fail to do it, "over and over," till you can do it quickly and neatly. These exercises require much care in order to get good figures.]

---

59. *Theorem.*—*The circumference of a circle is about* 3.1416 *times its diameter. The Greek letter* π (called p) *is used to represent this number; and hence the circumference is said to be* π *times the diameter.*

Ill.—The pupil can illustrate this fact by taking any wheel which is a true circle, and measuring the diameter with a narrow band of paper (something that will not stretch), and then wrapping this measure about the circumference. He will find that it takes *a little more* than three diameters to go around. Of course he cannot tell exactly how much more. In fact, nobody knows exactly. But the number given above is near enough for most purposes. For many purposes $3\frac{1}{7}$ is sufficiently accurate.

Fig. 35.

By drawing a circle very carefully, say 1 inch in diameter, as in the margin, and dividing the diameter into 10ths inches, a nice pair of dividers can be opened one 10th inch and made to step around the circumference. If it is all done with nicety, it will be found to be a little over 31 steps around, when it is 10 across.

Ex. 1. The distance across a wagon-wheel (the diameter) is 4 feet, how long a bar of iron will it take to make the tire?

Ex. 2. Suppose the crown of your hat is a circular cylinder 7 inches in diameter, how much ribbon will it take for a band, allowing $\frac{1}{4}$ of a yard for the knot?

Ex. 3. How many times will the driving-wheel of an engine, which is 6 feet in diameter, revolve in going from Detroit to Chicago, a distance of 288 miles, allowing nothing for slipping?

Ex. 4. A boy's hoop revolved 200 times in going around a city-square, a distance of 140 rods. What was the diameter of his hoop?

Ex. 5. What is the radius of a circle whose semi-circumference is π? In a circle whose radius is 1, what part of the circumference does $\frac{\pi}{2}$ represent? What part $\frac{\pi}{4}$? What part does 2π represent?

SECTION III.

ABOUT ANGLES.

60. Prob.—*To show how angles are generated and measured.*

Ill.—An angle is generated by a line revolving about one of its extremities. Thus, suppose OB to have started from coincidence with OA, and, O remaining fixed, the line to have revolved to the position OB, the angle BOA would have been generated. When the revolving line has passed one-quarter the way around, as to DO, it has generated a right angle; when one-half way around, as to FO, two right angles; when entirely around, four right angles.

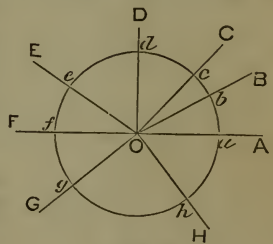


Fig. 36.

Now, if *any circle* be described from O as a centre, *the arc included by the sides of any angle having its vertex at* O, *is the same part of a quarter of this circumference as the angle is of a right angle.* Hence the angle is said to be measured by the arc included by its sides. Thus, the angle COA is measured by the arc *ac*; *i. e.*, it is the same part of a right angle that arc *ac* is of arc *ad*. (See Trigonometry, ***3–10***.)

61. Theorem.—*The relative lengths of arcs described with the same radius can be found in a manner altogether similar to that given in* (***36***) *for comparing straight lines.*

Ill.—If I wish to compare the two arcs *ab* and *cd* described *with the same radii*, I take the dividers, and placing the sharp point on *d* (one end of the shorter arc), open them till the other point is at *c*. I then measure this distance off on *ab* as many times as I can,—in this case 2 times, with a remainder *fb*. This remainder, *fb*, I measure off in the same way upon *dc*, and find it goes *once* with a remainder *gc*. This remainder, *gc*, I apply to the arc *fb*, and find it goes once with a remainder *hb*. This last remainder I find is contained in the last preceding, *gc*, 2 times. Then, counting up the parts, I find that *dc* is made up of 5 parts each equal to *hb*, and *ab* of 13 such parts. Therefore, *ab* is $2\frac{3}{5}$ times as long as *dc*. [The angle O is therefore $2\frac{3}{5}$ times the angle C.]

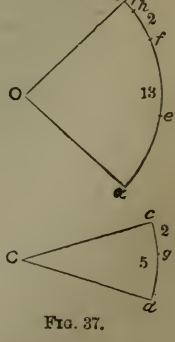


Fig. 37.

Ex. 1. Draw an acute angle and also an obtuse angle, and then compare them as above.

Ex. 2. Draw a small acute angle and a large acute one, and then compare them as above.

Ex. 3. Draw a small acute angle, and then draw another angle 3 times as large.

Ex. 4. Draw an acute angle, and also a right angle, and compare them as above.

Sug.—Article (*39*) shows how to draw a right angle.

Ex. 5. Draw any angle, and then draw another equal to it.

Ex. 6. Show that the angles *a*, *b*, and *c* are respectively $\frac{1}{5}$, $\frac{2}{5}$, and .6 of a right angle.*

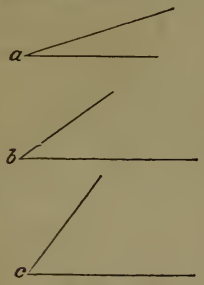


Fig. 38.

Fig. 39.

Ex. 7. Show that angles *a* and *b*, *Fig.* 39, are respectively $1\frac{1}{3}$ and $1\frac{1}{2}$ times a right angle.

Ex. 8. Draw a regular inscribed hexagon, as in *Fig.* 31, and then comparing any one of its angles with a right angle, find that it is $1\frac{1}{3}$ times a right angle.

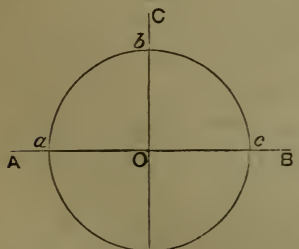


Fig. 40.

Ex. 9. Draw an equilateral triangle, as in *Fig.* 31, and find that any angle of it is $\frac{2}{3}$ of a right angle.

Ex. 10. Show that a right angle is measured by $\frac{1}{4}$ of a circumference.

Solution.—If CD is perpendicular to AB, the four angles formed are equal, and each is a right angle. But, as all of them taken together are measured by the whole circumference, one of them is measured by $\frac{1}{4}$ of the circumference.

* Of course, absolute accuracy is not to be expected in such solutions.

62. An Inscribed Angle is an angle whose vertex is in the circumference of a circle, and whose sides are chords, as A, *Fig.* 41.

63. Theorem.—*An inscribed angle is measured by one-half the arc included between its sides.*

Ill.—The meaning of this is that an inscribed angle like A, which includes any particular arc, as *cd*, is only half as large as an angle would be at the centre, as *c*O*d*, whose sides included the same arc, *cd*, or an equal arc. Thus, in this case, drawing the arc *ab* from A as a centre, with the same radius, O*d*, as *cd* is drawn with, I find that *ab* which measures A is ½ of *cd* which measures *c*O*d*.

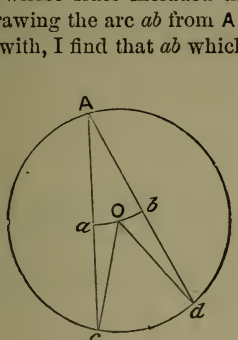


Fig. 41.

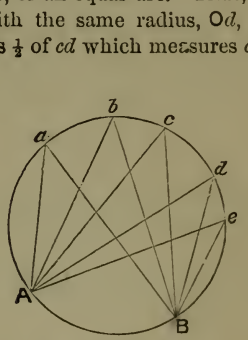


Fig. 42.

Ex. 1. Which of the angles *a*, *b*, *c*, *d*, *e* is the largest? What is *a* measured by? What *b*? What *c*? What *d*? What *e*? *Fig.* 42.

Ex. 2. Which is the greatest angle, *a*, *b*, or *c*, *Fig.* 43? By what is *a* measured? By what *b*? By what *c*? What is the measure of a right angle? [See Example 10 in the preceding set.]

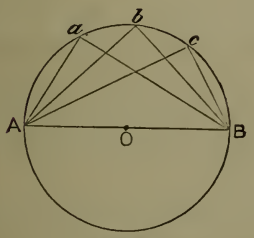


Fig. 43.

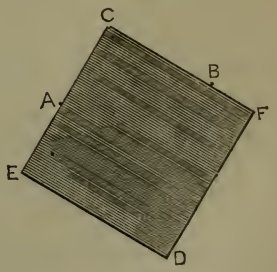


Fig. 44.

Ex. 3. Suppose I take a square card like CEDF, with a hole in one corner as at C, and sticking two pins firmly in my paper, as at A and B, place the corner of the card between them, as in *Fig.* 44, and then, keeping the sides of the card snug against the pins, put a

pencil through the hole C and move it around to A and then back to B; what kind of a line will the pencil trace? Will it make any difference whether C is a right angle or not? If any difference, what?

Ex. 4. By what part of a circumference is an angle of a regular inscribed hexagon measured? See (*55*), and *Fig.* 31. How many right angles is the angle of the hexagon equal to? What is the sum of the six angles equal to? *Ans. to last,* 8 right angles.

Ex. 5. Show, from the way in which an equilateral triangle is constructed in *Fig.* 31, that one of its angles is measured by $\frac{1}{6}$ of a circumference, and hence is $\frac{2}{3}$ of a right angle.

---

64. Theorem.— *When two lines intersect, they form either four right angles, or two equal acute and two equal obtuse angles.*

Ill.—[The pupil can illustrate this for himself by drawing lines and noticing what angles are equal.]

Ex. 1. Having a carpenter's square, an instrument represented by MON, I wish to test the angle O and ascertain whether it is, as it should be, a right angle. I draw an indefinite right line AB, and placing the angle O at some point C on this line with ON extending to the right on CB, I draw a line along OM. Turning the square over so that ON shall lie on CA, I draw another line along OM. Three cases may occur.—1st. Suppose the first line drawn along OM is CF, and the second CE; what kind of an angle is O? 2d. Suppose the first line drawn is CE and the second CF; what kind of an angle is O? 3d. Suppose the first and second lines drawn along OM coincide and are CD; what kind of an angle is O?

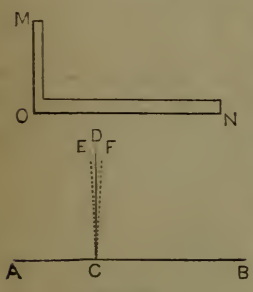


Fig. 45.

Ex. 2. Show that the sum of all the angles formed by drawing lines on one side of a given line, and to the same point in the line, is two right angles.

---

65. Prob.—*To bisect a given angle.*

Solution.—I wish to divide the angle AOB into two equal parts, *i. e.*, to

bisect it. With O, the vertex, as a centre, and any convenient radius, as O*a*, I strike an arc, as *ba*, cutting the sides of the angle. Then from *a* and *b* as centres, with the same radius in each case, I strike two arcs intersecting as at P. Drawing a line through P and O, it bisects the angle; *i. e.*, the angle POA = angle BOP. [Let the pupil try this by cutting out the angle AOB, and then folding the paper along the line P, or cutting it through in the line OP, and then putting one angle on the other, and thus see if they do not fit.]

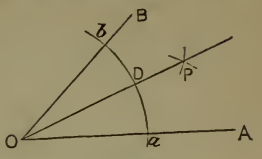


FIG. 46.

Ex. 1. Draw an angle equal to $\frac{1}{2}$ of a right angle.

SUG.—First draw a right angle and then bisect it.

Ex. 2. Draw an angle equal to $\frac{1}{3}$ of a right angle.

SUG.—Draw a circle. Inscribe an equilateral triangle. [Do it neatly, by rule, as in (*55*).] Then bisect any angle of this triangle. This will be $\frac{1}{3}$ of a right angle, since the whole angle is $\frac{2}{3}$. See Ex. 9 (*61*).

Ex. 3. How does it appear that the angle EDF, *Fig.* 31, is $\frac{1}{3}$ of a right angle?

---

66. Parallel Straight Lines are such as, lying in the same plane, will not meet how far soever they are produced either way.

ILL.—The sides of this page are parallel lines, as are also the top and bottom. The lines in *Fig.* 47 are parallel.

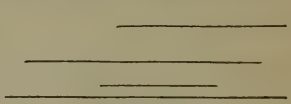


FIG. 47.

67. Prob.—*To draw a line through a given point and parallel to a given line.*

SOLUTION.—I wish to draw a line through the point O and parallel to the line AB. [The pupil should first draw some line, as AB, and mark some point, as O.] I take O as a centre, and with a radius * greater than the shortest distance to AB, as O*a*, draw an indefinite arc *a*P. Then with *a* as a centre, and *the same* radius, I draw an arc from O to the line AB at *b*. Taking the distance O*b* (the chord) in the dividers, I put the sharp point on *a* and strike a small arc intersecting this indefinite arc, as at P. Finally, drawing a line through O and P, it is the parallel sought.

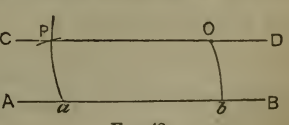


FIG. 48.

* This means "put the sharp point of the dividers on O and open them till the distance between the points (the radius) is more than the distance from O to AB."

68. Theorem.—*Two parallel lines are everywhere the same distance apart.*

ILL.—Let AB and CD be two parallel lines. I will examine them at the two points O and P. To find how far apart the lines are at these points I draw the perpendiculars OM and PN. [The pupil should not guess at these, but actually draw them as instructed in (*44*).] Measuring these, I find them equal.

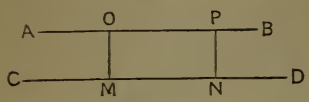


FIG. 49.

We can understand that this proposition must be true, since the lines could not approach each other for awhile and then separate more and more without being crooked; or, if they kept on approaching each other, they would meet after awhile, and so not be parallel.

---

69. Theorem.—*Parallel lines make no angle with each other.*

ILL.—Let AB be a straight line, and suppose CD another straight line passing through the point O. Now let CD turn around, first into the position D′C′, then into D″C″, etc., all the time passing through O. It is evident that the angle which this line makes with the line AB is all the time growing less, *i. e.*, $a' < a$, and $a'' < a'$. It is also evident that this angle will become 0 when the lines become parallel; for it becomes less and less all the time, but is always something so long as the lines are not parallel.

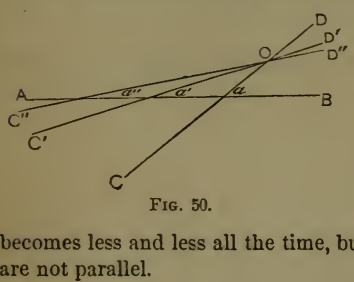


FIG. 50.

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70. Theorem.—*Parallel lines have the same direction with each other.*

ILL.—Thus, in *Fig.* 47, the parallel lines all extend to the right and left, *i. e.*, in the same direction.

EX. 1. How shall the farmer tell whether the opposite sides of his farm are parallel?

EX. 2. If we wish to cross over from one parallel road to another, is it of any use to travel farther in the hope that the distance across will be less?

EX. 3. If a straight line intersects two parallel lines, how many angles are formed? How many angles of the same size? May they *all* be of the same size? When? When will they not be all of the same size?

SECTION IV.

ABOUT TRIANGLES.

71. A Plane Triangle, or simply *A Triangle,* is a plane figure bounded by three straight lines.

72. With respect to their sides, triangles are distinguished as *Scalene, Isosceles,* and *Equilateral.* A scalene triangle has no two sides equal. An isosceles triangle has two sides equal. An equilateral triangle has all its sides equal.

73. With respect to their angles, triangles are distinguished as *acute* angled, *right* angled, and *obtuse* angled. An acute angled triangle has three acute angles. A right angled triangle has *one* right angle, and the side opposite the right angle is called the *hypotenuse.* An obtuse angled triangle has *one* obtuse angle.

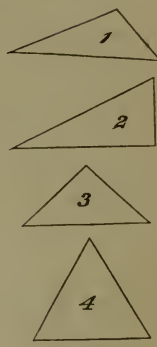

FIG. 51.

Ex. *Fig.* 51 affords illustrations of all the different kinds of triangles. Let the pupil point them out until he is perfectly familiar with the terms. He should also practise drawing the different kinds of triangles, for the purpose of familiarizing the names applied to the different kinds.

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74. Theorem.—*The sum of the angles of a triangle is two right angles.*

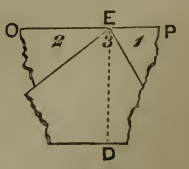

FIG. 52.

ILL.—Cut out any triangle from a piece of paper. Then cut off two of the angles, as 1 and 2, and turn them about and place them by the side of the other angle, as in the lower figure. You will then see that *the line* OP *is straight*, and that the three angles of the triangle just make up the two right angles OED and PED.

Ex. 1. If one angle of a triangle is a right angle, what is the sum of the other two?

Ex. 2. Can a triangle have more than *one* right angle? If two of its angles were right angles, what would the third angle be?

Ex. 3. Can a triangle have more than one obtuse angle?

Sug.—Try and see if you can draw a triangle with two right angles, or two obtuse angles.

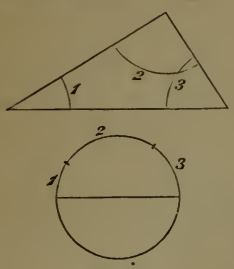

Fig. 53.

Ex. 4. Construct any triangle, and draw arcs measuring its angles. Then draw a circle *with the same radius* as the one used to measure the angles, and lay off upon the circumference the arcs measuring the angles. The sum of these arcs will always make up just a semi-circumference. What does this show?

Ex. 5. If two angles of one triangle are equal to two angles of another, can the third angles be unequal? Why?

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75. Prob.—*To make two triangles just alike.*

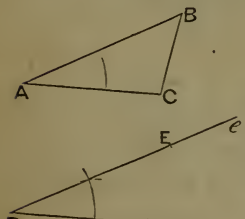

Fig. 54.

Solution.—There are three ways of doing this:

1*st Way.*—Suppose I have any triangle, as ABC, and want to make another just like it. I first draw an arc measuring any one of the angles, as A, of the given triangle. Then I make an angle D equal to the angle A, and draw the sides D*e* and D*f*. Now I measure DE = AB, and DF = AC. If I now draw EF, the triangle DEF will be just like ABC, so that, were I to cut them out, I could apply one like a pattern to the other, and it would just fit.

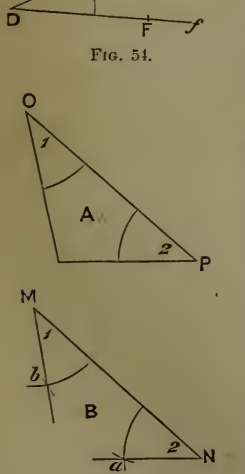

Fig. 55.

2*d Way.*—I have a triangle A, and wish to make another just like it. I draw arcs measuring any *two* of its angles, as O and P. Then, making a line MN equal to OP, I make an angle at M equal to O, and one at N, on the same side of MN, equal to P. Now making these two sides M*b* and N*a* long enough to meet (or, as we say, "producing them till they meet"), I have a second triangle, B, just like the first triangle, A. Were I to cut out the first triangle, it would fit on the second just like a pattern.

3*d Way.*—I have a triangle ACB, and want to make another just like it. I make a line DE equal to some side of the given triangle, as AB. Then taking AC as radius, I describe an arc from D as a centre, and in like manner, with BC as radius and E as a centre,

describe another arc. Through the intersections of these arcs, as F, I draw DF and EF. The triangle DEF is just like ABC. [Try it by drawing as described, and then cutting out one triangle, and seeing if you cannot fit it as a pattern on the other.]

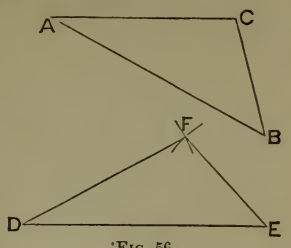

FIG. 56.

Ex. 1. In any triangle, which side is opposite the greatest angle? Which opposite the least angle?

Ex. 2. If you have two triangles with an angle in each equal, but the sides about this angle longer in one triangle than in the other, can you make one fit on the other as a pattern? Cut out two such triangles and try it.

Ex. 3. Can you make a triangle so that one of its sides shall be as long as both the others, or longer than both?

Ex. 4. Can you make a triangle so that one of its sides shall be less than the difference between the other two, or equal to the difference?

Ex. 5. If you have two triangles with *only* one side and one angle in the one equal to one side and one angle in the other, can you apply one as a pattern and make it fit on the other? Cut out two such triangles and try it.

Ex. 6. If you have two triangles with *only* two sides of one respectively equal to two sides of the other, can you make one fit as a pattern on the other? Try it.

Ex. 7. If you have two triangles with two sides in one equal respectively to two sides in the other, and the included angle in one greater than in the other, how is it with the third sides of the triangles?

---

76. Theorem.—*The lines which bisect the angles of a triangle meet within the triangle at a common point.*

ILL.—Try it, by drawing a triangle, and then bisecting its angles, as taught in (***65***). You will need to do it very neatly, or the lines will not meet. It is a delicate operation. Try it in various forms of triangles, as equilateral, right angled, scalene, obtuse angled, etc.

FIG. 57.

77. Theorem.—*The lines drawn from the vertices of a triangle to the middle of the opposite sides meet in a common point within the triangle.*

FIG. 58.

ILL.—Draw a triangle. Bisect each of the sides as taught in (***39***). Then join each angle and the middle of its opposite side with a straight line. If you do the work well, the three lines will cross each other at a common point within the triangle.

---

78. Theorem.—*The perpendiculars which bisect the sides of a triangle meet at a common point, which may be within or without the triangle, or in one of its sides, according to the form of the triangle.*

ILL.—Draw *an acute angled triangle*, and bisect its sides by perpendiculars. If you do it with accuracy, they will meet at a common point *within* the triangle.

Draw an *obtuse angled triangle*, bisect its sides with perpendiculars, and they will meet at a common point *without* the triangle.

Draw a *right angled triangle*, and the perpendiculars will meet in the side opposite the right angle (the hypotenuse).

FIG. 59.

Ex. 1. Draw an *equilateral* triangle, and find the three points characterized in the last three articles. Are they all in one place, or are they in different places?

Ex. 2. Draw a scalene triangle, and find the three points as above. Are they all in the same place, or are they in different places?

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79. Prob.—*To inscribe a circle in a given triangle.*

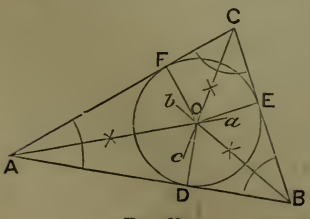

FIG. 60.

SOLUTION.—I wish to inscribe a circle in the triangle ABC; that is, a circle to which the sides of the triangle shall be tangents. [First draw the triangle.] I bisect the angles as taught in (***65***); and then from the point O, where these intersect, I let fall perpendiculars upon the sides, as taught in (***45***). Then from O as a centre, with a radius equal

to one of these perpendiculars (they are all equal), I draw a circle, and it is the circle required.

***80. Prob.**—To circumscribe a circle about a given triangle.*

SOLUTION.—I wish to circumscribe a circle about the triangle ABC which I have drawn. To do this, I *bisect the sides* with perpendiculars, and find their common intersection O, as taught in (*78*). With O as a centre and a radius equal to OB, the distance from O to the vertex of any one of the angles, as these distances are all equal, I draw a circle. This is the circumscribed circle, that is, the circle in whose circumference the vertices of the triangle lie. [This is really the same as PROB. (*58*).]

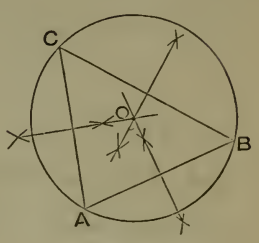

FIG. 61.

SECTION V.

ABOUT EQUAL FIGURES.

81. Equal, in geometry, signifies alike in all respects, *i. e.*, of the same shape and the same size.

82. Equivalent figures are such as have the same area, *i. e.*, are of the same size, irrespective of their form.

Ex. 1. Can a triangle be *equal* to a circle? Can it be *equivalent?* Can a circle be equivalent to a square? Can it be equal to a square?

Ex. 2. Can a right angled triangle be equal to an equilateral triangle? Can a right angled triangle be equal to an isosceles triangle? If either is possible, construct figures illustrating it.

***83. Prob.**—To apply one straight line to another.*

SOLUTION.—[*Applying* figures to each other is a very important thing in geometry, and may seem a little curious at first; but it is, in reality, very simple. The pupil must become perfectly familiar with it.] We will first apply the line AB to the equal line CD. Take the line AB,* and placing the end A upon the end C of the line CD, make the line AB take the same direction as CD, and put the former upon the latter. Now, since the lines are equal, the

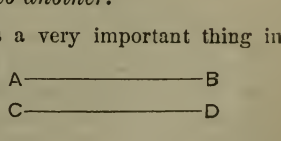

FIG. 62.

* That is, *think* about it just as if it were a little rod which you could pick up and handle.

extremity (or the point) B will fall upon D, and the two lines will coincide throughout their whole extent.

Again, we will apply the line EF to the line GH. Taking the line EF (*think* of it as a little rod which you can pick up and handle), put the point E upon G, and making the line EF take the same direction as GH, put the former upon the latter. Now, since EF is shorter than GH, the point (extremity) F will fall somewhere on the line GH, as at I. Therefore the lines do not coincide throughout their whole extent, and are not equal.

---

84. *Prob.*—*To apply one plane angle to another.*

Solution.—First we will apply one angle to another equal angle. Thus, to apply BAC to the equal angle EDF. Take the angle BAC (*think* of it as if it were two little rods put firmly together at this angle, and so that you could pick them up and handle them), and placing the vertex (point) A upon the vertex (point) D, make the side AC take the *direction* DF. As AC happens to be longer than DF, the extremity C will fall beyond F, at some point, as O. But we do not care for this, as the size of an angle does not depend upon the length of the sides. Now, while A lies on D, and the line AC on DF, let the line AB be conceived as lying in the plane of the paper also (*i. e.*, on it). Since the angle BAC is equal to EDF, the line AB will take the direction DE, and will fall on it, though the point B will fall somewhere beyond E, as at N, as AB chances to be longer than DE. The two angles therefore coincide, and are equal. [Notice carefully just what is meant by saying that the angles are equal. We do not mean that the sides are of the same length, but that the *opening* between them is the same, *i. e.*, that one is just as sharp a corner as the other.]

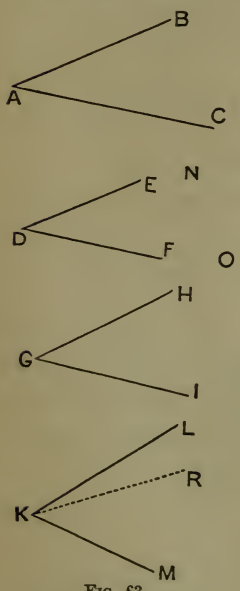

Fig. 63.

Queries.—If BAC were greater than EDF, and we should begin by putting A upon D, and make AC fall upon DF, where would AB fall, without the angle EDF or within it? If BAC were less than EDF, and we proceed as before, placing the vertex A on D, and AC on DF, would AB fall without EDF or within it?

Again, let us attempt to apply the angle HGI to LKM. Placing the vertex G on the vertex K, making the side GI take the direction KM, and then bringing GH into the plane of the paper, the side GH will fall within the angle LKM (as in the line KR), since the angle HGI is less than LKM. The angles, therefore, do not coincide.

85. Prob.— *When two triangles have two sides and the included angle of one equal to two sides and the included angle of the other, to apply one triangle to the other.*

SOLUTION.—In the two triangles ABC and DEF, let the angle A be equal to the angle D, the side AB = the side DE, and AC = DF. We will apply the triangle ABC* to DEF. Take the triangle ABC and place the vertex A upon the vertex D, making the side AC take the direction DF. Since AC = DF, the extremity C will then fall on F.† Now bring the triangle ABC into the plane of DEF, keeping AC in DF, and the line AB will take the direction DE, since the angle A = the angle D. Again, as AB = DE, the extremity B will fall upon E. Thus we have placed ABC upon DEF, so that A falls upon D, C upon F, and B upon E, and find that they exactly coincide.

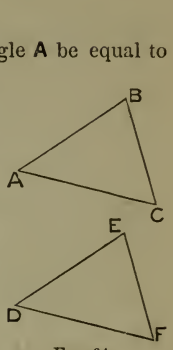

FIG. 64.

Ex. 1. Suppose you attempt to apply ABC in the last figure to DEF by placing B on D, and letting BC fall upon DF. Where will C fall? Measure it and find out. Which side will then fall nearly or quite on DE? Will it fall exactly on it? On which side will it fall? Can you make the triangles coincide (fit) in this way?

Ex. 2. Can you make the triangles in the last figure coincide by placing C upon D, and letting CA fall upon DF? Where will A fall? What line will fall on or near DE? Will it fall without DE, or within?

Ex. 3. Construct two isosceles triangles,‡ as ACB and DEF, in which AC = CB = DE = EF. Can you apply DEF to ABC by putting D upon A? Describe the process. Can you put D upon A and DE upon AB, and make the triangles coincide? Can you make the triangles coincide by putting F upon A? If so, describe the process. Can you make them coincide by putting E upon A? If not, point out the difficulties.

FIG. 65.

* Think of ABC as made of little rods, so that you can pick it up and place it upon DEF. in the manner described.

† It will make it clearer if the pupil thinks of ABC, at this stage of the operation, as having the side AC on DF, but the angle B not down on the paper; just as if he were to cut out ABC, and set the edge AC on the line DF, and *afterward* bring the triangle ABC down on to DEF, keeping the edge AC on the line DF.

‡ The teacher must insist upon the figures being drawn, and that accurately, according to rule.

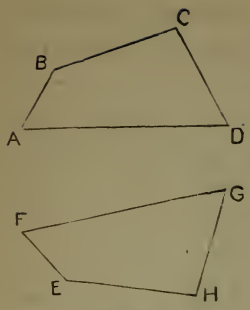

Fig. 66.

Ex. 4. Construct two equal trapeziums,* as ABCD and EFGH, and describe the process of applying one to the other.

Solution.—I will apply EFGH to ABCD. As the angle E is equal to the angle B, I will begin by putting the vertex E on B, and making EH fall upon BC. Since EH = BC, H will fall on C. Now, as angle H = angle C, HG will take the direction (fall on) CD; and since HG = CD, G will fall on D. Again, as G = D, GF will take the direction DA; and since GF = DA, F will fall on A. Finally, as F = A, FE will take the direction AB; and since FE = AB, E will fall on B, as it ought, since I started by conceiving E as placed on B.

Ex. 5. Describe the application of ABCD in the last figure to EFGH, by beginning with C upon H.

Ex. 6. Having two equal equilateral triangles, can you apply one to the other by beginning indifferently with any one angle of one upon any one angle of the other? Draw two such triangles, and go through with the details of the application.

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86. Prob.—*Given two triangles with two angles and the included side of the one respectively equal to two angles and the included side of the other, to apply one triangle to the other.*

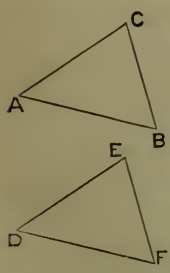

Fig. 67.

Solution.—[The pupil should first draw any triangle, as ABC. Then make a line DF equal to AB, and at the extremities D and F make angles, as D and F, respectively equal to A and B. This is preliminary.] Having the two triangles ABC and DEF, in which A = D, B = F, and AB = DF, I propose to apply one to the other. I will apply ABC to DEF. Taking ABC, I place A upon D, and make AB take the direction and fall upon DF. Since AB = DF, B will fall upon F. Now keeping the line AB in DF, I conceive the triangle ABC to come into the plane of DEF. Since A = D, the side AC will take the direction DE, and the extremity C of AC will fall somewhere in the line DE, or in DE produced. Also, since B = F, the line BC will take the direction FE, and the extremity C of BC will fall somewhere in FE or FE produced. Finally, as C falls in DE and

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* The teacher must insist upon the figures being drawn, and that accurately, according to rule.

FE both, it must be at E, their intersection. Thus I find that the triangle ABC, when applied to DEF, coincides with it throughout.

Ex. 1. Given the two triangles DEF and ABC, in which DE = AB, D = A, but E > B; show how an attempt to apply one to the other fails.

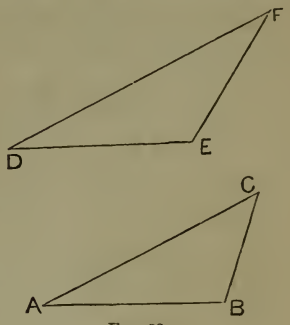

Fig. 68.

Solution.—Since angle D = angle A,* I apply the vertex D to the vertex A, and make DE take the direction AB. As DE = AB, E will fall on B, and the sides DE and AB will coincide. Again, since D = A, the side DF will take the direction AC when the planes of the triangles coincide; and the extremity F will fall in AC, or in AC produced (really in AC produced, in this case). Finally, since E > B, EF will fall to the right of BC, and the application fails.

Ex. 2. Construct two trapeziums with their respective sides equal, as AC = HE, AB = HG, BD = GF, and CD = EF, but with their angles unequal; and show how an attempt to apply one to the other fails.

Ex. 3. If the sides of two trapeziums, as in the last figure, are equal, and two of the angles including a side in one are respectively equal to the corresponding angles in the other, as A = H, and B = G, can one be applied to the other? If so, give the details of the process.

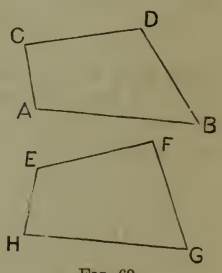

Fig. 69.

SECTION VI.

ABOUT SIMILAR FIGURES, ESPECIALLY TRIANGLES.

87. Similar Figures are such as are shaped alike—*i. e.*, have the same form.

A more scientific definition is, *Similar Figures* are such as have their angles respectively equal, and their homologous (corresponding) sides proportional.

* Be careful to distinguish between the vertex, which is a point, and the angle, which is the *opening* between the lines.

88. Homologous, or *Corresponding* ***Sides*** of similar figures, are those which are included between equal angles in the respective figures.

IN SIMILAR TRIANGLES, THE HOMOLOGOUS SIDES ARE THOSE OPPOSITE THE EQUAL ANGLES.

ILL.—The triangles ABC and DEF are similar, for they are of the same shape. But it is easy to see that ABC is not similar to IHK or MON. The pupil should notice that $A = D$, $C = F$, and $B = E$. Also, side e is $1\frac{1}{2}$ times b, side f is $1\frac{1}{2}$ times c, and side d is $1\frac{1}{2}$ times a; so that $f : c :: e : b$, and $f : c :: d : a$, and $d : a :: e : b$. Now there are no such relations existing between the parts of ABC and IHK. The angles B and K are nearly equal, but A is much larger than H, and C is smaller than I. So these triangles are *not* mutually equiangular, *i. e.*, each angle in one has not an equal angle in the other. Again, as to their sides, IH is a little less than AC, but HK is greater than AB. These two triangles are, therefore, not similar.

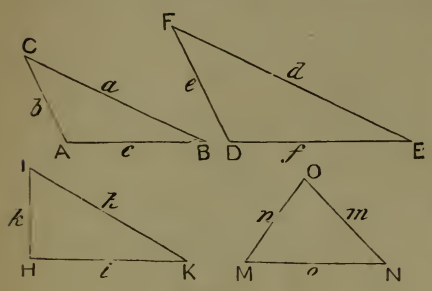

FIG. 70.

In the similar triangles ABC and DEF, b is homologous with e, since they are opposite the equal angles B and E. For a like reason a is homologous with d, and c with f. It may also be observed, that the shortest sides in two similar triangles are homologous with each other; the longest sides are also homologous with each other, and the sides intermediate in length are homologous with each other.

Ex. 1. Can a scalene triangle be similar to an isosceles triangle? Can an obtuse angled triangle be similar to a right angled triangle?

Ex. 2. Are all squares similar figures?

SUG.—First, are the angles equal? Second, is any one side of one square to some side of another square as a second side of the first is to a second side of the second, etc.?

Ex. 3. A farmer has two fields, each of which has 4 sides and 4 right angles. The first field is 20 rods by 50, and the other 40 by 80. Are they similar?

SUG.—Are they mutually equiangular? Then are the lengths in the same ratio as the widths? If they are not similar, how long would the second have to be in order to make them similar? Draw two such figures, and see if they look alike in shape.

89. Prob.—*To find a fourth proportional to three given lines.*

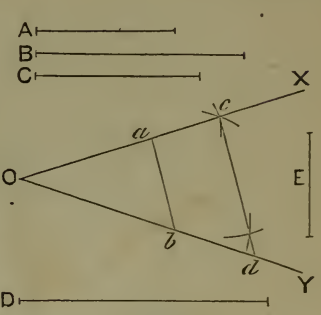

Fig. 71.

Solution.—I have the three given lines A, B, and C, and wish to find a fourth line such that

A shall be to B as C is to the *fourth line, i. e.*,

A : B :: C : *fourth line.*

To do this, I draw two indefinite lines OX and OY, from a common point O. On one of these, as OX, I lay off $Oa = A$, and $Oc = B$. Then on the other I make $Ob = C$, and draw *ab*. Finally, drawing a parallel to *ab* through the point *c* (**67**), I have *Od* as the line sought. Thus, calling *Od*, D, the proportion is

$$Oa : Oc :: Ob : Od, \text{ or}$$
$$A : B :: C : D.$$

N.B.—*The order in which the lines are taken, and the way of drawing the lines ab and cd, are essential.* The following directions will insure correctness: *Lay off the* first *and* second *on the* same line, as on OX; *and the* third *on the* other line, as on OY. *Then join the extremities of the* first *and* third, *and draw the parallel through the extremity of the* second.

Ex. 1. Show that if the order of the proportionals in *Fig.* 71 is B : A :: C : *fourth line,* the fourth proportional is E, *Fig.* 71.

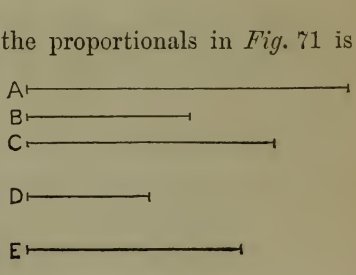

Fig. 72.

Ex. 2. Show that a fourth proportional to A, B, and C is D. Also, that a fourth proportional to C, A, and B is E. Show that, if the order be A : C :: B : *fourth line,* D is still the fourth proportional. Show that B : C :: A : 2C, nearly.

Ex. 3. Solve the proportion $3 : 8 :: 5 : x$, and find x geometrically.

Sug.—Using the scale of 100ths of a foot, the figure is that in the margin. OD is the fourth proportional, or $x = OD$, which is found *by measurement* to be $13\frac{1}{3}$, as it is by arithmetic.

Fig. 73.

90. Prob.—*To draw a triangle similar to a given triangle, and having a given side.*

Solution.—*1st Method.*—I have a triangle ACB, and want to make another similar to it, but having the side homologous to BC equal to *a*. I draw an indefinite line, and on it take EF, equal to *a*. Then at F I make an angle equal to C, and make the side indefinite. Now I find a fourth proportional to BC, EF, and AC. Having found this, as in the last article, I lay it off from F, as FD. Drawing DE, I have DEF, the triangle required.

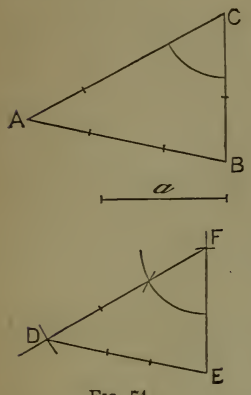

Fig. 74.

I can readily satisfy myself that DEF is similar to ABC, for besides the fact that it looks as if it were of the same shape, by measuring the other two angles, I find that E = B, and D = A. Moreover, I know that BC, EF, AC, and DF are proportional, because I made them so. And, by finding a fourth proportional to BC, EF, and AB, I find it exactly equal to DE. In like manner constructing a fourth proportional to AC, DF, and AB, I find it to be DE. So that the two triangles are mutually equiangular, and have their homologous sides (those opposite the equal angles) proportional. Hence, the triangles are similar.

2d Method.—But an easier way to construct DEF is to make the angle F = C as before, and then make E = B, and produce the sides till they meet in D. The triangles will then be similar, and the proportionality of the sides can be tested.

Ex. 1. Given a triangle whose sides are 7, 11, and 15, to construct a similar triangle having the side corresponding to the one which is 11 in the given triangle, 8.

Ex. 2. Construct two triangles with equal angles, and then compare the sides, and see whether you can make two triangles whose angles shall be respectively equal, and their sides not be proportional.

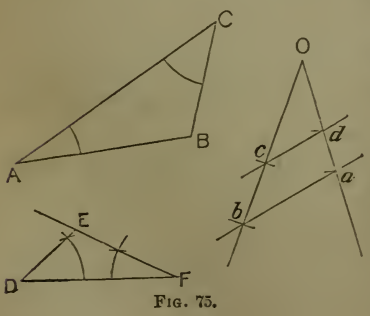

Fig. 75.

Sug.—Having the triangle ABC, make DEF equiangular with it, and then compare the homologous sides. In the figure D is made equal to C, and F to A; whence E = B. DE and BC are homologous sides, because opposite the equal angles F and A. DF is homolo-

gous with AC, because it is opposite angle E, which equals B. For a similar reason EF is homologous with AB. Now, taking two sides of ABC, as BC and AB, and a side of DEF homologous with one of them, as DE, and finding a fourth proportional Oc, it will be found exactly equal to EF; so that

$$BC : DE :: AB : EF \ (= Oc).$$

Ex. 3. Make two triangles, two of whose angles shall be, one $\frac{2}{3}$ and the other $\frac{1}{4}$ of a right angle; but make the side included between these angles twice as great in the second triangle as in the first. What will be the ratio of the side opposite the angle $\frac{2}{3}$ in the first triangle to the homologous side in the second? What the relation of the sides opposite the angles $\frac{1}{4}$?

Ex. 4. If you make one triangle whose sides are 5, 8, and 3; and a second whose sides are 15, 24, and 9, will they be mutually equiangular? Which angles are the equal ones? Which are the homologous sides?

Ex. 5. There are three pairs of similar triangles in *Fig.* 76. Can you point them out? Also point out their homologous parts. Are all the triangles which you can make out from the figure similar to each other?

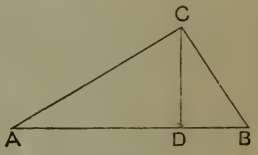

FIG. 76.

Ex. 6. Wishing to know the height EC of a house, I set up a stake DB 5 feet long; and putting my eye close to the ground, I moved back from the stake to A, so that the top of the stake and the top of the house were just in range (in a line). Then by measuring I found AB = 10 feet, and AC = 80 feet. What was the height of the house?

FIG. 77.

Ex. 7. If you take three sticks of different lengths and put them together by joining their ends two and two, so as to represent a triangle; can you, by putting together the same sticks in a different order, make a triangle of different form from the first? Will the angles opposite the same sticks always be the same?

Ex. 8. If you take more than three sticks (say 4), and make of them the boundary of a figure, by putting their ends together two and two, can you put them together so as to make another figure of different form? Can you make figures having different angles?

Ex. 9. If you take three sticks, A 3 inches long, B 5 inches,

and C 6 inches; and also three other sticks, D 9 inches long, E 15 inches, and F 18 inches;* can you place them together so as to make dissimilar triangles? Will the corresponding angles of the two triangles be equal however you may arrange the sticks? If the sides of two triangles are proportional, will their angles be equal and the triangles similar?

Ex. 10. If you take four sticks, A 3 inches long, B 5 inches, C 6 inches, and K 4 inches; and also four other sticks, D 9 inches long, E 15 inches, F 18 inches, and L 12 inches;* can you place them together so as to make four-sided figures which shall be dissimilar (*i. e.*, not of the same shape)? Will the corresponding angles of the two figures be necessarily equal? If the sides of a four-sided figure are proportional, does it follow that the corresponding angles are equal, and the figures similar?

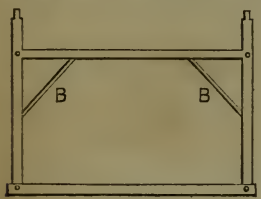

FIG. 78.

Ex. 11. Why do the braces in the frame of a building stiffen it? Is a four-sided figure stiff? *i. e.*, are its angles incapable of change while its sides remain of the same length? Can the angles of a triangle be changed while the sides remain unchanged?

SECTION VII.

ABOUT AREAS.

91. A Quadrilateral is a plane surface inclosed by *four* right lines.

92. There are three *Classes* of quadrilaterals, viz., *Trapeziums*, *Trapezoids*, and *Parallelograms*.

93. A Trapezium is a quadrilateral which has no two of its sides parallel to each other.

94. A Trapezoid is a quadrilateral which has but two of its sides parallel to each other.

* Notice that the sides are proportional, *i. e.*, in the same ratio taken two and two.

95. A Parallelogram is a quadrilateral which has its opposite sides parallel.

96. A Rectangle is a parallelogram whose angles are right angles.

97. A Square is an equilateral rectangle.*

98. A Rhombus is a parallelogram whose angles are not right angles, and all of whose sides are equal.

99. A Rhomboid is a parallelogram whose angles are not right angles, and two of whose sides are greater than the other two.

ILL.—The figures in the margin are all quadrilaterals. A is a trapezium. (Why?) B is a trapezoid. (Why?) C, D, E, and F are parallelograms. (Why?) D and E are rectangles, although D is the form usually referred to by the term rectangle. So C is the form usually referred to when a parallelogram is spoken of, without saying what kind of a parallelogram. C is also a rhomboid. (Why?) E is a square. (Why?) F is a rhombus. (Why?) This page is a rectangle; so also are the common panes of glass.

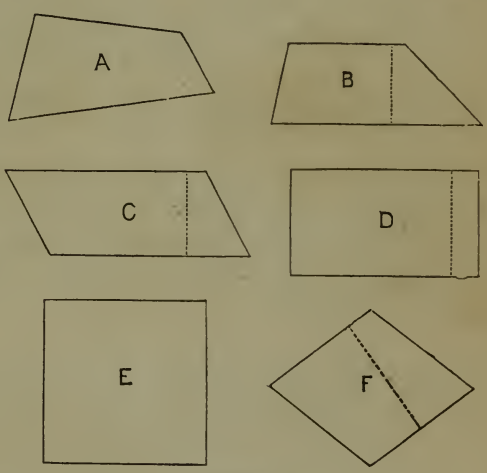

FIG. 79.

100. A Diagonal is a line joining two angles of a figure, not adjacent.

ILL.—In common language, a diagonal is a line running "from corner to corner."

Ex. 1. To construct a square, having given a side; or, in other words, to construct a square on a given line.

* The pupil should be able to give this and all similar definitions *at length*. Thus, A Square is a surface inclosed by four equal right lines making right angles with each other.

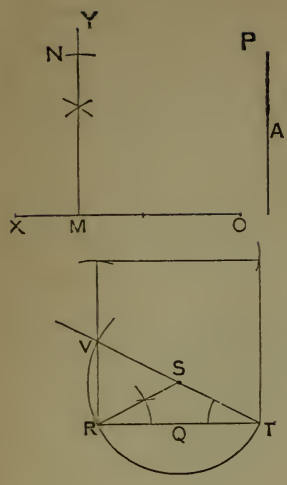

FIG. 80.

1st Method.—Let A be the given side. Draw the indefinite line OX, and lay off OM = A. At M erect a perpendicular MY, as taught in (**44**). On this take MN = A. From N and O as centres, with a radius equal to A, describe arcs intersecting, as at P. Draw NP and PO.

2d Method.—Let Q be the given side. Construct equal angles at the extremities of Q, and produce the sides till they meet, and one of them till it will meet another side of the square proposed. With S as a centre, and ST or SR as radius, describe a semicircle. Draw RV, and it forms a right angle at R. The construction can now be finished as before.

Ex. 2. Construct a rhombus whose side is 2 inches, and one of whose acute angles is $\frac{2}{3}$ of a right angle.

Ex. 3. Construct a rectangle whose adjacent sides are 3 and 5.*

Ex. 4. Construct a rhomboid whose adjacent sides are 3 and 7, and their included angle $\frac{1}{2}$ a right angle.

Ex. 5. How many diagonals has a triangle? How many has a quadrilateral? How many has a figure with five sides (a pentagon)? Of six? Of eight?

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101. The Area of a surface is the number of times it contains some other surface taken as a unit of measure; or it is the ratio of one surface to another assumed as a standard of measure.

102. The *Unit of Area* usually assumed is a square, a side of which is some linear unit: thus, a *square inch*, a *square foot*, a *square yard*, a *square mile*, etc. By these terms is meant a square 1 inch on a side, one foot on a side, one yard on a side, etc.

The acre is an exception to the general rule of assuming the square on some linear unit as the unit of area, there being no linear unit in use whose length is the side of a square acre.

ILL.—The area of a board is the number of squares 1 foot on a side which it would take to cover it. The area of a floor may be spoken of in square yards, and is the same as the number of square yards of carpeting it would take to cover it.

* Take any convenient unit, as $\frac{1}{4}$ inch, 1 inch.

103. The Altitude of a parallelogram is the distance between its opposite sides; of a trapezoid, it is the distance between its parallel sides; of a triangle, it is the distance from any vertex to the side opposite or to that side produced.

104. The Bases of a parallelogram or of a trapezoid are the sides between which the altitude is conceived as taken; of a triangle, it is the side to which the altitude is perpendicular.

ILL.—The dotted lines in B, C, D, and F, *Fig.* 79, represent altitudes. When the altitude is the distance between two parallels, the figure has two bases. The altitude of a parallelogram may be reckoned between either pair of parallel sides, but it is most common to conceive it as the distance between the two longer sides. The altitude of a rectangle is the same as either side to which it is parallel. A triangle may have three altitudes, and any side of a triangle may be conceived as its base. In *Fig.* 81, AB is conceived as the base in each case, and CD the altitude.

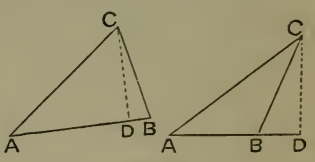

FIG. 81.

Ex. What side of a triangle must you conceive as the base, in order that the altitude shall fall upon it, and not upon its prolongation? From what angle will the altitude be reckoned in such a case?

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105. Theorem.—*The area of a rectangle is the product of its two adjacent sides; or, what is the same thing, the product of its altitude and base.*

ILL.—Let ABCD represent a rectangle, of which AB is 8 units long, and AC 5. Now, let us conceive a square *a* constructed on one of these units. Using this surface as the unit of area, it is evident that in the rectangle *c*AB*d* there will be 8 such. Hence, the area of this rectangle is 8 (square units). Now, drawing parallels to the base through the several points of division of the altitude, it is evident that the whole rectangle ABCD is made up of as many rectangles like *c*AB*d* as there are units in the altitude—in this case 5. Hence the whole area is 5 times the area of *c*AB*d*, *i. e.*, 5 times 8 (square units) = 40 (square units).

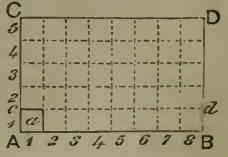

FIG. 82.

N.B.—*The pupil should be careful to observe that the language "product of base into altitude," is only a convenient form of abbreviated expression.* It is

just as absurd to talk about multiplying a line by a line, as to talk about multiplying dollars by dollars. Thus 8 inches in length can be taken 5 times, and makes 40 inches in length. But what does 8 inches in length, multiplied by 5 inches *in length* mean? Or what is 8 dollars taken 5 *dollars* times? The multiplier must always be an abstract number, and the product be like the multiplicand, from the very nature of multiplication. With this the explanation given above agrees. When we say that the area of ABCD $= 8 \times 5$, we mean 5 times 8 square units, which equals 40 square units.

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106. Theorem.—*The area of any parallelogram is the same as the area of a rectangle having the same base and altitude as the parallelogram, and hence is the product of its base and altitude.*

ILL.—This truth is easily illustrated by cutting out a parallelogram, as ABCD. Then, cutting off the triangle DEC, being careful to make DE perpendicular to BC, and placing DC upon AB so as to bring the triangle DEC into the position AFB, the two parts will just make up the rectangle AFED. Hence we see that the area of ABCD is the same as the area of AFED, which latter is a rectangle having the same base AD, and the same altitude ED, as the given parallelogram.

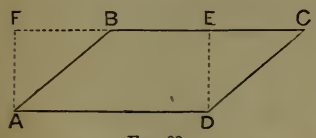

FIG. 83.

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107. Theorem.—*The area of a triangle is half the product of its base and altitude.*

ILL.—To illustrate this truth, cut out two triangles A and B *just alike*. By placing them together, a parallelogram can be formed whose base and altitude are the same as the base and altitude of the triangle. The area of the parallelogram is the product of its base and altitude. Hence the area of one of the triangles is one-half the product of its base and altitude.

FIG. 84.

In fact, by cutting one of the triangles, as A, into two triangles, its parts can be put with B so as to make a *rectangle* having the same base and altitude as the triangles. [The pupil should do it.]

---

108. Theorem.—*The area of a trapezoid is the product of its altitude into the line joining the middle points of its inclined sides.*

ILL.—To illustrate this truth, cut out any trapezoid, as ABCD, and through

the middle of the inclined sides, as *a* and *b*, cut off the triangles A*am* and B*bn*, being careful to cut in lines *am* and *bn* perpendicular to the base. These can be applied as indicated in the figure, so as to fill out the rectangle *omnp*. Hence we see that the area of the trapezoid is just equal to the product of its altitude into the line joining the middle points of its inclined sides, as *ab*.

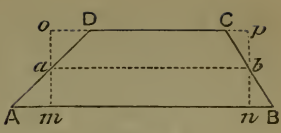

FIG. 85.

Ex. 1. How many square yards of plastering in the walls of a room 20 feet by 30, and 15 feet high, including the ceiling?

Ans. 233⅓.

Ex. 2. A salesman is selling a piece of velvet which is worth $8 per yard. The velvet is cut "on the bias," as the technical phrase is, *i.e.*, obliquely, instead of square across. The piece he is selling is measured along the selvedge in the usual way half a yard. He is disposed to charge the customer somewhat more than $4. Is he right? The customer claims that he is getting but half a yard of velvet, and so ought to pay but $4. Is he right?

Ans. Both are right,—the salesman in his demand, and the customer in his statement. How is it?

Ex. 3. There are two parallel roads one mile apart. A has a farm which extends along one of the roads half a mile, and the lines run perpendicularly from one road to the other. B has a farm lying between the same roads, and half a mile front on each road, but running obliquely across. Which has the larger farm?

Ex. 4. Of the four triangles ACB, ADB, AEB, and AFB, *Fig.* 86, which has the greatest area, CF being parallel to AB?

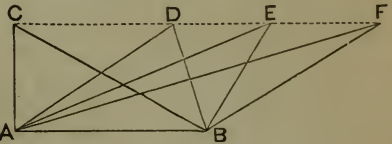

FIG. 86.

Ex. 5. Which is the largest triangle which can be inscribed in a semicircle, having the diameter for its base?

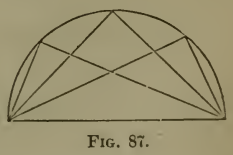
FIG. 87.

Ex. 6. Can you vary the area of a triangle while the sides remain of the same length? Can you vary the area of a quadrilateral while the sides remain of the same length?

Ex. 7. If you have two lines each 5 inches long, and two each 3 inches long; into what kind of a parallelogram must you form them in order to have its area the greatest?

Ex. 8. Rough boards are usually narrower at one end than at the other, for which reason the lumberman measures their width in the middle. What is the number of square feet in the following:

12 boards 16 feet long, 10 inches wide (in the middle);
15 boards 11 feet long, 9 inches wide " " ;
8 boards 10 feet long, 13 inches wide " " ?

What principle is involved in such measurement?

Ex. 9. What is the area of a triangle whose altitude is 6 feet, and base 10 feet? Are these elements sufficient to fix the *form* of the triangle?

Ex. 10. If a line be drawn from any angle of a triangle to the middle of the opposite side, what is the relation of the areas of the two partial triangles? Why?

THE PYTHAGOREAN PROPOSITION.

109. Theorem.—*The square described on the hypotenuse of a right angled triangle is equivalent to the sum of the two squares described on the other two sides.*

Ill.—The meaning of this proposition may be illustrated thus: Let ABC be a right angled triangle, right angled at C, and the sides AC and CB be 4 and 3 respectively. Then measuring AB, it will be found to be 5, and we observe that $4^2 + 3^2 = 5^2$. This is also seen from the figure, in which the square on AC contains $4^2 = 16$ square units, and that on CB $3^2 = 9$; while that on AB contains $5^2 = 25$, *i. e.*, as many as on both the other sides. We cannot so readily *illustrate* the truth of the proposition when the ratio of the sides is any other, than that of 3, 4, and 5, but it is equally true in all cases, as will be proved in the next part of this book.

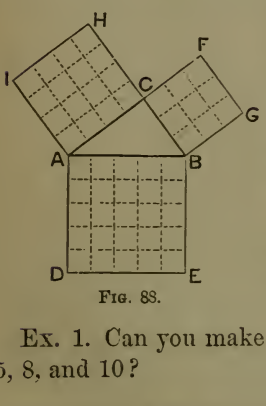

Fig. 88.

Ex. 1. Can you make a right angled triangle whose sides shall be 5, 8, and 10?

Sug.—As 10 is the longest side, it will have to be the hypotenuse. Now $5^2 + 8^2 = 25 + 64 = 89$. But $10^2 = 100$. Hence, 10 is too long for the hypotenuse of a right angled triangle whose other sides are 5 and 8.

Ex. 2. Can you make a right angled triangle whose sides shall be 9, 12, and 15?

Ex. 3. A carpenter has framed the four sills of a building together, and placed them on the foundation. He then wishes to adjust them so that the angles shall be right angles. He places one end of his ten foot pole ab at a, 6 feet from c; and, holding it in position, orders his attendants to move the sill AB to the right. How far will the end b of the pole be from c when the angle B is a right angle?

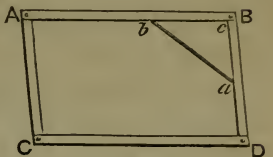

FIG. 89.

Ex. 4. A gate is to be 10 feet long and 4 feet high. How long must the brace be to go in as a diagonal and hold the gate in the form of a rectangle?

Ex. 5. The angles of a room are all right angles, and its dimensions are 20 feet by 30 on the floor, and 15 feet high. What is the length of the longest diagonal extending from one corner on the floor to the opposite corner in the ceiling?

Ans. A little more than 39 feet.

Ex. 6. The numbers 3, 4, and 5 are much used by artizans as parts of a right angled triangle. Will any equi-multiples of them answer the same purpose, as twice them, *i. e.*, 6, 8, and 10; or three times them, as 9, 12, and 15, etc.?

Ex. 7. In an obtuse angled triangle, is the square of the side opposite the obtuse angle greater or less than the sum of the squares of the other two sides? How is it with the square of the side opposite an acute angle?

SUG.—In the right angled triangle ABC, $\overline{AC}^2 = \overline{CB}^2 + \overline{AB}^2$. In the obtuse angled triangle C′B is equal to CB in the right angled triangle. But $\overline{AC'}^2$ is greater than $\overline{AC}^2$; hence $\overline{AC'}^2 > \overline{BC'}^2 + \overline{AB}^2$. By a similar inspection the other case may be determined.

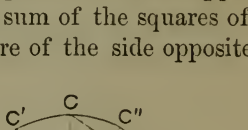

FIG. 90.

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110. Prob.—*To find a mean proportional between two lines.*

SOLUTION.—I wish to find a mean proportional between the lines M and N, *i. e.*, a line x, such that

$$M : x :: x : N, \text{ whence } x^2 = M \times N, \text{ and}$$
$$x = \sqrt{M \times N}.$$

I draw a line AB equal to the sum of M and N, making DB = M, and AD = N. I draw a

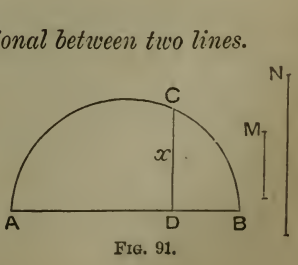

FIG. 91.

semicircumference on AB, and at D erect CD perpendicular to AB. CD is x, the mean proportional required.

Ex. 1. To construct a square which shall be equal in area to a given rectangle.

Sug.—Draw any rectangle. Then find a mean proportional between its adjacent sides as described above. A square constructed on this line will be equal in area to the rectangle; since, if x is the side of the square, and M and N are the adjacent sides of the rectangle, $x^2 = M \times N$. But x^2 is the area of the square, and $M \times N$ is the area of the rectangle.

Ex. 2. To find the square root of 15 by means of the ruler and compasses.

Sug.—Since $15 = 3 \times 5$, if DB = 3 and AD = 5, *Fig.* 91, x (CD) $= \sqrt{3 \times 5} = \sqrt{15}$. Therefore, making a figure having DB and AD of these lengths, CD can be measured, and thus the square root of 15 obtained, approximately, in numbers.

N. B.—*In such a case* CD *represents exactly the required root, although we may not be able to express the value exactly in numbers.* In this case geometry does exactly what arithmetic can only do approximately.

Ex. 3. Draw a line which shall represent, exactly, the square root of 5.

Sug.—Make DB = 1, and AD = 5.

Ex. 4. Draw a rectangle whose adjacent sides are 2 and 3, and then draw a square of the same area.

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111. Theorem.—*The areas of similar triangles are to each other as the squares of their homologous sides.*

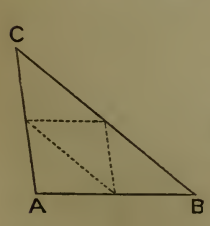

Fig. 92.

Ill.—The meaning of this is, that if ABC and DEF are similar, and any side of ABC is 2 times as great as the homologous side of DEF (as is the case in the figure, CB being = 2FE, CA to 2FD and AB to 2DE) the area of ABC is 4 times the area of DEF. In fact, in a simple case like this, we can divide ABC into four triangles exactly equal to DEF, as is done by the dotted lines.

Ex. 1. A and B have triangular pieces of land, which are similar to each other, and similarly situated. But A's front is to B's as 5 to 3; how much more land has A than B?

Ans. $2\frac{7}{9}$ times as much.

Ex. 2. In order that one triangle may be similar to and 4 times as great as another, how must any side of the first compare with the homologous side of the second?

Ex. 3. In order that the areas of two similar triangles may be to each other as 4 to 9, what must be the ratio of their homologous sides?

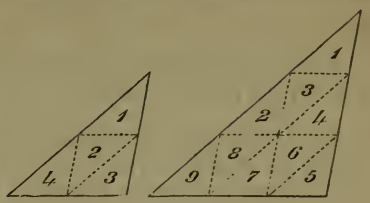

Fig. 93.

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112. Theorem.—*The homologous sides of similar triangles are to each other as the square roots of their areas.*

This theorem is involved in the theorem that the areas of similar triangles are to each other as the squares of their homologous sides. It is illustrated in the preceding examples.

Ex. Construct a triangle with one of its sides 2 in length. Then construct a similar triangle $1\frac{1}{2}$ times as large. What must be the length of the side of the second triangle which is homologous with the side 2 of the first.

Solution.—Let CAB be the given triangle, whose side AB is 2. Since the second is to be $1\frac{1}{2}$ times as great as the first, the ratio of the areas is 2 : 3. Hence, $\sqrt{2} : \sqrt{3}$ :: AB (or 2) : x, the side of the required triangle homologous with side 2 of the given triangle. Construct the square roots of 2 and 3, as ab and ac in the figure, and then find a fourth proportional to ab, ac, and AB. This is found to be ay. Taking DE = ay, construct on it a triangle DEF similar to ABC, and it will be $1\frac{1}{2}$ times as large.

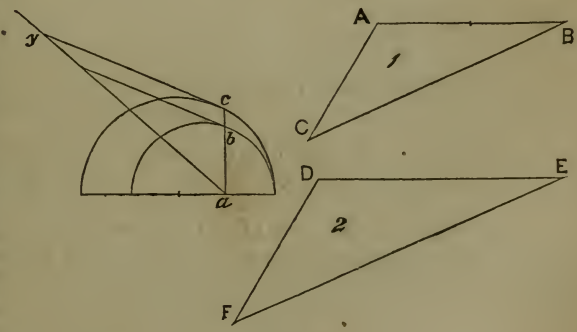

Fig. 94.

THE AREA OF A CIRCLE.

113. Theorem.—*The area of a circle whose radius is r, is* πr^2, *i. e.,* 3.1416 *times the square of its radius.*

Ill.—If we take a circle whose radius is r and circumscribe about it a square ABCD, we observe that the area of this square is $4r^2$. Hence we see that the area of a circle is *less* than 4 times the square of its radius. Again, drawing two diameters EF and GH at right angles to each other, and joining their extremities, we have the inscribed square GEHF. The area of this square is equal to the area of the two triangles GEF and EHF. But area GEF $= \frac{1}{2}$GO $\times$ EF $= \frac{1}{2}r \times 2r = r^2$; and in like manner EHF $= r^2$. Hence area GEHF $= 2r^2$. We thus see that the area of a circle is *more* than two times the square of the radius. The area of a circle is therefore somewhere between two and four times the square of its radius. Just how many times r^2 the area is, we do not propose to find in this place, but only say that it has been found to be 3.1416 times r^2. We must also remark that this is not *exact;* but it is near enough for practical purposes. In fact, nobody knows exactly how many times the square of the radius the area of a circle is.

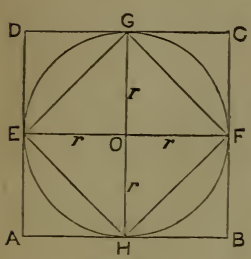

Fig. 95.

Ex. 1. If you cut from a square the largest possible circle, show that you cut away a little less than $\frac{1}{4}$ of the square, or more exactly .2146 of it.

Ex. 2. What is the area in acres of a circle whose diameter is 3 miles? *Ans.* 4523.904.

Ex. 3. A horse is so tied to a tree that he can graze on every side of it to a distance of 100 feet. What is the area in acres over which he can graze? *Ans.* A little less than $\frac{3}{4}$ of an acre.

Ex. 4. What is the area of a circle whose radius is 1?

[Remember this result.]

Ex. 5. What is the area of a circle whose radius is 2? 3? 4? How do these areas compare with the area of a circle whose radius is 1?

---

114. Theorem.—*The areas of circles are to each other as the squares of their radii.*

Ill.—This is readily seen from the last theorem. Thus the area of a circle whose radius is 5 is 25π; and of one whose radius is 6, the area is 36π. Now,

the ratio of these areas $25\pi : 36\pi$ is the same as 25 : 36, *i. e.*, as the squares of the radii of the two circles.

Ex. 1. In the figure the radius of the outer circle is twice that of the inner. How do their areas compare? How do the 4 parts into which the larger circle is divided compare with each other?

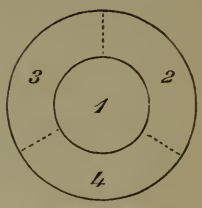

FIG. 96.

Ex. 2. The radii of 2 circles are 3 and 5 respectively; what is the relation of their areas?

Ans. 9 : 25; or one is $2\frac{7}{9}$ times as large as the other.

Ex. 3. I have a circle whose radius is 5, and wish to make another whose area is twice as great; what must be its radius?

Ans. $\sqrt{50}$, or 7.071 nearly.

Ex. 4. Can we compare the areas of circles by means of the squares of their diameters as well as by means of the squares of their radii? How much greater is the square of the diameter of any circle than the square of the radius?

Ex. 5. Two 5-inch stovepipes run together into one 7-inch pipe. Is the capacity of the one pipe equal to that of the two?

Ex. 6. Two men bought grindstones of equal thickness. The stones cost \$4 and \$9 respectively. One was 2 feet in diameter and the other 3. What was the difference in the rates paid?

SECTION VIII.

OF POLYGONS.

115. A Polygon is a portion of a plane bounded by straight lines.

The word polygon means many-angled; so that with strict propriety we might limit the definition to plane figures with five or more sides. This limitation in the use of the word is frequently made.

116. A polygon of three sides is a *triangle;* of four, a *quadrilateral;* of five, a *pentagon;* of six, a *hexagon;* of seven, a *heptagon;* of eight, an *octagon;* of nine, a *nonagon;* of ten, a *decagon;* of twelve, a *dodecagon.*

117. A Regular Polygon is a polygon whose sides are equal each to each, and whose angles are equal each to each.

118. The Perimeter of a polygon is the distance around it, or the sum of the bounding lines.

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119. Theorem.—*Any polygon may be divided by diagonals drawn from any angle, into as many triangles as the polygon has sides, less two sides.*

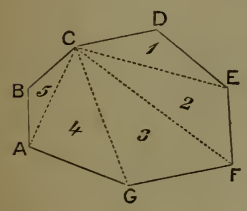

FIG. 97.

ILL.—In the figure the polygon has 7 sides. By drawing the diagonals from C to the other angles, we divide the polygon into 5 (7—2) triangles.

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120. Theorem.—*The sum of the angles of any polygon is twice as many right angles as the polygon has angles (or sides), less four right angles.*

ILL.—Draw a polygon, as ABCDEFG, and the arcs a, b, c, d, e, f, g, measuring its angles. *With the same radius* draw a circle. Beginning at some point, as O, lay off OA = a, AB = b, BC = c, CD = d, DE = e, EF = f, and FG = g. It is found in this case that the sum of these measures is two circumferences and a half. Now, one circumference is the measure of 4 right angles. Hence, $2\frac{1}{2}$ circumferences measure $2\frac{1}{2} \times 4 = 10$ right angles. Thus it appears that the sum of all the angles of the polygon is 10 right angles. This agrees with the theorem; for, by that, the sum should be 2 *right angles* × 7 — 4 right angles, which is 10 right angles.

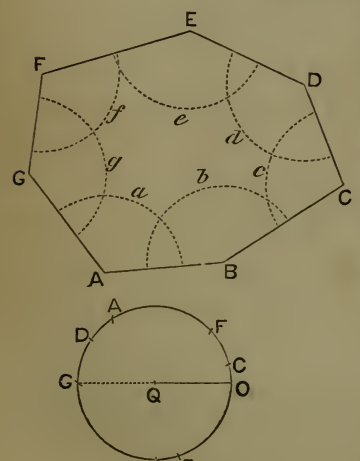

FIG. 98.

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121. Prob.—*To draw a regular polygon.*

SOLUTION.—Draw a circle, and divide the circumference into as many *equal arcs* as the polygon has sides. The chords of these arcs will constitute the perimeter of the polygon.

The practical difficulty lies in dividing the circumference as required. The circumference can be divided into 6 equal arcs by (*55*). Drawing radii to these points of division, and bisecting the included angle, a division into 12 equal parts is effected. These can be again bisected, and the division into 24 equal parts effected, etc. Again, the circumference can be divided into 4 equal parts by drawing two diameters at right angles to each other (see *Fig.* 95). These arcs can be bisected as indicated above, and the division into 8 equal parts effected. Bisecting the latter arcs, we have 16 equal parts, etc. There is also a way to divide the circumference into 10 equal parts, but it is too difficult to be given here. For all regular polygons except those of 3, 6, 12, 24, etc., and 4, 8, 16, etc., sides, the pupil, at this stage of his progress, is expected to effect the division *by trial.*

EXERCISES.

1. By drawing diagonals from any one angle, into how many triangles can a pentagon* be divided? Show it with a figure. Into how many an octagon? A dodecagon? A nonagon? A hexagon?

2. What is the sum of the angles of a hexagon? Determine the number mentally, and then measure the angles geometrically, as in the solution of (*120*), observing that the latter result verifies the former. In like manner determine the sum of the angles of a pentagon. Of an octagon. Of a decagon. Of a nonagon. Of a triangle. Of a quadrilateral.

3. If the angles of a hexagon are equal each to each—that is, if the hexagon is equiangular—what is the value of any one angle?

Ans. $1\frac{1}{3}$ right angles.

[Note.—A regular polygon is equiangular.]

4. What is the value of any angle of a regular octagon? Of a regular pentagon? Of a regular dodecagon?

Answer to the last, $1\frac{2}{3}$ right angles.

5. Construct a regular dodecagon.

6. Construct a regular heptagon.

Sug's.—Observing that as the chord for the *hexagon* is the radius, and hence the chord for the heptagon is *a little less*, we can readily find *by trial* just how wide to open the dividers so that they shall step around the circumference at 7 steps. This is not a very scientific way of constructing a figure, it is true, but it is the only way we can get the chord in this case.

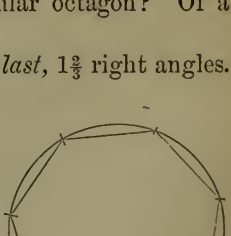


Fig. 99.

* Polygons are not to be assumed regular unless they are so designated.

7. Construct a regular octagon.

SUG.—See the general solution (*121*).

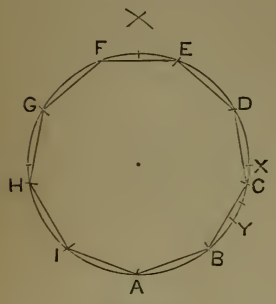


FIG. 100.

8. Construct a regular nonagon.

SOLUTION.—First get a quarter of the circumference by marking the points where two diameters at right angles to each other would cut the circumference. AX is an arc of 90°. Then from A take AY = 60° by using radius as a chord. YX is therefore an arc of 30°. Divide this into three equal parts *by trial.* Measure YB equal to two-thirds of YX, and AB and BC are arcs of 40°, and the chords AB and BC are chords of the regular nonagon.

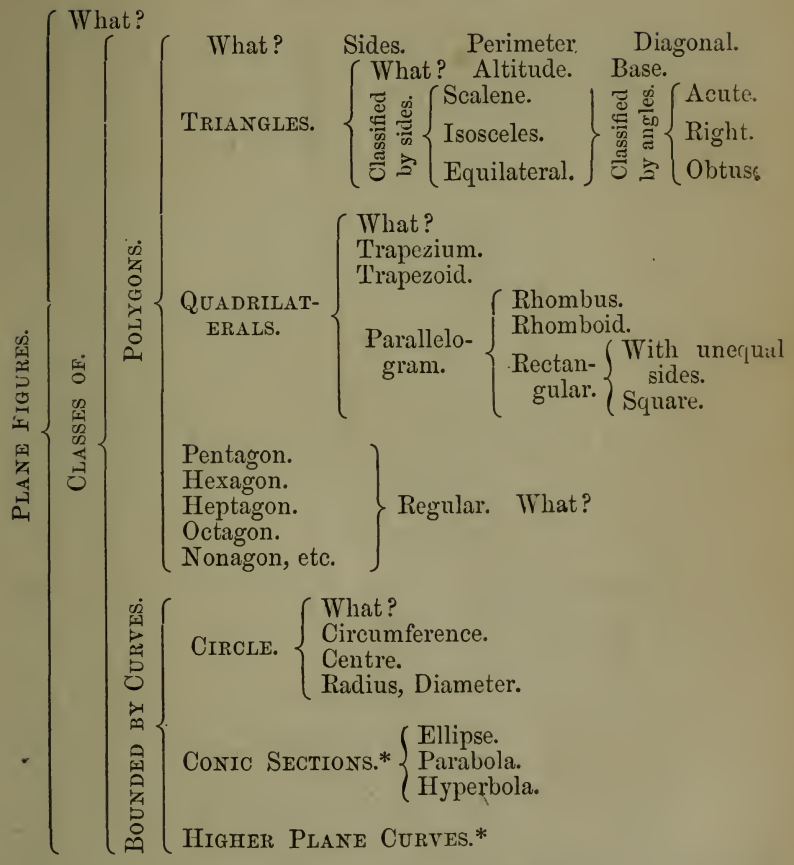
FIG. 101.

9. To draw a five-point star.

SOLUTION.—Draw a circle, and dividing the circumference into five equal parts, join the alternate points of division, as in the figure.

10. To circumscribe a square about a circle (*56*). Also an equilateral triangle, and a regular hexagon.

SYNOPSIS OF PLANE FIGURES.



* These are inserted simply to give completeness. Of course, the student is not expected to know more than their names.

PART II.

THE FUNDAMENTAL PROPOSITIONS OF ELEMENTARY GEOMETRY, DEMONSTRATED, ILLUSTRATED, AND APPLIED.

CHAPTER I.

PLANE GEOMETRY.

SECTION I.

OF PERPENDICULAR STRAIGHT LINES.

PROPOSITION I.

122. Theorem.—*At any point in a straight line, one perpendicular can be erected to the line, and only one, which shall lie on the same side of the line.*

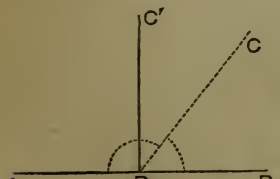


FIG. 102.

DEM.—Let AB* represent any line, and P be any point therein; then, on the same side of AB there can be one and only one perpendicular erected at P. For from P draw any oblique line, as PC, forming with AB the two angles CPB and CPA. Now, while the extremity P, of PC, remains at P, conceive the line PC to revolve so as to increase the less of the two angles, as CPB, and decrease the greater, as CPA. It is evident that for a certain position of CP, as C′P, these

* In class recitation the pupil should go to the blackboard, after having had his proposition assigned him, and first draw the figure required for the demonstration. This should be done neatly, accurately, with dispatch, and *without any aids*. The figure being complete, he stands at the board, pointer in hand, enunciates the proposition, and then gives the demonstration as it is in the text, pointing to the several parts of the figure as they are referred to.

angles will become equal. In this position C′P becomes perpendicular to AB (*26*).* Again, if the line C′P revolve from the position in which the angles are equal, one angle will increase and the other diminish; hence there is *only one* position of the line on this side of AB in which the adjacent angles are equal. Therefore there can be one and only one perpendicular erected to AB at P, which shall lie on the same side of AB. Q. E. D.

123. COR. 1.—*On the other side of the line a second perpendicular, and only one, can be drawn from the same point in the line.*

124. COR. 2.—*If one straight line meets another so as to make the angle on one side of it a right angle, the angle on the other side is also a right angle, and the first line is perpendicular to the second.*

125. COR. 3.—*If two lines intersect so as to make one of the four angles formed a right angle, the other three are right angles, and the lines are mutually perpendicular to each other.*

DEM.—Thus, if CEB is a right angle, CEA, being equal to it, is also a right angle. Then, as AEC is a right angle, the adjacent angle AED is a right angle, since they are equal. Also, as CEB is a right angle, and BED equal to it, BED is a right angle. Hence CD being perpendicular to AB, AB is perpendicular to CD, as it meets CD so as to make the adjacent angles AEC and AED, or CEB and BED equal to each other (*43*).

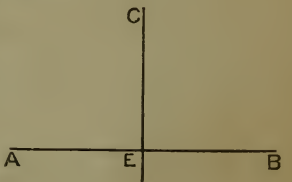


FIG. 103.

PROPOSITION II.

126. Theorem.—*When two straight lines intersect at right angles, if the portion of the plane of the lines on one side of either line be conceived as revolved on that line as an axis until it coincides with the portion of the plane on the other side, the parts of the second line will coincide.*

FIG. 104.

DEM.—Let the two lines AB and CD intersect at right angles at E; and let the portion of the plane of the lines on the side of CD on which B lies be conceived to revolve on the line CD as an axis,† until it falls in the

* When a preceding principle is referred to, *it should be accurately quoted by the pupil.*

† As if the paper, which may represent the plane of the lines, were folded in the line CD. It is important that this process be clearly conceived, as it is to be made the basis of many subsequent demonstrations.

portion of the plane on the other side of CD. Then will EB fall in and coincide with AE.

For, the point E being in CD, does not change position in the revolution; and, as EB remains perpendicular to CD, it must coincide with EA after the revolution, or there would be two perpendiculars to CD on the same side and from the same point, E, which is impossible (*122*). Hence EB coincides with EA. Q. E. D.

PROPOSITION III.

127. Theorem.—*From any point without a straight line, one perpendicular can be let fall upon that line, and only one.*

DEM.—Let AB be any line, and P any point without the line; then one perpendicular, and only one, can be let fall from P upon AB.

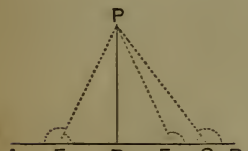


FIG. 105.

For, conceive any oblique line, as PC, drawn, making the angle PCB > PCA. Now, while the extremity P of this line remains fixed, conceive the line to revolve so as to make the greater angle PCB decrease, and the less angle PCA increase. At some position of the revolving line, as PD, the two angles which it makes with the line AB will become equal. When these adjacent angles are equal, the line, as PD, is perpendicular to AB (*26, 43*). Moreover, there is *only one* position of the line in which these angles are equal; hence, only one perpendicular can be drawn from a given point to a given line. Q. E. D.

PROPOSITION IV.

128. Theorem.—*From a point without a straight line, a perpendicular is the shortest distance to the line.*

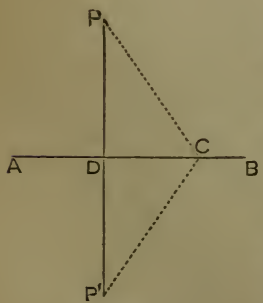


FIG. 106.

DEM.—Let AB be any straight line, P any point without it, PD a perpendicular, and PC any oblique line; then is PD < PC.

Produce PD, making DP′ = PD, and draw P′C. Then let the portion of the plane of the lines above AB be revolved upon AB as an axis, until it coincides with the portion below AB. Since PP′ and AB intersect at right angles, PD will fall in DP′ (*126*); and, since PD = DP′, P will fall at P′, and PC = P′C, since they coincide when applied. Finally, PP′ being a straight line, is shorter than

PCP′, which is a broken line, since a straight line is the shortest distance between two points. Hence PD, the half of PP′, is less than PC, the half of the broken line PCP′. Q. E. D.

PROPOSITION V.

129. Theorem.—*If a perpendicular be erected at the middle point of a straight line,*

1st. *Any point in the perpendicular is equally distant from the extremities of the line.*

2d. *Any point without the perpendicular is nearer the extremity of the line on its own side of the perpendicular.*

DEM.—1st. Let PD be a perpendicular to AB at its middle point D. Then, O being any point in the perpendicular, OA = OB.

For, revolve the figure OBD upon OD as an axis until it falls in the plane on the other side of PD. Since ODB and ODA are right angles, DB will fall in DA (*126*); and, since DB = DA, B will fall at A. Hence, OA and OB coincide, and OA = OB.

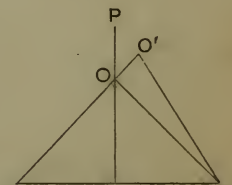


FIG. 107.

2d. O′ being any point without the perpendicular on the same side as B, O′B < O′A.

For, drawing O′A and O′B, let O be the point at which O′A cuts the perpendicular. Draw OB. Now O′B < BO + OO′, since O′B is a straight and O′OB is a broken line. But, as OA = OB, we may substitute it in the inequality, and have O′B < OA + OO′, which sum = O′A.

130. COR.—*If each of two points in one line is equally distant from the extremities of another line, the former line is perpendicular to the latter at its middle point.*

DEM.—Every point equally distant from the extremities of a straight line lies in a perpendicular to that line at its middle point, by the proposition. But, two points determine the position of a straight line. Hence, two points, each equally distant from the extremities of a straight line, determine the position of the perpendicular at the middle point of the line.

EXERCISES.

1. ***Prob.***—*To erect a perpendicular to a given line at a given point in the line.*

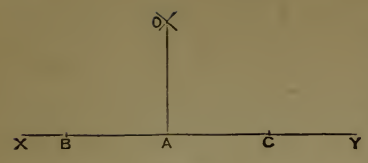


Fig. 108.

Solution.—[The *process* is given in (***44***), and should be repeated here exactly as given there, with the *reasons for the solution*, as follows.] A is one point in the line OA, which is equally distant from B and C, by construction, and O is another. Hence OA is perpendicular to BC at A, by (***130***).

2. ***Prob.***—*To bisect a given line.*

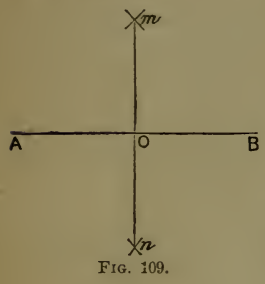


Fig. 109.

Solution.—[For the *process* see (***39***). The student should first do it as he did then. The reason why this process bisects AB is as follows.] Since *m* is one point equally distant from the extremities A and B, and *n* another, there are two points in *mn* each equally distant from the extremities of AB. Hence *mn* is perpendicular to AB at its middle point O, by (***130***). [The reason for the process in *Fig.* 20 is the same. Let the student give this method, and show how the corollary (***130***) applies.]

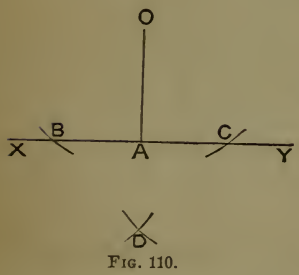


Fig. 110.

3. ***Prob.***—*From a point without a given line, to let fall a perpendicular upon the line.*

Solution.—[Repeat the process as in (***45***), and give the reason for it as follows.] O is one point equally distant from B and C, and D is another. Hence a line drawn from O to D is perpendicular to BC by (***130***).

4. Wishing to erect a line perpendicular to AB at its centre, I take a cord or chain somewhat longer than AB, and, fastening its ends at A and B, take hold of the middle of the cord or chain and carry it as far from AB as I can, first on one side and then on the other, sticking pins at the most remote points, as at P and P′. These points determine the perpendicular sought. What is the principle involved?

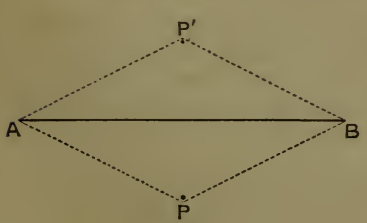


Fig. 111.

5. Two boys are skating together on the ice, and both start from

the same point at the same time, one skating directly to the shore and the other obliquely. They both reach the shore at the same time. Which skates the faster? What principle is involved?

6. Several persons start at different times from the same point in a straight road that runs along a wood, and each travels directly away from the road. Will they come out at the same, or at different points on the opposite side of the wood? What principle is involved? What is the geometrical language for the colloquial phrase "Directly away from the road"?

7. If I go from A to B, *Fig.* 111, by first passing over AP, will I gain or lose in distance by going on a little farther in the direction of AP before I turn and go straight to B? What principle is involved? Would I gain or lose by stopping short of P on the line AP? Why?

SECTION II.

OF OBLIQUE STRAIGHT LINES.

PROPOSITION I.

131. Theorem.—*When an oblique line meets another straight line forming two adjacent angles, the sum of these angles is two right angles.*

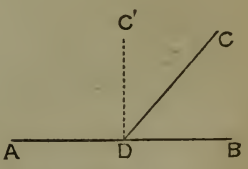


FIG. 112.

DEM.—Let the oblique line CD meet the straight line AB forming the two adjacent angles CDB and CDA; then CDB + CDA equals two right angles.

For suppose CD to revolve toward the position of the perpendicular C'D; the angle CDB will increase at the same rate that CDA diminishes; hence their *sum* will remain constant (*i. e.*, the same). But, when CD becomes perpendicular, the sum of the adjacent angles formed with AB is two right angles by definitions (*26*, *43*). Therefore CDB + CDA = two right angles. Q. E. D.

132. COR.—*The sum of all the consecutive angles formed by any number of lines meeting a given line on the same side and at a given point is two right angles.*

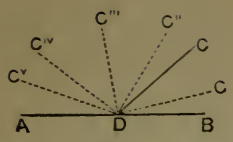


FIG. 113.

DEM.—Thus $ADC^{v} + C^{v}DC^{iv} + C^{iv}DC''' + C'''DC'' + C''DC' + C'DC + CDB = ADC' + C'DB$, which sum is two right angles by the proposition. Or, in general terms, the angles thus formed can always be united into two groups, constituting respectively the two adjacent angles formed by one line meeting another.

133. DEF.—Two angles whose sum is two right angles, are called *Supplemental Angles*. Hence, the *Supplement* of an angle is what remains after subtracting it from two right angles.

PROPOSITION II.

134. Theorem.—*When any two straight lines intersect, the opposite or vertical angles are equal to each other, and the sum of the four angles formed is four right angles.*

DEM.—Let AB and CE intersect at D; then CDA = the opposite angle BDE, ADE = the opposite or vertical angle CDB, and ADC + CDB + BDE + EDA = four right angles. For, since CD meets AB, ADC is the supplement of CDB (***131, 133***). Also, since BD meets CE, BDE is the supplement of CDB. Hence ADC = BDE. In a similar manner ADE can be proved equal to CDB. [The student should give the proof.]

FIG. 114.

Again, since ADC + CDB = two right angles, and BDE + EDA = two right angles, by adding the corresponding members together, we have ADC + CDB + BDE + EDA = four right angles.

135. COR.—*The sum of all the consecutive angles formed by any number of lines meeting at a common point is four right angles.*

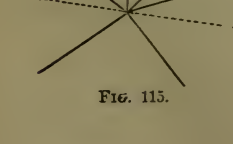


FIG. 115.

DEM.—The truth of this corollary is rendered apparent by drawing a line through the common vertex, and observing that the sum of all the angles on each side thereof is two right angles; whence the sum of all the angles on both sides, which is the same as the sum of all the consecutive angles formed by the line, is four right angles. [Let the student put letters on the figure, and demonstrate by means of it.]

PROPOSITION III.

136. Theorem.—*If two supplemental angles are so situated as to be adjacent to each other, the two sides not common will fall in the same straight line.*

DEM.—Let the sum of the two angles BOA and CO′D be two right angles. Prolong CO′, forming the angle DO′E. Then is DO′E supplemental to CO′D (*131, 133*), and hence equal to BOA, which is supplemental to CO′D by hypothesis. Now, if AOB be placed adjacent to CO′D, the vertex O being at O′, and the side OA falling in O′D, OB will fall in O′E, since BOA = DO′E. Hence, when the angles are so situated, OB becomes the prolongation of CO′. Q. E. D.

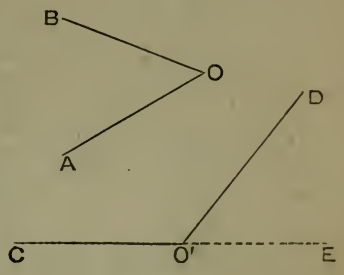


FIG. 116.

PROPOSITION IV.

137. Theorem.—*If from a point without a straight line a perpendicular be drawn, oblique lines from the same point cutting the line at equal distances from the foot of the perpendicular are equal to each other; the angles which they form with the perpendicular are equal to each other; and the angles which they form with the line are equal to each other.*

DEM.—Let AB be any straight line, P any point without it, PD a perpendicular, and PC and PE oblique lines cutting AB at C and E, so that DC=DE; then PC=PE, angle CPD = angle DPE, and angle PCD = angle PED.

Revolve the figure PDE upon PD as an axis, until it falls in the plane on the other side of PD. Since AB is perpendicular to PD, DB will fall in DA; and, since DE = DC, E will fall at C. Now, as P remains stationary, the triangles PDE and PDC coincide. Hence, PC = PE, angle CPD = angle DPE, and angle PCD = angle PED. Q. E. D.

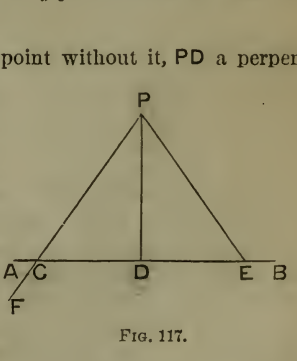


FIG. 117.

QUERY.—How does the equality of PE and PC follow from (*129*)?

PROPOSITION V.

138. Theorem.—*If from a point without a line a perpendicular be drawn to the line, and also from the same point two oblique*

*lines making equal angles with the perpendicular, the oblique lines are equal to each other, cut the line at equal distances from the foot of the perpendicular, and make equal angles with it.**

DEM.—PD being a perpendicular to AB, and angle CPD equal to angle DPE, PC equals PE, CD equals DE, and PCD equals PED.

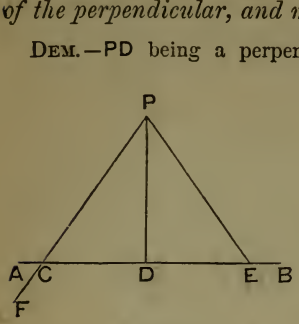


FIG. 118.

Revolve the figure PDE upon PD as an axis, till it falls in the plane of PDC. Since angle EPD = angle CPD, PE will take the direction PC, and E will fall somewhere in the indefinite line PF. But, since PDE and PDC are right angles, DE will fall in DA (*126*) and E will fall somewhere in the indefinite line DA. Now, as E falls at the same time in PF and DA, it must fall at their intersection C. Hence, PE coincides with PC, and DE with DC. Therefore PE = PC, DE = DC, and angle PED = PCD. Q. E. D.

PROPOSITION VI.

139. Theorem.—*If from a point without a line a perpendicular be let fall on the line, and two oblique lines be drawn, the oblique line which cuts off the greater distance from the foot of the perpendicular is the greater.*

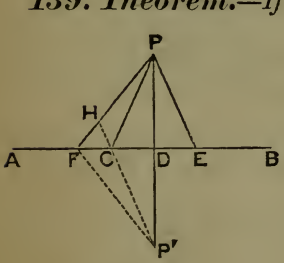


FIG. 119.

DEM.—Let AB be any straight line, P any point without it, and PC and PF two oblique lines of which PF cuts off the greater distance from the foot of the perpendicular; that is DF > DC. Then is PF > PC.

Produce PD, making DP′ = PD, and draw P′F and P′C, producing the latter until it meets PF in H. Revolve the figure FPD upon AB as an axis, until it falls in the plane on the opposite side of AB. Since PP′ is perpendicular to AB, PD will fall in DP′; and, since PD = DP′, P will fall at P′. Then P′C = PC and P′F = PF. Now the broken line PCP′ < than the broken line PHP′, since the straight line PC < the broken line PHC. For a like reason the broken line PHP′ < PFP′, since HP′ < HFP′. Hence PCP′ < PFP′, and PC the half of PCP′ < PF the half of PFP′. Q. E. D.

SCH.—If the two oblique lines to be compared lie on different sides of the perpendicular, as PF and PE, DF being greater than DE, lay off DC = DE, and draw PC. Then since PC = PE, if it is found less than PF, as in the demonstration, PE is less than PF.

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* This proposition is the *converse* of the last. The significance of this statement will be more fully developed farther on (*154*).

140. COR. 1.—*From a given point without a line, there can not be two equal oblique lines drawn to the line on the same side of a perpendicular from the point to the line.*

141. COR. 2.—*Two equal oblique lines drawn from the same point in a perpendicular to a given line, cut off equal distances on that line from the foot of the perpendicular.*

DEM.—For, if the distances cut off were unequal, the lines would be unequal.

EXERCISES.

1. Having an angle given, how can you construct its supplement? Draw any angle on the blackboard, and then construct its supplement.

FIG. 120.

2. The several angles in the figure are such parts of a right angle as are indicated by the fractions placed in them. If these angles are added together by bringing the vertices together and causing the adjacent sides of the angles to coincide, how will MA and GN lie? Construct seven consecutive angles of these several magnitudes. How do the two sides not common lie? Why?

3. If two times A, B, two times D, three times E, three times C, three times C, two times F, in the last figure, are added in order, how will AM and GN lie with reference to each other? Why?

Ans. They will coincide.

4. If you place the vertices of any two equal angles together so that two of the sides shall extend in opposite directions and form one and the same straight line, the other two sides lying on opposite sides thereof, how will the latter sides lie? By what principle?

5. Upon what principle in this section may the common method of erecting a perpendicular at the middle of a straight line (***39, 44***) be explained? Upon what the method of letting fall a perpendicular upon a straight line from a point without (***45***)?

6. A and B start at the same time, from the same point in a certain road; A travels directly to a point in another road at right angles to the first, and at ten miles from their intersection, and B travels directly toward a second point in the second road, which point is seven miles from the intersection. Both reach their destination at the same time. Which travels the faster? What principle is involved?

SECTION III.

OF PARALLELS.

PROPOSITION I.

142. Theorem.—*Two straight lines lying in the same plane and perpendicular to a third line are parallel to each other.*

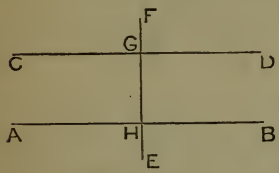

FIG. 121.

DEM.—Let AB and CD be two straight lines lying in the same plane and each perpendicular to FE; then are they parallel.

For if AB and CD are not parallel, they will meet at some point if sufficiently produced (*66*). But, if they could meet, we should have two straight lines from one point (their point of meeting), perpendicular to the same straight line, which is impossible (*127*). Therefore, as the lines lie in the same plane and cannot meet how far soever they be produced, they are parallel. Q. E. D.

143. COR. 1.—*Through the same point one parallel can always be drawn to a given line, and only one.*

DEM.—Let AB be the given line, and G the given point, there can be one and only one perpendicular through G to AB (*127*.) Let this be FE. Now through G one and only one perpendicular can be drawn to FE. Let this be CD. Then is CD parallel to AB by the proposition. That there is only one such parallel, we shall assume as axiomatic.*

144. COR. 2.—*If a straight line is perpendicular to one of two parallels, it is perpendicular to the other also.*

DEM.—If FE is perpendicular to AB it is perpendicular to CD. For, if through G where FE intersects CD, a perpendicular be drawn to FE, it is par-

* Nous regarderons cette proposition comme ÉVIDENTE. P.-F. COMPAGNON. So also CHAUVENET.

allel to AB by the proposition. But, by *Cor.* 1, there can be but one line through C parallel to AB. Hence the perpendicular to FE at C coincides with, or is, the parallel CD.

PROPOSITION II.

145. Theorem.—*Two straight lines which are parallel to a third, are parallel to each other.*

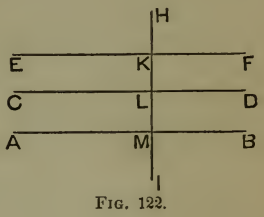

Fig. 122.

Dem.—Let AB and CD be each parallel to EF; then are they parallel to each other.

For draw HI perpendicular to EF; then will it be perpendicular to CD because CD is parallel to EF. For a like reason HI is perpendicular to AB. Hence CD and AB are both perpendicular to HI, and consequently parallel. Q. E. D.

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146. Definitions.—When two lines are cut by a third line the angles formed are named as follows:

Exterior Angles are those without the two lines, as 1, 2, 7, and 8.

Interior Angles are those within the two lines, as 3, 4, 5, and 6.

Alternate Exterior Angles are those without the two lines and on different sides of the secant line, but not adjacent, as 2 and 7, 1 and 8.

Alternate Interior Angles are those within the two lines and on different sides of the secant line but not adjacent, as 3 and 6, 4 and 5.

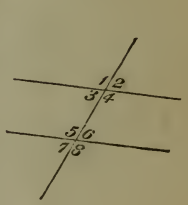

Fig. 123.

Corresponding Angles are one without and one within the two lines, and on the same side of the secant line but not adjacent, as 2 and 6, 4 and 8, 1 and 5, 3 and 7.

PROPOSITION III.

147. Theorem.—*If two lines are cut by a third line, making the sum of the interior angles on the same side of the secant line equal to two right angles, the two lines are parallel.*

Dem.—Let AB and CD be met by the line EF, making EGD + FKB = two right angles; then are AB and CD parallel.

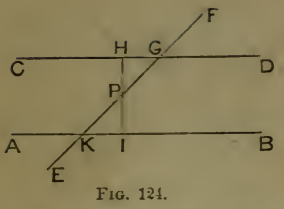

Fig. 124.

For, through P, the middle of GK, draw HI perpendicular to AB. Since HPG and KPI are vertical angles, they are equal by (*134*). Also, since GKB and CGK are both supplements of DGK, the former by hypothesis, and the latter by (*133*), GKB = CGK. Now, conceive the portion of the figure below P, while remaining in the same plane (the plane of the paper), to revolve upon P (as a pivot) from right to left till PK falls in PC.* Since PK = PG, K will fall at G. Again, since KPI = GPH PI will take the direction PH, and I will fall in PH, or PH produced; and, since PKI = PGH, KI will take the direction GH, and I will fall somewhere in GC. Hence, as I falls in both PH and GC, it must fall at their intersection H; and KIP coincides with, and is equal to PHG. But KIP is a right angle by construction; hence GHP is a right angle. Therefore, AB and CD are both perpendicular to HI, and consequently parallel by (*142*). Q. E. D.

148. Cor. 1.—*If two lines are cut by a third, making the sum of the two exterior angles on the same side of the secant line equal to two right angles, the two lines are parallel.*

Dem.—For, if FGD + EKB = two right angles, EKB is the supplement of FGD; so also is KGD (*131, 133*). Hence EKB = KGD. Again, for like reasons, FGD and GKB are both supplements of EKB and therefore equal. Hence, when FGD + EKB = two right angles, GKB + KGD = two right angles, and the lines are parallel by the proposition. The same is true for FGC and AKE. [Let the student prove it.]

149. Cor. 2.—*If two lines are cut by a third, making either two alternate interior, or either two alternate exterior, or either two corresponding angles, equal to each other, the lines are parallel.*

Dem.—If CGK = GKB, KGD + GKB = two right angles, since CGK + KGD = two right angles. Hence the lines are parallel by the proposition. So also if KGD = AKG, or FGD = AKE, or CGF = EKB, or FGD = GKB, or CGF = AKG, the two lines are parallel. [Let the student show the truth in each case.]

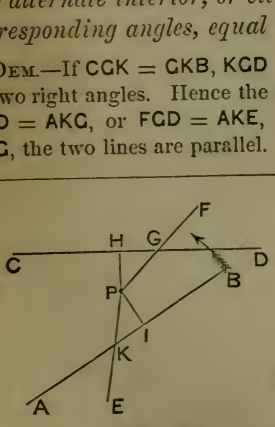

Fig. 125.

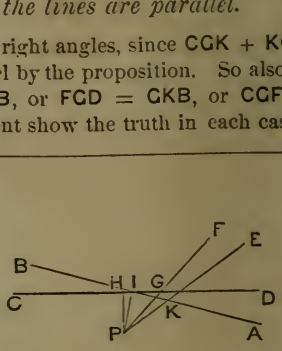

Fig. 126.

* The accompanying figures will aid the student in getting this conception. *Fig.* 125 represents the position of the lines after the revolution has gone about half a right angle, and *Fig.* 126 when the revolution is almost completed.

PROPOSITION IV.

150. Theorem.—*If two parallel lines are cut by a third line, the sum of the interior angles on the same side of the secant line is equal to two right angles.*

DEM.—Let the parallels AB and CD be cut by EF, then is DGK + GKB = two right angles.

For, if DGK is not the supplement of GKB, let LM be drawn through G so as to make MGK that supplement. Then, by the preceding proposition, LM is parallel to AB; and we have two parallels to AB through the point G, which is impossible (***143***). Hence, as no line but a parallel can make this interior angle the supplement of the other, the parallel makes it so. Q. E. D.

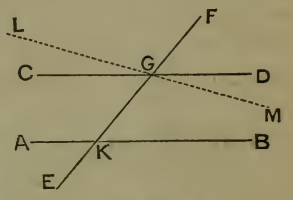

FIG. 127.

[Let the student demonstrate this proposition as the preceding was demonstrated. In this case CD and AB are parallel by hypothesis, and HI being drawn perpendicular to one is perpendicular to the other also. When K falls at G, KI falls on CG, since from a point without a line only one perpendicular can be drawn to that line.]

151. COR. 1.—*If two parallel lines are cut by a third line, the sum of either two exterior angles on the same side of the secant line is equal to two right angles.*

DEM.—FGD + EKB = two right angles. For FGD and GKB are both supplements of DGK (***133, 150***), and therefore equal to each other. For like reasons, EKB = KGD. Therefore, FGD + EKB = GKB + DGK = two right angles, by the proposition.

152. COR. 2.—*If two parallel lines are cut by a third line, either two alternate interior, or either two alternate exterior, or either two corresponding angles, are equal to each other.*

DEM.—If CD and AB are parallel, CGK = GKB. For each is the supplement of KGD, the former by (***131***), the latter by (***150***). [Let the student show in like manner that AKG = KGD, FGD = AKE, CGF = EKB, FGD = GKB, and CGF = AKG.]

153. COR. 3.—*Of the eight angles formed when one line cuts two parallels, the four acute angles are equal each to each, and the four obtuse angles; or, in case any one angle is a right angle, all the others are right angles.*

154. SCH.—The last two propositions and their corollaries are the *converse* of each other; *i. e.*, the hypotheses or data and the conclusions or things proved are exchanged. Thus, in PROP. III., the hypothesis is, that *The sum of the two interior angles on the same side of the secant line is equal to two right angles;* and the conclusion is, that *The two lines are parallel.* Now, in PROP. IV., the hypothesis is, that *The two lines are parallel;* and the conclusion is, that *The sum of the two interior angles on the same side of the secant line is two right angles.** [A clear conception of this scholium will save the student from confounding these propositions.]

PROPOSITION V.

155. Theorem.—*If two straight lines are cut by a third line making the sum of the interior angles on one side of the secant line less than two right angles, the two lines will meet on this side of the secant line, if sufficiently produced.*

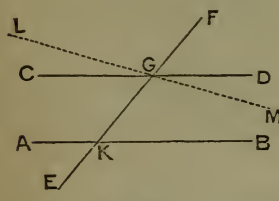

FIG. 128.

DEM.—Let AB and LM be cut by EF making MGK + FKB < two right angles; then will AB and LM meet on the side of EF on which MGK and FKB lie, if sufficiently produced.

For the angle which a parallel to AB through G makes with EF is the supplement of FKB. But by hypothesis MGK is less than this supplement. Hence the portion GM, of the line LM, lies within GD, and will meet KB if sufficiently produced. Q. E. D.

PROPOSITION VI.

156. Theorem.—*Two parallels are everywhere equally distant from each other.*

DEM.—Let E and F be any two points in the line CD, and EG and FH perpendiculars measuring the distances between the parallels CD and AB at these points; then is EG = FH.

For, let P be the middle point between E and F, and PO a perpendicular at

---

* The learner may think that, if a proposition is true, its converse is necessarily true; and hence, that when a proposition has been proved, its converse may be assumed as also proved. Now this is by no means always the case. Although in a great variety of mathematical propositions, it happens that the proposition and its converse *are* both true, we never assume one from having proved the other; and we shall occasionally find a proposition whose converse is not true.

this point. Revolve the portion of the figure on the right of PO, upon PO as an axis, until it falls upon the plane of the paper at the left. Then, since FPO and EPO are right angles, PD will fall in PC; and, as PF = PE, F will fall on E. As F and E are right angles, FH will take the direction EG, and H will lie in EG or EG produced. Also,

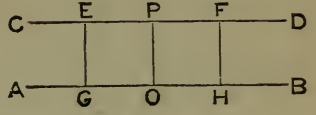

Fig. 129.

as POH and POG are right angles, OB will fall in OA, and H falling at the same time in EG and OA is at their intersection G. Hence FH coincides with and is equal to EG. Q. E. D.

EXERCISES.

1. ***Prob.***—*Through a given point to draw a line parallel to a given line, by the principle contained in* Prop. I. *of this section.*

Sug's.—Draw a straight line on the blackboard. Designate with a dot some point without the line. To draw a line through the designated point and parallel to the given line, is the problem. Let fall a perpendicular upon the line from the point. Then through the given point draw a line perpendicular to this perpendicular. The latter line will be parallel to the given line. (By what proposition?)

2. ***Prob.***—*Through a given point to draw a parallel to a given line by* Prop. III.

Sug's.—Through the given point draw an oblique line cutting the given line. Then draw a line through the given point making an angle with the oblique line equal to the supplement of the angle which is included between the oblique line and the given line, and on the same side of the former. [Of course the student will be required to *do the work on* the blackboard, *guessing at nothing.*]

3. ***Prob.***—*Through a given point to draw a line parallel to a given line, upon the principle that the alternate angles made by a secant line are equal* (***152***).

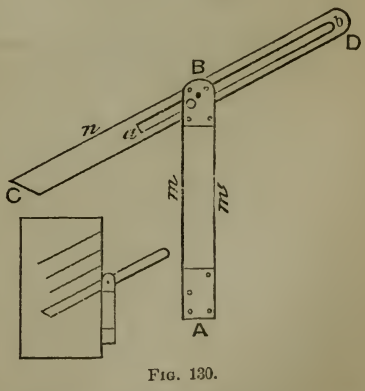

Fig. 130.

4. A *bevel* is an instrument much used by carpenters, and consists of a main limb AB, in which a tongue CD is placed, so as to open and shut like the blade of a knife. This tongue turns on the pivot O, which is a screw, and can be tightened so as to hold the tongue firmly at any angle with the limb. The tongue can also be adjusted so as

to allow a greater or less portion to extend on a given side, as CB, of the limb. Now, suppose the tongue fixed in position, as represented in the figure, and the side m of the limb to be placed against the straight edge of a board, and slid up and down, while lines are drawn along the side n of the tongue. What will be the relative position of these lines? Upon what proposition does their relative position depend? How can the carpenter adjust the bevel to a right angle upon the principle in PROP. I., Sec. 1? At what angle is the bevel set, when, drawing two lines from the same point in the edge of the board, one with one edge m of the bevel against the edge of the board, and the other with the other edge m', these lines are at right angles to each other?

5. Are the two walls of a building which are carried up by the plumb line exactly parallel? Why?

6. Pass a circumference through three given points, as in (*58*), and show from principles contained in one of the preceding sections, that O is equally distant from A, B, and C; and hence that, if a circumference be drawn from O as a centre with a radius OA, it will pass through A, B, and C.

7. Construct two triangles of unequal sizes, but having the sides of the one respectively parallel to the sides of the other. Are they shaped alike?

8. Construct two triangles of unequal sizes, but having the sides of the one respectively perpendicular to the sides of the other. Are they shaped alike?

9. Construct a parallelogram, two of whose sides are 6 and 10. Can you construct different-shaped figures with the same sides?

SYNOPSIS OF THE THREE PRECEDING SECTIONS.

RELATIVE POSITIONS OF STRAIGHT LINES.

PERPENDICULARS.

- Definition (*43*).
- PROP. I. One and only one to a given line at a given point.
  - *Cor.* 1. Second perp.
  - *Cor.* 2. If one angle is right.
  - *Cor.* 3. One of 4 angles right.
- PROP. II. Revolved perpendicular.
- PROP. III. From a point without a line.
- PROP. IV. Shortest distance from a point to a line.
- PROP. V. Point in. Without.
  - *Cor.* Two points equally distant from extremities of a line.
- EXERCISES.
  - *Prob.* To erect a perpendicular.
  - *Prob.* To bisect a line.
  - *Prob.* To let fall a perpendicular.
  - Other exercises.

OBLIQUE LINES.

- PROP. I. Sum of adjacent angles.
  - *Cor.* Sum of consec. angles on one side of line.
  - *Def.* Supplement.
- PROP. II. Opp. angles equal.
  - *Cor.* Angles about a point.
- PROP. III. Supplemental angles made adjacent.
- PROP. IV. Cutting equal distances from foot of perpendicular.
- PROP. V. Making equal angles with perpendicular.
- PROP. VI. Cutting unequal distances from the foot of perpendicular.
  - *Cor.* 1. Not two equal on same side of perpendic.
  - *Cor.* 2. Two equal oblique lines.
- EXERCISES.

PARALLELS.

- Definition (*66*).
- PROP. I. Two perpendiculars to a line.
  - *Cor.* 1. One parallel through a point.
  - *Cor.* 2. A perp. to one of two parallels.
- PROP. II. Two lines parallel to a third.

A SECANT TO.

- Def's of angles formed.
  - Exterior, Interior, Alternate Exterior, Alternate Interior, Corresponding.
- PROP. III. Sum of Inter. angles, two right angles.
  - *Cor.* 1. Sum of two Exterior angles, two right angles.
  - *Cor.* 2. Two Alt. Inter., Alt. Exter., or Correspond'g angles equal.
- PROP. IV. Converse of III.
  - *Cor.* 1. Converse of Cor. 1., Prop. III.
  - *Cor.* 2. Converse of Cor. 2., Prop. III.
  - *Cor.* 3. Of the eight angles.
  - *Sch.* Meaning of Converse.
- PROP. V. Sum of Inter. angles < 2 right angles.

- PROP. VI. Everywhere equidistant.
- EXERCISES.—*Probs.* 1, 2, 3. Methods of drawing.

SECTION IV.

OF THE RELATIVE POSITIONS OF STRAIGHT LINES AND CIRCUMFERENCES.

PROPOSITION I.

158. Theorem.—*Any diameter divides a circle, and also its circumference, into two equal parts.*

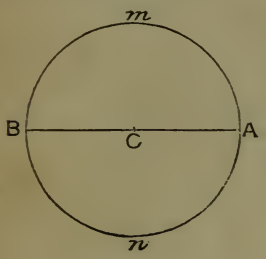

FIG. 131.

DEM.—Let AB be any diameter of the circle A*m*B*n*; then is the figure A*m*B equal to A*n*B.

For revolve A*n*B upon AB as an axis until it falls on the plane of A*m*B. Then, since every point in A*n*B is at the same distance from the centre C, as every point in A*m*B, the figures will coincide, and are, consequently, equal. Hence surface A*n*B = surface A*m*B, and arc A*n*B = arc A*m*B. Q. E. D.

PROPOSITION II.

159. Theorem.—*A radius which is perpendicular to a chord bisects the chord and also the subtended arc.*

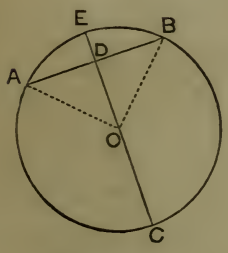

FIG. 132.

DEM.—Let AB be any chord and OE a radius perpendicular to it at D; then AD = BD, and AE = BE.*

For, drawing the radii OA and OB, revolve the semicircle CBE upon the diameter CE until it falls on CAE. The semicircles will coincide (*158*); and since AB is perpendicular to OE, DB will fall in DA. Moreover, as there cannot be two equal oblique lines from a point to a line on the same side of a perpendicular, OB and OA must coincide. Hence BD coincides with AD, and BE with AE. Therefore AD = BD, and AE = BE. Q. E. D.

* To avoid confusing the pupil by a multiplicity of details, the demonstrations in this section are generally limited to the consideration of arcs less than a semi-circumference. All the propositions, except PROP. V., are equally true whatever the arcs, and the demonstrations can easily be applied to cases in which the arcs are greater than semi-circumferences. But this had better not be done till a review is taken, for the reason given above.

160. COR. 1.—*A radius which is perpendicular to a chord bisects the angle subtended by the arc of that chord.*

Thus OE bisects AOB, since BOE is found to coincide with AOE in the demonstration above.

161. COR. 2.—Conversely, *A radius which bisects an arc is perpendicular to the chord of that arc at its middle point.*

DEM.—If OE bisects arc AB at E, when semicircle CBE is revolved on CE till it falls on CAE, EB will coincide with EA; and as D remains fixed and B falls on A, BD coincides with DA. Hence OE has two points, O and D, each equidistant from the extremities of AB, and is, consequently, perpendicular to it at its middle point.

162. COR. 3.—Also, conversely, *A radius which bisects a chord is perpendicular to the chord and bisects the subtended arc.*

For it has two points, each equidistant from the extremities of the chord.

163. COR. 4.—*The line* OD *measures the distance of the chord* AB *from the centre;* since by the distance from a point to a line is always meant the *shortest* distance.

PROPOSITION III.

164. Theorem.—*In the same or in equal circles, equal chords are equally distant from the centre.*

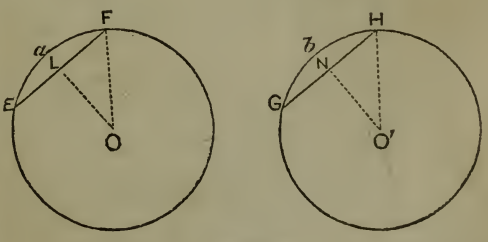

FIG. 133.

DEM.—Let O and O′ be two equal circles, and chord EF = chord GH; then are the perpendiculars LO and NO′, which measure the distances of the chords from the centre (***163***), equal.

For, since FE is perpendicular to LO and GH to NO′, and LF = NH (***159***), the equal oblique lines FO and HO′ cut off equal distances from the foot of each perpendicular (***141***). Therefore LO = NO′. Q. E. D.

PROPOSITION IV.

165. Theorem.—*In the same or in equal circles, equal arcs have equal chords; and conversely, equal chords subtend equal arcs.*

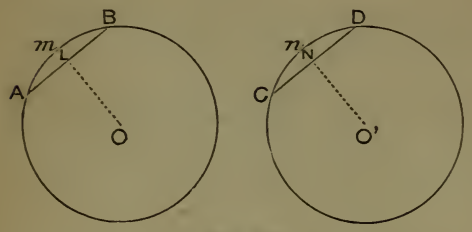

Fig. 134.

Dem.—Let O and O′ be the centres of two equal circles, and arc A*m*B = arc C*n*D; then chord AB = chord CD.

Apply the circle O′ to the circle O, with O′ at O, and C at A. Since the circumferences coincide, all the points in each being equally distant from the centre, and since arc A*m*B = arc C*n*D by hypothesis, D will fall at B. Hence AB = CD.

Conversely, if chord AB = chord CD, arc A*m*B = arc C*n*D. Draw the perpendiculars OL and O′N from the centres to the chords. Conceive the plane of circle O′ placed upon circle O, so that CD shall fall upon its equal AB, and O′ be on the same side of AB as O. Since L and N are the middle points of the equal chords, they will coincide; and as LO and NO′ are perpendiculars to the respective chords, and equal (***164***), O′ will fall at O. As the circles are equal, the circumferences will coincide, and consequently the arc A*m*B coincides with C*n*D.

PROPOSITION V.

166. Theorem.—*In the same or in equal circles, the less of two arcs has the shorter chord; and, conversely, the shorter chord subtends the less arc.*

Dem.—Let O and O′ be the centres of two equal circles, and arc A*m*B be less than arc C*n*D; then is chord AB < chord CD. Drawing the diameters AL and CM, place circle O′ upon circle O, with CM upon AL. Take arc AD′ = arc C*n*D and draw AD′, OB, and OD′. AD′ = CD by (***165***). Now AB < AN + NB. Also OD′ < ND′ + NO; or, as OD′ = OB, OB < ND′ + NO. Subtracting NO from both members, OB − NO (or NB) < ND′. Hence, we may substitute ND′ for NB in the inequality AB < AN + NB and have AB < AN + ND′ or AD′, which equals CD.

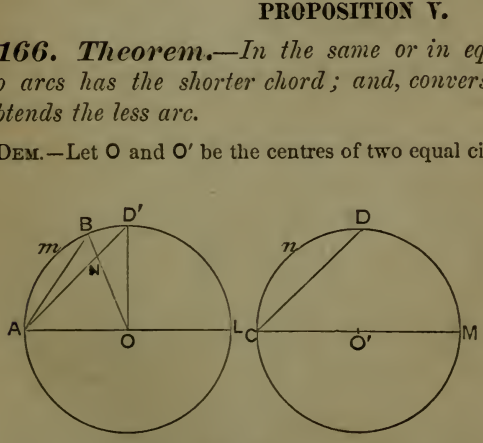

Fig. 135.

Conversely, if chord AB is less than chord CD, arc AmB is less than arc CnD. For if arc AmB = arc CnD, chord AB = chord CD (*165*). And, if arc AmB > arc CnD, chord AB > chord CD. But both of these conclusions are contrary to the hypothesis. Hence, as arc AmB can neither be equal to nor greater than arc CnD, it must be less.

PROPOSITION VI.

167. Theorem.—*In the same or in equal circles, of two unequal chords, the less is at the greater distance from the centre.*

DEM.—Let CE < AB, then is the perpendicular OD, which measures the distance of CE from the centre, greater than OD′ which measures the distance of AB from the centre.

From A lay off AE′ = CE, and draw the perpendicular OD″. Then OD″ = OD, since equal chords are equally distant from the centre. As arc AE′ < arc AB, AB cuts OD″ in some point as H. Now OH > OD′ since the former is oblique and the latter perpendicular to AB. Also OD″ > OH. Much more then is OD″ > OD′. Therefore OD (which equals OD″) > OD′. Q. E. D.

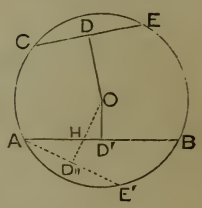

FIG. 136.

168. COR.—Conversely, *Of two chords which are unequally distant from the centre, that which is at the greater distance is the less.*

DEM.—Thus, if CE is at a greater distance from the centre than AB, CE < AB. For, if CE were equal to AB, it would be equally distant from the centre. And if CE were greater than AB, it would be at a less distance from the centre. Hence, as CE cannot be at an *equal* distance from the centre with AB, nor at a *less* distance, it must be at a greater.

PROPOSITION VII.

169. Theorem.—*A straight line can intersect a circumference in only two points.*

DEM.—The distances from the centre to the intersections, being radii, are equal. Hence, as there can be only two equal straight lines drawn from a point to a straight line, there can be only two intersections. Q. E. D.

PROPOSITION VIII.

170. Theorem.—*A straight line which intersects a circumference in one point intersects it also in a second point.*

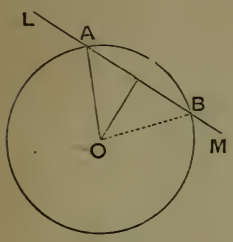

FIG. 137.

DEM.—Let LM intersect the circumference at A; then does it intersect at some other point, as B.

For, since LM intersects the circumference, it passes within it, and has points nearer to O than A. The radius OA is, therefore, an oblique line. Now two equal oblique lines can be drawn from O to the straight line LM. But all points in the plane at the distance OA from O, are in the circumference. Hence there is a second point, as B, common to LM and the circumference. Q. E. D.

171. COR.—*Any line which is oblique to a radius at its extremity, is a secant line.*

PROPOSITION IX.

172. Theorem.—*A straight line which is perpendicular to a radius at its extremity is tangent to the circumference.*

DEM.—The line touches the circumference because the extremity of the radius is in the circumference. Moreover, it does not intersect the circumference, since, if it did, it would have points nearer the centre than the extremity of the radius; but these it cannot have, as the perpendicular is the shortest distance from a point to a line. Hence, as a line which is perpendicular to a radius at its extremity touches the circumference but does not intersect it, it is a tangent (***53***). Q. E. D.

173. COR.—Conversely, *A tangent to a circumference is perpendicular to a radius at the point of contact.*

For, as a tangent to a circumference does not pass within, the point of contact is the nearest point to the centre, and hence is the foot of a perpendicular from the centre.

PROPOSITION X.

174. Theorem.—*Two parallel secants intercept equal arcs.*

DEM.—Let the parallels LM and RS intersect the circumference AECF; then are the intercepted arcs AB and DC equal.

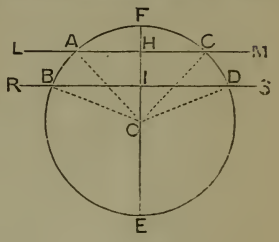

FIG. 138.

Draw the diameter EF perpendicular to one of the parallels, as LM, whence it will be perpendicular to the other (***144***). Draw the radii OB and OD. Revolve the portion of the figure on the right of EF, upon EF until it falls on the plane on the left of EF. Then, since RS and LM are perpendicular to EF, IS will fall in IR, and HM in HL. Moreover, as there cannot be two equal oblique lines on the same side of a perpendicular, and from the same point (***140***), OD and OB must coincide, and D fall at B. In like manner C falls at A, and CD coincides with AB. Therefore CD = AB. Q. E. D.

PROPOSITION XI.

175. Theorem.—*If a secant be parallel to a tangent, the arcs intercepted between the intersections and the point of tangency are equal.*

DEM.—Let the secant LM be parallel to the tangent RS; then is CP = EP.

For, draw the radius OP to the point of tangency; it will be perpendicular to the tangent (***173***), and also to the parallel LM (***144***). But a radius which is perpendicular to a chord, as OP to CE, bisects the subtended arc (***159***), hence CP = EP. In like manner, if VU is parallel to LM, CB = EB. Q. E. D.

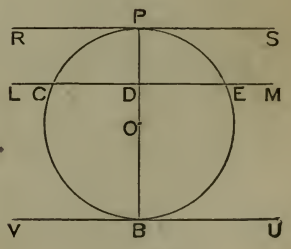

FIG. 139.

176. COR.—*Two parallel tangents include equal arcs between the points of tangency; and these arcs are semi-circumferences.*

EXERCISES.

1. Draw a circle and divide it into two equal parts. What proposition is involved?

2. Given a point in a circumference, to find where a semi-circumference reckoned from this point terminates. What proposition is involved?

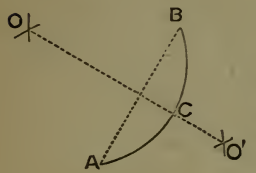

Fig. 140.

3. ***Prob.***—*To bisect a given arc.*

Solution.—Let AB be an arc which we wish to bisect.* Draw its chord AB, and erect OO′ bisecting the chord, by (***130***). Now, as OO′ is perpendicular to the chord at its middle point, it bisects the arc by (***162***), since there can be but one perpendicular at the middle point of the chord. The arc AB is, therefore, bisected at C, *i. e.*, AC = CB.

4. ***Prob.***—*To bisect a given angle.*

Sug.—The method of solving this is given in Part I. The student should *do it* as there directed, and then point out the principle upon which the method depends.

5. In a circle whose radius is 11 there are drawn two chords, one at 6 from the centre, and one at 4. Which chord is the greater? By what proposition?

6. In a certain circle there are two chords, each 15 inches in length. What are their relative distances from the centre? Quote the principle.

7. There is a circular plat of ground whose diameter is 20 rods. A straight path in passing runs within 4 rods of the centre. What is the position of the path with reference to the plat? What is the position of a straight path whose nearest point is 10 rods from the centre? One whose nearest point is 11 rods from the centre?

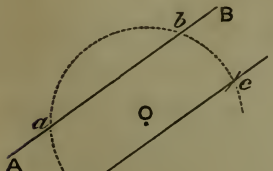

Fig. 141.

8. Pass a line through a given point, and parallel to a given line, by the principles contained in (***174***) and (***165***).

* This solution and many others are given, not so much that it is feared that the student will not be able to solve the problems, as to afford models for describing the process. In this case an arc should be drawn first, and all trace of the centre obliterated. *Then* proceed as directed.

9. ***Prob.***—*To draw a tangent to a circle at a given point in the circumference.*

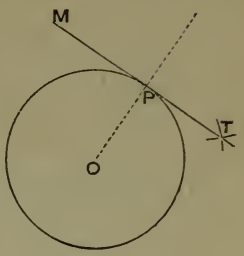

Fig. 141*.

Solution.—Let P be the point at which a tangent is to be drawn. Draw the radius OP to the given point of tangency, and produce it any convenient distance beyond the circle. Erect a perpendicular to this line at P, as MT; then is MT a tangent to the circle (*172*).

10. ***Prob.***—*To find the centre of a circle whose circumference is known, or of any arc of it.*

Sug.—The *process* is given in Part I. Do the work as there directed, and then show upon what proposition in this section it is founded.

SYNOPSIS.

Relative Positions of Straight Lines and Circumferences.

- **Diameters.** Prop. I. How divide circles and circumferences.
- **Chords.**
  - Prop. II. Radius perp. to chord. — *Cor.* 1. Bisects angle. *Cor.* 2. Converse of Prop. *Cor.* 3. " " " *Cor.* 4. Dist. from centre.
  - Prop. III. Distance of equal chords from centre.
  - Prop. IV. Equal arcs, and converse.
  - Prop. V. Unequal arcs.
  - Prop. VI. Unequal chords. Distance from centre. — *Cor.* Converse.
- **Secants.**
  - Prop. VII. Intersect in only two points.
  - Prop. VIII. If a line intersect in one point, it intersects also in another. — *Cor.* Line oblique to radius at extr.
- **Tangents.** Prop. IX. Line perpendicular to radius at extremity. — *Cor.* Converse.
- **Parallels.**
  - Prop. X. Parallel secants intercept equal arcs.
  - Prop. XI. Secant par. to tangent. — *Cor.* Two parallel tangents.
- **Exercises.**
  - *Prob.* To bisect an arc.
  - *Prob.* To bisect an angle.
  - *Prob.* To draw a tangent at a point in circumference.
  - *Prob.* To find centre of circumference or arc.

SECTION V.

OF THE RELATIVE POSITIONS OF CIRCUMFERENCES.

PROPOSITION I.

177. Theorem.—*All the circumferences which may be passed through three points not in the same straight line coincide, and are one and the same.*

DEM.—Let A, B, and C be three points not in the same straight line; then all the circumferences which can be passed through them will coincide.

For join the points, two and two, by straight lines, as AB and BC. Bisect these lines with perpendiculars, as DF and EH. Since AB and BC are not in the same straight line, DF and EH will meet when sufficiently produced, at one and only one point, as O, because they are straight lines Now, every point in FD is equally distant from A and B, and every point in HE is equally distant from B and C (***129***). Hence O is equally distant from the three points A, B, and C; and, if a circumference be drawn with O as a centre, and a radius AO, it will pass through the three points. Moreover, every circumference passing through these points must have O for its centre, since the centre must be in FD (otherwise it would be unequally distant from A and B), and also in HE (***129***). But these lines intersect only in O. Also, every circumference with O as its centre, and passing through A, must have AO for its radius. Hence, as all circles having the same centre and the same radius coincide, all those passing through three points, A, B, and C, coincide. Q. E. D.

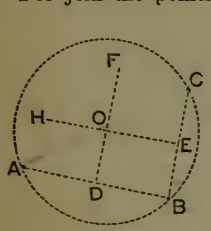

FIG. 142.

178. COR. 1.—*Through any three points not in the same straight line a circumference can be passed.*

179. COR. 2.—*Three points not in the same straight line determine a circumference as to position and extent;* i.e., *in all respects.*

180. COR. 3.—*Two circumferences can intersect in only two points.*

For, if they have three points common, they coincide, and form one and the same circumference.

PROPOSITION II.

181. Theorem.—*Two circumferences which intersect in one point, intersect also in a second point.*

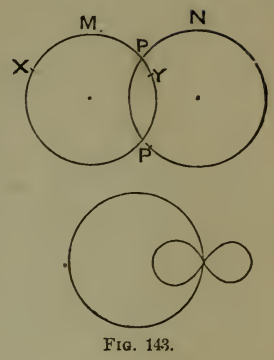

FIG. 143.

DEM.—Let M intersect N at P. As M passes from without to within the circle N, it has points both without and within. Now, for M to return into itself from any point within N, as Y, to any point without, as X, it must intersect N; but it cannot intersect in P, for a circumference does not intersect itself. Hence, it intersects in a second point, as P'. Q. E. D.

PROPOSITION III.

182. Theorem.—*If a straight line be drawn through the centres of two circles, of the intersections of either circumference with that line, the one on the side toward the centre of the other circle is the nearest point in this circumference to that centre, and the one on the opposite side is the farthest point from that centre.*

DEM.—Let M and N, or M′ and N′, be two circumferences whose centres are C and O′. Draw an indefinite line through these centres. Let A and H be the intersections of M or M′ with this line, of which A is on the side of M or M′ toward the centre O′, and H is on the opposite side. Then is A the nearest point in M or M′ to O′, and H the farthest point from O′.

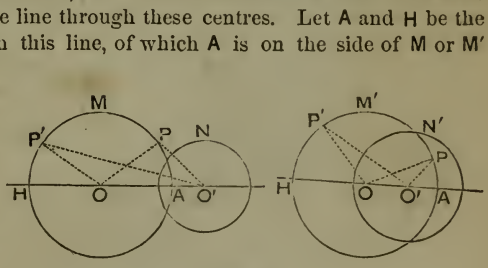

FIG. 144.

First, To show that A is nearer O′ than any other point in the circumference. A will lie between O and O′, in O′, or beyond O′. When A lies between O and O′, as in M, let P be any other point in M, and draw OP and O′P. Now OO′ being a straight line, is less than OPO′, a broken line. Subtracting OA from the former, and its equal OP from the latter, we have AO′ < PO′. When A falls at O′ the truth is self-evident. When A lies beyond O′, as in M′, let P be any other point in M′, and draw OP and O′P. Now O′P + OO′ > OP (= OA). Subtracting OO′ from both, we have O′P > OA − OO′ (= O′A). Hence, in any case, A is the nearest point in M or M′ to O′.

Second, To show that H is the farthest point in M or M′ from O′. In either

figure, let P′ be any other point in the circumference than H, and draw OP′ and O′P′. Now, P′O + OO′ > P′O′. But P′O = HO. Hence HO + OO′ (= HO′) > P′O′

PROPOSITION IV.

183. Theorem.—*When the distance between the centres of two circles is greater than the sum of their radii, the circumferences are wholly exterior the one to the other.*

Dem.—Let M and N be the circumferences of two circles whose centres are O and O′. Let OO′ be greater than the sum of the radii. Then are M and N wholly exterior the one to the other.

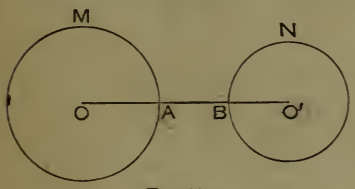

Fig. 145.

For A, the intersection of M with OO′, is between O and O′, since OA < OO′. Now, by hypothesis, OO′ > OA + BO′. Subtracting OA from both, we have AO′ > BO′. Hence, as the nearest point in M is farther from O′ than the circumference of the latter circle, M lies wholly exterior to N. Q. E. D.

184. Cor.—Conversely, *When two circumferences are exterior the one to the other, the distance between their centres is greater than the sum of their radii.*

Dem.—For, join the centres OO′ with a straight line. Now the point A where this line cuts the circumference M is the nearest point in this circumference to the centre O′. But, by hypothesis, this (and every other point in circumference O) is without circle O′. Hence, AO′ > BO′. To each add OA, and OA + AO′ (or OO′) > OA + BO′.

PROPOSITION V.

185. Theorem.—*When the distance between the centres of two circles is equal to the sum of their radii, the circumferences are tangent to each other externally.*

Dem.—Let M and N be two circumferences, and OO′, the distance between their centres, be equal to OC + O′C′, the sum of their radii; then are the circumferences tangent to each other externally.

The point A, where M cuts the line joining the centres, is between O and O′, since OA < OO′ by hypothesis. Moreover, A is the nearest point in M to the centre O′. Again, as OO′ = OC + O′C′, subtracting OA from the first member, and its equal OC from the other, we have O′A = O′C′; that is, A is in the circumference N. Hence, as A lies in N, and all other points in M are more distant from O′ than the length of the radius O′C′, M is entirely without N, except the point A, and the circles are tangent to each other externally. Q. E. D.

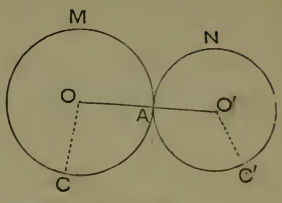

FIG. 146.

186. COR. 1.—Conversely, *When two circumferences are tangent to each other externally, the distance between their centres is equal to the sum of their radii.*

DEM.—M being tangent to N externally, the point in M nearest the centre O′ must be in N, while all other points in M are exterior to N. Now, the point in M nearest to O′ is A on the line joining their centres (***182***). A is therefore the point of tangency, and OO′ = OA + O′A.

187. COR. 2.—*When two circumferences are tangent to each other externally, the point of tangency is in the line joining their centres.*

PROPOSITION VI.

188. Theorem.— *When the distance between the centres of two circles is less than the sum and greater than the difference of their radii, the two circumferences intersect.*

DEM.—Let M and N be the circumferences of two circles whose centres are O and O′. Let the radius of M be equal to or greater than the radius of N. Now, if OO′ <OA + O′B, and > OA—O′B, M and N intersect.

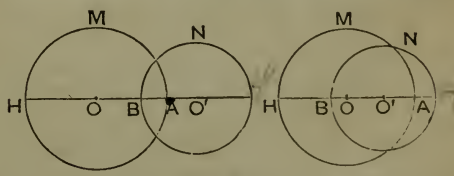

FIG. 147.

For, when OO′ > OA, OO′ < OA + O′B gives OO′ − OA (= AO′) < O′B; and when OO′ $\overline{\leq}$ OA, OO′ > OA − O′B gives O′B > OA − OO′ (= AO′). Hence the nearest point in M to O′ lies within N. Again, to the first member of OO′ > OA − O′B add HO, and to the second its equal OA, and we have OO′ + HO (= HO′) > 2OA − O′B. Now, since O′B $\overline{\leq}$ OA,* by hypothesis, the difference 2OA − O′B $\overline{\geq}$ O′B. Hence, HO′ > O′B, and H lies without N. As,

---

* Read "O′B is equal to or less than OA."

therefore, M has one point at least within N and one without, M and N intersect. Q. E. D.

189. COR.—Conversely, *When two circumferences intersect, the distance between their centres is less than the sum and greater than the difference of their radii.*

DEM.—Let the radius of N be equal to or less than the radius of M. As the circumferences intersect the farthest point H′ of N from O must be farther from O than the length of the radius of M, *i. e.*, must lie without that circle. So we have by hypothesis $H'O > OA$. Subtracting $H'O'$ from the first member and its equal BO' from the second, we have $H'O - O'H' (= OO') > OA - BO'$; that is, the distance between the centres is greater than the difference of the radii. Again, as the nearest point in M to O′ must lie within N, we have $AO' < BO'$, and adding OA to both members, $OA + AO' (= OO') < OA + BO'$; that is, the distance between the centres is less than the sum of the radii.

PROPOSITION VII.

190. Theorem.—*When the distance between the centres of two unequal circles is equal to the difference of their radii, the less circumference is tangent to the other internally.*

DEM.—Let M and N be the circumferences of two circles whose centres O and O′ are so situated that $OO' = OC - O'C'$; then are the circles tangent to each other internally.

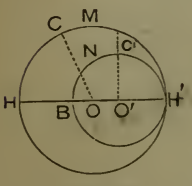

FIG. 148.

For, let N be the circumference of the less circle, so that $OC > O'C'$. Let HH′ be a diameter of M. By hypothesis $OO' = OC - O'C'$. Now, subtracting each member of this equality from OH′, we have $OH' - OO' (= O'H') = O'C'$. Whence it appears that H′, the point in N at the greatest distance from O, is in M; and, consequently, that every other point in N is within M. Hence, N is tangent to M internally. Q. E. D.

191. COR. 1.—Conversely, *When a less circumference is tangent to a greater internally, the distance between their centres equals the difference of their radii.*

DEM.—The less circumference N being tangent to the greater M, internally, the point in N at the greatest distance from the centre O of M, must be in M, while all other points of N lie within M. Now H′ in the line passing through the centres is the point of N at the greatest distance from O. Hence we observe that $OO' = OH' - O'H'$, *i. e.*, the difference between the radii.

192. Cor. 2.—*When one circumference is tangent to another internally, the point of tangency is in the line passing through their centres.*

193. Sch.—If the radii are equal the two circumferences coincide.

PROPOSITION VIII.

194. Theorem.—*When the distance between the centres of two unequal circles is less than the difference of their radii, the less circumference is wholly within the greater.*

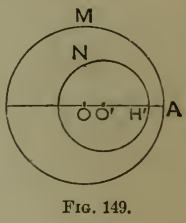

Fig. 149.

Dem.—Let N be a less circumference than M, and OO', the distance between their centres, be less than $OA - O'H'$, the difference of their radii; then is N wholly within M.

For, to each member of $OO' < OA - O'H'$ add $O'H'$, and we have $OO' + O'H' < OA$. But $OO' + O'H' = OH'$. Hence $OH' < OA$, and H', the farthest point in N from O, is within M, and consequently N lies wholly within M. Q. E. D.

195. Cor.—Conversely, *When a less circumference is wholly within a greater, the distance between their centres is less than the difference of their radii.*

Dem.—If N lies wholly within M, the farthest point in N from O, the centre of M, must be nearer O than is any point in M, *i. e.*, $OH' < OA$. Now, subtract $O'H'$ from each member, and we have $OH' - O'H' \ (= OO') < OA - O'H'$. Q. E. D.

196. Sch.—If the centres coincide so that $OO' = 0$, the circumferences are said to be *concentric*. If, at the same time, their radii are equal, they are *coincident*.

PROPOSITION IX.

197. Theorem.—*When two circumferences intersect, the line which passes through their centres is perpendicular to their common chord at its middle point.*

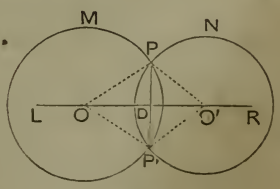

Fig. 150.

Dem.—Let the circumferences M and N intersect in the points P and P' (*181*); let PP' be the common chord, and LR the line passing through the centres O and O'; then is LR perpendicular to PP'.

For O′ is equally distant from the extremities P and P′, and O is also equally distant from P and P′. Hence, as LR has two points equally distant from the extremities of PP′, it is perpendicular to PP′ at its middle point. Q. E. D.

PROPOSITION X.

198. Theorem.— *When one circumference is tangent to another, either externally or internally, they have a common rectilinear* tangent at their common point.*

DEM.—Since the radii of the two circles drawn to the common point form one and the same straight line (***187, 192***), a line perpendicular to one at its extremity is perpendicular to the other also. And a line which is perpendicular to a radius at its extremity is tangent to the circle (***172***). Q. E. D.

199. COR.—*All circumferences having their centres in the same line, and having but one common point, are tangent to each other, and have a common rectilinear tangent at the common point.*

EXERCISES.

1. ***Prob.***—*To pass a circumference through three given points not in the same straight line.*

SUG.—The process should be gone through with as learned from PART I., and then the reasons for the process given as furnished by this section.

2. To pass a circumference through two given points, whose center shall be in a given line.

3. ***Prob.***—*To circumscribe a circumference about a given triangle, and give the reasons for the process.*

4. The centres of two circles whose radii are 10 and 7, are at 4 from each other. What is the relative position of the circumferences? What if the distance between the centres is 17? What if 20? What if 2? What if 0? What if 3?

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* Straight line.

5. Given two circles O and O′, to draw two others, one of which shall be tangent to these externally, and to the other of which the two given circles shall be tangent internally. Give *all* the principles involved in the construction. Give other methods.

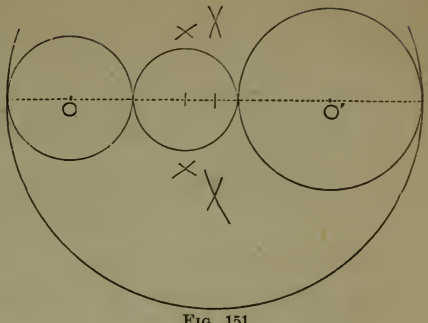


FIG. 151.

6. Given two circles whose radii are 6 and 10, and the distance between their centres 20. To draw a third circle whose radius shall be 8, and which shall be tangent to the two given circles? Can a third circle whose radius is 2 be drawn tangent to the two given circles? How will it be situated? Can one be drawn tangent to the given circles, whose radius shall be 1? Why?

SYNOPSIS.

RELATIVE POSITIONS OF CIRCUMFERENCES.

- PROP. I. Through three points. — *Cor.* 1. A circf. can be passed. *Cor.* 2. A circf. determined by. *Cor.* 3. Intersections of two circf's.
- PROP. II. Two circumferences which intersect in *one* point.
- PROP. III. Points in one circumference nearest to and farthest from the centre of another.
- DISTANCE BETWEEN CENTRES OF TWO CIRCLES.
  - PROP. IV. Greater than sum of radii. — *Cor.* Converse.
  - PROP. V. Equal to sum of radii. — *Cor.* 1. Converse. *Cor.* 2. Point of tangency.
  - PROP. VI. Less than sum and greater than difference of radii. — *Cor.* Converse.
  - PROP. VII. Equal to diff. of radii. — *Cor.* 1. Converse. *Cor.* 2. Point of tangency. *Sch.* Radii equal.
  - PROP. VIII. Less than diff. of radii. — *Cor.* Converse. *Sch.* Concentric, Coincident.
- PROP. IX. Perpendicular to common chord.
- PROP. X. Common tangent to two circles tangent to each other. — *Cor.* To all.
- EXERCISES. — *Prob.* To pass circumference through three points. *Prob.* To circumscribe a triangle with a circumference.

SECTION VI.

OF THE MEASUREMENT OF ANGLES.

200. Angles are said to be measured by arcs, according to the principles developed in the three following propositions.

PROPOSITION I.

201. *Theorem.*—*In the same or in equal circles, equal arcs subtend equal angles at the centre.*

DEM.—In the equal circles M and N, let *arc* AB = *arc* DC; then will the angles O and O′, called angles at the centre, be equal. For, placing N upon M so that O′ shall fall on O, and O′D on OA, since the circles are equal, D will fall on A; and since, by hypothesis, *arc* DC = *arc* AB, C will fall on B. Hence, O′C will coincide with OB, and *angle* O′ = *angle* O, because they coincide when applied. Q. E. D.

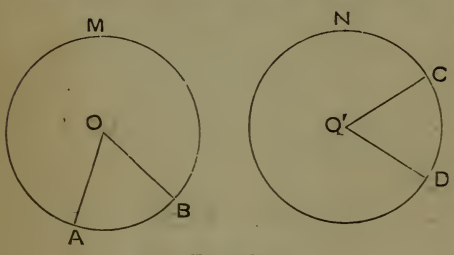


FIG. 152

202. COR. 1.—Conversely, *In the same or in equal circles, equal angles at the centre intercept equal arcs.*

DEM.—If, by hypothesis, *angle* O′ = *angle* O, in the equal circles M and N, *arc* DC = *arc* AB. For, placing circle N upon M, so that O′ shall fall on O, and O′D on its equal OA, D falls on A, and, since *angle* O′ = *angle* O. O′C takes the direction OB, and, being equal to it, C falls on B. Hence, DC and AB coincide and are equal.

203. COR. 2.—*A right angle at the centre intercepts a quarter of a circumference, and is said to be measured by it. Hence, a semicircumference is the measure of two right angles, and a whole circumference of four.*

PROPOSITION II.

204. Theorem.—*In the same or in equal circles, arcs which are in the ratio of two whole numbers subtend angles at the centre which have the same ratio, whence the angles are to each other as the arcs which subtend them.*

DEM.—In the equal circles M and N, let the arcs EF and IH, which subtend the angles O and O′ at the centre, be in the ratio of 5 to 8; then are the angles O and O′ in the ratio of 5 to 8, and we have

angle O : *angle* O′ :: *arc* EF : *arc* IH.

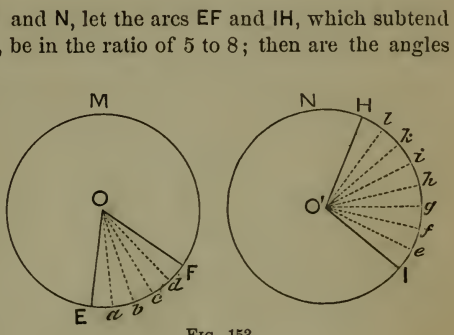


FIG. 153.

For, divide EF into 5 equal parts, as E*a*, *ab*, etc., then IH can be divided into 8 such parts, I*e*, *ef*, etc. Draw the radii O*a*, O*b*, O*c*, etc., and O′*e*, O′*f*, O′*g*, etc.; and, since these partial arcs are equal, the partial angles which they subtend are equal, by the preceding proposition. Now, O is composed of 5 of these angles, and O′ of 8; whence

angle O : *angle* O′ :: 5 : 8.
But, *arc* EF : *arc* IH :: 5 : 8.

Hence, the two ratios being equal, we have

angle O′ : *angle* O :: *arc* IH : *arc* EF.

As the same method could be pursued in case the arcs were to each other as any other two whole numbers, the argument is general.

205. COR.—Conversely, *In the same or in equal circles, angles at the centre which are in the ratio of two whole numbers are to each other as their intercepted arcs.*

DEM.—Thus, let *angle* O′ be to *angle* O in the ratio of 8 to 5. Conceive O′ divided into 8 equal partial angles, then will O be divisible into 5 such partial angles. Now, the partial angles being equal, their intercepted arcs are equal, by the preceding proposition, *Cor.* 1. Whence,

arc IH : *arc* EF :: 8 : 5.
But, *angle* O′ : *angle* O :: 8 : 5, by hypothesis.
Hence, *arc* IH : *arc* EF :: *angle* O′ : *angle* O.

And the same method could be pursued with angles having the ratio of any other whole numbers.

PROPOSITION III.

206. Theorem.—*In the same circle or in equal circles, two incommensurable arcs are to each other as the angles which they subtend at the centre.*

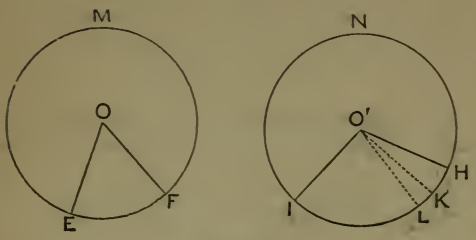


FIG. 154.

DEM.—In the equal circles M and N, let EF and IH be incommensurable arcs. Now there is some arc to which EF bears the same ratio as *angle* O to *angle* O′. If that arc is not IH let it be IL, an arc *less* than IH, so that

angle O : *angle* O′ : : *arc* EF : *arc* IL.*

Conceive EF divided into equal parts, each of which is less than LH,† the assumed difference between IH and IL. Then conceive one of these equal parts to be applied to IH as a measure, beginning at I. Since the measure is less than LH, at least one point of division must fall between L and H. Suppose K to be such a point. Draw O′K. Now, the arcs EF and IK are commensurable, and by the last proposition

angle O : *angle* IO′K : : *arc* EF : *arc* IK. But we assumed that

angle O : *angle* IO′H : : *arc* EF : *arc* IL.

In these proportions the antecedents being alike, the consequents should be proportional, so that

angle IO′K should be to *angle* IO′H : : *arc* IK : *arc* IL.

But this proportion is false, since

angle IO′K < *angle* IO′H, whereas *arc* IK > *arc* IL.

In a manner altogether similar (the student should supply it) we can show that

angle O is not to *angle* O′ : : *arc* EF : any *arc greater* than IH.

Hence, as the fourth term of the proportion cannot be less or greater than IH, it must be IH itself; and

angle O : *angle* O′ : : *arc* EF : *arc* IH. Q. E. D.

207. COR.—Conversely, *In the same or in equal circles, two incommensurable angles at the centre are to each other as the arcs which they intercept.*

---

* This is a false hypothesis, and the object of the argument following is to show its falsity.

† This can be done by supposing EF bisected, then the halves bisected, then the fourths bisected, and this process of bisection continued until the parts are each less than LH.

DEM.—In the equal circles M and N, O and O′ being incommensurable angles at the centre, are to each other as the arcs EF and IH. If not, let us suppose

arc EF : *arc* IH : : *angle* O : *angle* IO′L, an angle less than O′.

Divide O into equal partial angles, each less than LO′H, the assumed difference between IO′H and IO′L. Also conceive this angle to be applied as a measure to IO′H, beginning at O′I. At least one line of division will fall between O′L and O′H. Let O′K be such a line. Now, as O and IO′K are commensurable, we have by (***205***),

arc EF : *arc* IK : : *angle* O : *angle* IO′K.

But by supposition

arc EF : *arc* IH : : *angle* O : *angle* IO′L.

Therefore, since the antecedents are the same,

arc IK should be to *arc* IH : : *angle* IO′K : *angle* IO′L.

But this is false, since

arc IK < *arc* IH, whereas *angle* IO′K > *angle* IO′L.

Whence we learn that the fourth term of the proportion cannot be less than *angle* IO′H. In a similar manner it can be shown (let the student do it) that it cannot be greater. Hence it must be IO′H itself; and

arc EF : *arc* IH : : *angle* O : *angle* IO′H.

208. SCH.—Out of the truths developed in the three preceding propositions grows the method of representing angles by degrees, minutes, and seconds, as given in Trigonometry (PART IV., ***3–6***). It will be observed, that in all cases, if arcs be struck *with the same radius*, from the vertices of angles as centres, the angles bear the same ratio to each other as the arcs intercepted by their sides. Hence the arc is said to measure the angle. Though this language is convenient, it is not quite natural; for we naturally measure a quantity by another of *like kind*. Thus, *distance* (length) we measure by *distance*, as when we say a line is 10 inches long. The line is *length;* and its measure, an inch, is *length* also. So, likewise, we say the area of a field is 4 acres : the quantity measured is *a surface;* and the measure, an acre, is *a surface* also. Yet, notwithstanding the artificiality of the method of measuring angles by arcs, instead of directly by angles, it is not only convenient but universally used; and the student must know just what is meant by it. For example, a circumference is conceived as divided into 360 equal arcs, called degrees. Hence, as a right angle at the centre is subtended by one-fourth of the circumference, it is called an angle of 90 degrees. 180 degrees is the measure of two right angles, 45 degrees, of half a right angle, etc. Thus we get a perfectly definite idea of the

magnitude of an angle from the statement of the number of degrees which measure it; and, for brevity, the angle is spoken of as an angle of the same number of degrees as the intercepted arc.

---

209. An Inscribed Angle is an angle whose vertex is in a circumference, and whose sides are chords, or a chord and diameter, of that circumference.

PROPOSITION IV.

210. Theorem.—*An inscribed angle is measured by half the arc intercepted between its sides.*

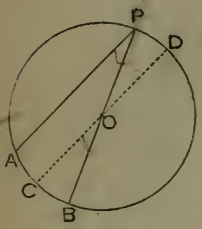


FIG. 155.

DEM.—*First, when one side is a diameter.* Let APB be an inscribed angle, and PB a diameter; then is APB measured by one-half of arc AB. For, through the centre O, draw the diameter DC parallel to the chord PA; then COB = POD (***134***), whence *arc* CB = *arc* PD (***202***), also COB = APB (***152***); and *arc* PD = *arc* AC (***174***), whence PD = CB = $\frac{1}{2}$AB. Now COB is measured by CB (***208***); hence APB is measured by CB = $\frac{1}{2}$AB. Q. E. D.

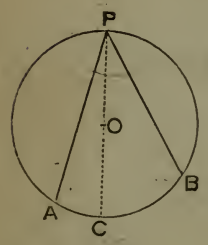


FIG. 156.

Second, when both sides are chords and the centre of the circle lies between them. Let APB be such an angle. Draw the diameter PC. Now, by the preceding part, APC is measured by $\frac{1}{2}$AC, and CPB by $\frac{1}{2}$CB. Hence APC + CPB, or APB, is measured by $\frac{1}{2}$AC + $\frac{1}{2}$CB, or $\frac{1}{2}$AB. Q. E. D.

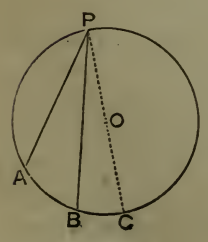
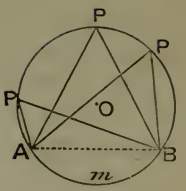
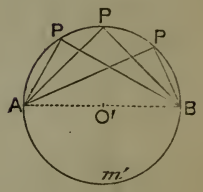


FIG. 157.

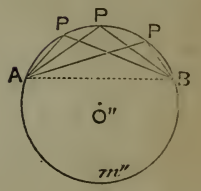
Third, when both sides are chords and the centre lies without the angle. Let APB be such an angle. Draw the diameter PC. Now APC is measured by $\frac{1}{2}$AC, and BPC by $\frac{1}{2}$BC. Hence APC − BPC, or APB, is measured by $\frac{1}{2}$AC − $\frac{1}{2}$BC, or $\frac{1}{2}$AB. Q. E. D.



211. COR.—*In the same or equal circles all angles inscribed in the same segment, intercept equal arcs, and are consequently equal. If the segment is less than a semicircle, the angles are obtuse; if a semicircle, right; if greater than a semicircle, acute.*



ILL.—In each separate figure the angles P are equal, for they are each measured by half the same arc. . . . In O, each angle P is acute, being measured by $\frac{1}{2}m$, which is less than a quarter of a circumference. . . . In O′, each angle P is a right angle, being measured by $\frac{1}{2}m'$, which is a quadrant (quarter of a circumference). . . . In O″, each angle P is obtuse, being measured by $\frac{1}{2}m''$, which is greater than a quadrant.

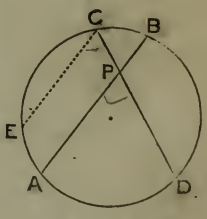


SCH.—The converse of this proposition is usually taken for granted; *i.e.*, that if the several angles P, P, etc., are equal and subtended by the same chord, their vertices lie in the circumference. This is readily proved rigorously after the next two propositions. Thus, if vertex P were without, the angle would be measured by $\frac{1}{2}$AB $-\frac{1}{2}$ the other intercepted arc; and if within, by $\frac{1}{2}$AB $+\frac{1}{2}$ the other intercepted arc.

FIG. 158.

PROPOSITION V.

212. Theorem.—*Any angle formed by two chords intersecting in a circle is measured by one-half the sum of the arcs intercepted between its sides and the sides of its vertical, or opposite, angle.*



DEM.—Let the chords AB and CD intersect at P; then is APD, or its equal CPB, measured by $\frac{1}{2}$(AD + CB); and APC, or its equal BPD, is measured by $\frac{1}{2}$(AC + BD).

For, through C draw CE parallel to BA; whence ECD = APD (***152***), and CB = EA (***174***). But ECD is measured by $\frac{1}{2}$ED (***210***), which equals $\frac{1}{2}$(AD + EA) = $\frac{1}{2}$(AD + CB).

FIG. 159.

That APC, or its equal BPD, is measured by $\frac{1}{2}$(AC + BD), appears from the fact that the sum of the four angles about P being equal to four right angles, is measured by a whole circumference (***208***). But APD + CPB is measured by AD + CB; whence APC + BPD, or 2 APC, is measured by the whole circumference minus (AD + CB); that is, by AC + BD. Then is APC measured by $\frac{1}{2}$(AC + BD).

213. Sch.—The case of the angle included between two chords passes into that of the inscribed angle in the preceding proposition, by conceiving AB to move parallel to its present position until P arrives at C and BA coincides with CE. The angle APD is all the time measured by half the sum of the intercepted arcs; but, when P has reached C, CB becomes 0, and APD becomes an inscribed angle measured by half its intercepted arc.

In a similar manner we may pass to the case of an angle at the centre, by supposing P to move toward the centre. All the time APD is measured by $\frac{1}{2}(AD + CB)$; but, when P reaches the centre, $AD = CB$, and $\frac{1}{2}(AD + CB) = \frac{1}{2}(2AD) = AD$; *i. e.*, an angle at the centre is measured by its intercepted arc.

PROPOSITION VI.

214. Theorem.—*An angle formed by two secants meeting without the circle is measured by one-half the difference of the intercepted arcs.*

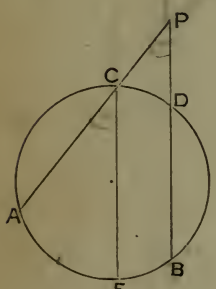


Fig. 100.

Dem.—Let APB be an angle formed by the two secants AP and PB; then is it measured by $\frac{1}{2}(AB - CD)$, *i. e.*, one-half the intercepted arcs.

For, draw CE parallel to PB; then $CD = EB$ (***174***), and ACE = its corresponding angle APB. But ACE is measured by $\frac{1}{2}\,AE = \frac{1}{2}(AB - EB) = \frac{1}{2}(AB - CD)$. Q. E. D.

215. Sch.—This case passes into that of an inscribed angle, by conceiving P to move toward C, thus diminishing the arc CD. When P reaches C, the angle becomes inscribed; and, as CD is then 0, $\frac{1}{2}(AB - CD) = \frac{1}{2}AB$. Also, by conceiving P to continue to move along PA, CD will reappear *on the other side of* PA, hence will change its sign,* and $\frac{1}{2}(AE - CD)$ will become $\frac{1}{2}(AE + CD)$, as it should, since the angle is then formed by two chords intersecting within the circumference.

PROPOSITION VII.

216. Theorem.—*An angle formed by a tangent and a chord drawn from the point of tangency is measured by one-half the intercepted arc.*

* In accordance with the law of positive and negative quantities as used in mathematics, whenever a continuously varying quantity is conceived as diminishing till it reaches 0, and then as reappearing by the same law of change, it must change its sign.

DEM.—Let TPA be an angle formed by TM tangent at P, and the chord PA; then is TPA measured by one-half the intercepted arc AP. For, draw any chord CD parallel to TM and cutting AP. Then CEA = TPA. But CEA is measured by $\frac{1}{2}$ (AC + PD). Hence, as PD = CP (*175*), TPA is measured by $\frac{1}{2}$ (AC + CP), or $\frac{1}{2}$ AP. Q. E. D.

Show that APM is measured by $\frac{1}{2}$ arc A*m*P.

Also, observe how the case of two secants (*214*) passes into this.

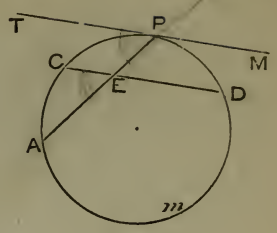


FIG. 161.

PROPOSITION VIII.

217. Theorem.—*An angle formed by two tangents is measured by one-half the difference of the intercepted arcs.*

DEM.—Let APB be an angle formed by the two tangents AP and PB; then is it measured by $\frac{1}{2}$ (*arc* C*m*D − *arc* C*n*D), *i. e.*, one-half the difference of the intercepted arcs. For, through one of the points of tangency, as C, draw a chord, as CE, parallel to the other tangent. Now, ACE is measured by $\frac{1}{2}$ *arc* CE, by the last proposition. But ACE = APB, and *arc* CE = C*m*D − D*m*E = C*m*D − C*n*D, since C*n*D = E*m*D (*175*). Hence, APB is measured by $\frac{1}{2}$ (C*m*D − C*n*D). Q. E. D.

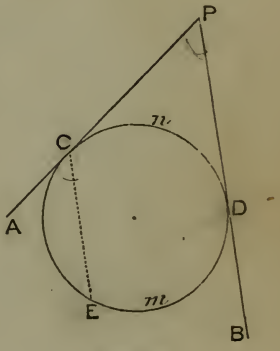


FIG. 162.

218. SCH.—The case of two secants (*214*) passes into this by supposing the secants to separate until they become tangents.

EXERCISES.

1. ***Prob.***—*Through a given point to draw a parallel to a given line, on the principles contained in* (*149*), (*201*), *and* (*165*).

SOLUTION.—Through P to draw a parallel to AB. From P as a centre, with any radius greater than the distance from P to AB, describe an arc cutting AB, as *ac*. From *a* as a centre, with the same radius, strike an arc through P, intersecting AB, as P*b*. Take the chord P*b* and apply it from *a* on the arc *ac*, as *a*O. These chords being equal, the arcs P*b* and

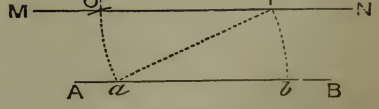


FIG. 163.

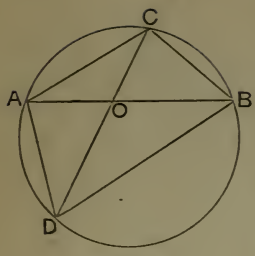


Fig. 164.

aO are equal (***165***). Again, angle Pab = angle OPa, since they are measured by equal arcs struck with the same radius (***201***). These alternate angles being equal, MN is parallel to AB (***149***).

2. In *Fig.* 164 there are 4 pairs of equal angles. Which are they, and why? Show also that COB = ABD + CDB, by (***210***), and (***212***). Show also that DOB = ABC + DAB.

3. ***Prob.***—*From a point without a circle to draw a tangent to the circle.*

Solution.—Let O be the given circle, and P the given point. Join P with the centre O, and upon PO as a diameter describe a circle. Let T and T′ be the intersections of the two circumferences. Now, if lines be drawn from P through T and T′, they will be tangent to the circle O. For OTP and OT′P, being inscribed in semi-circles, are right angles (***211***). Hence, PM is perpendicular to radius OT at its extremity T, and is therefore a tangent (***172***). In like manner PT′ is shown to be a tangent, and we see that *from a point without a circle two tangents can be drawn to the circle.*

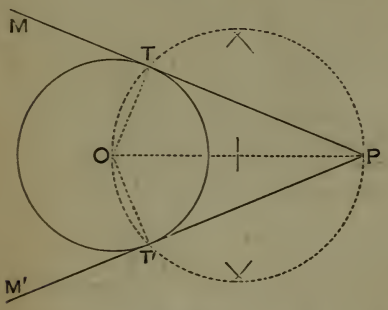


Fig. 165.

4. ***Prob.***—*On a given line, to construct a segment which shall contain a given angle.*

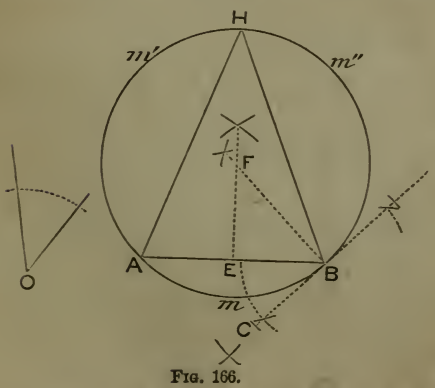


Fig. 166.

Solution. — Let AB be the given line, and O the given angle. At one extremity of the given line, as B, construct an angle ABC equal to the given angle O, which shall lie on the opposite of AB from that on which the required segment is to lie. Erect a perpendicular to the line CB at B, and also a perpendicular bisecting AB. Let FB and FE be

these perpendiculars, intersecting at F. From F as a centre, with a radius equal to FB, describe a circle. Then is AHB the segment required. For, CB being perpendicular to radius FB at its extremity, is tangent to the circle, and *angle* ABC (= *angle* O) is measured by ½ of arc A*m*B (*216*). Now, any angle inscribed in the segment A*m'm''*B, as AHB, has ½ A*m*B for its measure, and is, consequently, equal to O.

ANOTHER SOLUTION.—On the side of AB on which the segment is to lie, draw *any* line through either extremity of AB, *making an acute angle* with AB. Let CB be such a line. At *any* point in CB, as C, draw a line CE, making angle ECB = the given angle O, *Fig.* 166. Through A pass a parallel to CE (see *Ex.* 1), as AD. Pass a circumference through A, D, and B. Any angle inscribed in segment A*m*B is equal to O. [Let the student give all the reasons, and make the construction. The requisite marks for the construction are made in the figure. Why is it said, make CBA an *acute angle?* When would a right angle answer? When an obtuse angle?]

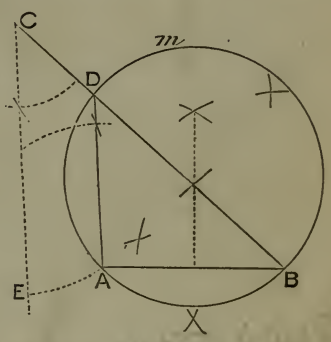


FIG. 167.

SYNOPSIS.

MEASUREMENT OF ANGLES.

- How angles are measured.
- FUNDAMENTAL PROPOSITIONS.
  - PROP. I. Equal arcs subtend equal angles at the centre. — *Cor.* 1. Converse. *Cor.* 2. Measure of 1, 2, and 4 right angles.
  - PROP. II. Commensurable arcs in the same ratio as their subtended angles. — *Cor.* Converse.
  - PROP. III. Incommensurable arcs. — *Cor.* Converse. *Sch.* Method of measuring angles.
- Inscribed angle, what?
- PROP. IV. Inscribed angle, how measured. — *Cor.* In segment, >, =, or < semicircle.
- PROP. V. Angle between two chords. — *Sch.* Compared with preceding.
- PROP. VI. Angle between two secants. — *Sch.* Compared with Prop. IV.
- PROP. VII. Angle between tangent and chord.
- PROP. VIII. Angle between two tangents. — *Sch.* Compared with Prop. VI.
- EXERCISES.
  - *Prob.* To draw a parallel through a given point.
  - *Prob.* To draw a tangent to a circle from a point without.
  - *Prob.* To construct a segment on a given line which shall contain a given angle.

SECTION VII.

OF THE ANGLES OF POLYGONS, AND THE RELATION BETWEEN THE ANGLES AND SIDES.

OF TRIANGLES.

PROPOSITION I.

219. Theorem.—*The sum of the three angles of a triangle is two right angles.*

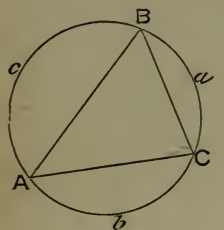


FIG. 168.

DEM.—Conceive a circumference passed through the vertices of the triangle, as *abc*, through the vertices of the triangle ABC (*58*). The angle A is measured by $\frac{1}{2}$ *arc a*, B by $\frac{1}{2}$ *b*, and C by $\frac{1}{2}$ *c*. Hence, A + B + C is measured by $\frac{1}{2}$ $(a + b + c)$, or a semi-circumference, and is equal to two right angles (*203*). Q. E. D.

220. COR. 1.—*A triangle can have only one right angle, or one obtuse angle.* Why?

221. COR. 2.—*Two angles of a triangle, or their sum, being given, the third may be found by subtracting this sum from two right angles,* i. e., *either angle is the supplement of the other two.*

222. COR. 3.—*The sum of the two acute angles of a right-angled triangle is equal to one right angle;* i.e., *they are complements of each other.*

223. COR. 4.—*If the angles of a triangle are equal each to each, any one is one-third of two right angles, or two-thirds of one right angle.*

PROPOSITION II.

224. Theorem.—*The sides of a triangle sustain the same* GENERAL *relation to each other as their opposite angles; that is, the greatest side is opposite the greatest angle, the second greatest side opposite the second greatest angle, and the least side opposite the least angle.*

DEM.—In the triangle ABC let C > B > A be the order of the values of the angles; then AB > AC > BC is the order of the values of the sides.

For, circumscribe the circumference *abc*. The angle C being greater than B, the arc *c*, the half of which measures C, is greater than the arc *b*, the half of which measures B. Now, the greater arc has the greater chord (***166***). Hence, AB > AC. In like manner, if B > A, arc *b* > arc *a*, and AC > BC. If either angle, as C, is obtuse, AB is greater than AC or BC, because it lies nearer the centre (***167***).

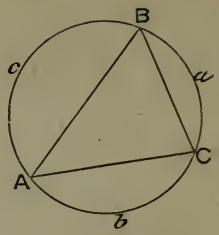


FIG. 169.

225. COR. 1.—Conversely, *The order of the magnitudes of the sides being* AB > AC > BC, *the order of the magnitudes of the angles is* C > B > A.

[Let the student give the demonstration in form.]

226. COR. 2.—*An equiangular triangle is also equilateral; and, conversely, an equilateral triangle is equiangular.*

DEM.—If A = B = C, arc *a* = arc *b* = arc *c*, and, consequently, chord BC = chord AC = chord AB. Conversely, if the chords are equal, the arcs are, and hence the angles subtended by these arcs.

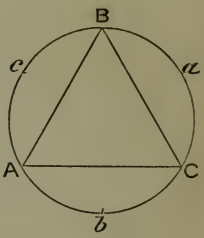


FIG. 170.

227. COR. 3.—*In an isosceles triangle the angles opposite the equal sides are equal; and, conversely, if two angles of a triangle are equal, the sides opposite are equal, and the triangle is isosceles.*

DEM.—If AB = BC, arc *a* = arc *c*; and hence, angle A, measured by ½ *a*, = angle C, measured by ½ *c*. Conversely, if A = C, arc *a* = arc *c*; and hence chord BC = chord AB.

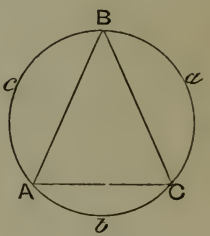


FIG. 171.

228. SCH.—It should be observed that the proposition gives only the *general* relation between the angles and sides of a triangle. It is not meant that the sides are *in the same ratio* as their opposite angles: this is not true. Thus in *Fig.* 172 angle *c* is twice as great as angle *a*; but side *c* is not *twice* as great as side *a*, although it *is greater*. Trigonometry discovers the *exact* relation which exists between the sides and angles.

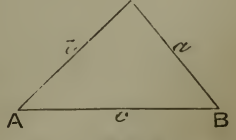


FIG. 172.

PROPOSITION III.

229. Theorem.—*If from any point within a triangle lines be drawn to the extremities of any side, the included angle is greater than the angle of the triangle opposite this side.*

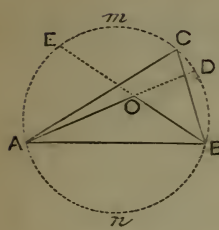


Fig. 173.

Dem.—Let OA and OB be two lines drawn from any point O within the triangle ABC, to the extremities of the side AB; then angle AOB > ACB.

For, circumscribe a circle about the triangle. Now, ACB is measured by $\frac{1}{2}$ A*n*B, but AOB is measured by $\frac{1}{2}$ (A*n*B + E*m*D). Therefore, AOB > ACB. Q. E. D.

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230. An Exterior Angle of a polygon is an angle formed by any side with its adjacent side produced, as CBD, *Fig.* 174.

PROPOSITION IV.

231. Theorem.—*An exterior angle of a triangle is equal to the sum of the two interior non-adjacent angles.*

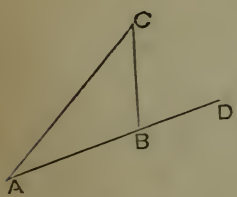

Fig. 174.

Dem.—Let ABC be any triangle, and CBD an exterior angle; then CBD = A + C.

For CBD is the supplement of CBA by (***131***), and CBA is the supplement of A + C by (***221***). Hence, CBD = A + C. Q. E. D.

232. Cor.—*Either angle of a triangle not adjacent to a specified exterior angle, is equal to the difference of this exterior angle and the other non-adjacent angle.*

Thus, since CBD = A + C, by transposition, CBD − A = C, and CBD − C = A.

OF QUADRILATERALS.

PROPOSITION V.

233. Theorem.—*The sum of the angles of a quadrilateral is four right angles.*

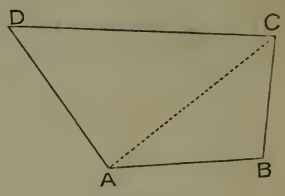

FIG. 175.

DEM.—Let ABCD be any quadrilateral; then DAB + ABC + BCD + CDA = four right angles.

For, draw either diagonal, as AC, dividing the quadrilateral into two triangles. Then, as the sum of the angles of the two triangles is the same as the sum of the angles of the quadrilateral, and the sum of the angles of the triangles is twice two right angles (***219***); the sum of the angles of the quadrilateral is four right angles. Q. E. D.

PROPOSITION VI.

234. Theorem.—*The opposite angles of any quadrilateral which can be inscribed in a circle are supplementary.*

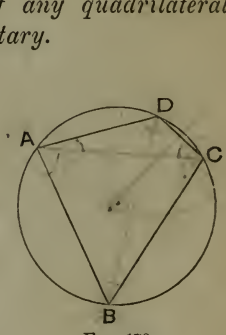

FIG. 176.

DEM.—Let ABCD be any inscribed quadrilateral; then A + C = two *right angles*, and D + B = two *right angles*.

For, A is measured by ½ *arc* BCD, and C is measured by ½ *arc* DAB (***210***). Hence, A + C is measured by one-half a circumference, and is, therefore, equal to two right angles (***203***). In like manner D is measured by ½ *arc* ABC, and B by ½ *arc* ADC. Consequently, D + B is measured by one-half a circumference, and is, therefore, equal to two right angles.

PROPOSITION VII.

235. Theorem.—*The opposite angles of a parallelogram are equal, and the adjacent angles are supplementary.*

DEM.—ABCD, *Fig.* 177, being any parallelogram, A = C, B = D, and B + C, C + D, D + A, and A + B, each = *two right* angles.

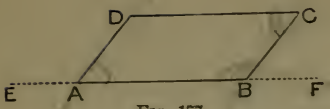

FIG. 177.

For, produce any side, as AB, forming the two exterior angles EAD and CBF. Since CB and DA are parallel, and FE cuts them, the corresponding angles, CBF and DAB are equal (*152*). Again, since EF and DC are parallel, and CB cuts them, the alternate interior* angles CBF and C are equal (*152*). Hence, as DAB and C are each equal to CBF, they are equal to each other. In a similar manner D can be proved equal to CBA. [Let the student give the proof.]

That the angles B and C of the parallelogram are supplemental is evident from (*150*), which proves that the sum of two interior angles on the same side of a secant cutting two parallels is two right angles. For a like reason A + D = *two right angles*, etc.

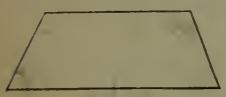
FIG. 178.

236. COR. 1.—*The two angles of a trapezoid adjacent to either one of the two sides not parallel are supplemental.*

[Let the student show why.]

237. COR. 2.—*If one angle of a parallelogram is right, the others are also, and the figure is a rectangle.*

PROPOSITION VIII.

238. Theorem.—Conversely to the last, *If the adjacent angles of a quadrilateral are supplementary, or the opposite angles equal, the figure is a parallelogram.*

DEM.—If A + D = two right angles, AB and DC are parallel by (*147*). For a like reason, if D + C = two right angles, DA and CB are parallel. Again, if A = C and D = B, by adding we have A + D = C + B. But A + D + C + B = four right angles (*233*). Hence, A + D = two right angles, and AB and CD are parallel. So, also, A + B can be shown to be equal to two right angles; and, consequently, AD and CB are parallel.

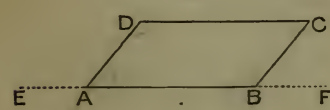

FIG. 179.

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* Interior with reference to the parallels (146).

PROPOSITION IX.

239. Theorem.—*If two opposite sides of a quadrilateral are equal and parallel, the figure is a parallelogram.*

DEM.—In (*a*) let DC be equal and parallel to AB; then is ABCD a parallelogram.

For, drawing the diagonal AC, it makes the angles ACD and CAB equal, since they are alternate interior angles (***152***). Conceive the quadrilateral divided in this diagonal into two triangles, as in (*b*). Reverse the triangle ACB and place it as in (*c*). Draw DB. Since angle DCA = angle CAB, and DC = BA, if CBA be revolved upon AC, AB will take the direction CD, B will fall in D, and CBA will coincide with ADC. Hence, angle ACB = angle DAC. But in (*a*) these are alternate interior angles made by AC with AD and BC. Therefore, AD is parallel to BC (***149***).

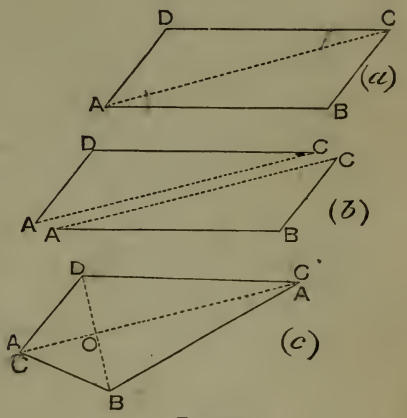

FIG. 180

Q. E. D.

PROPOSITION X.

240. Theorem.—*If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.*

DEM.—In (*a*) let AB = DC, and AD = BC; then is ABCD a parallelogram.

For, divide the quadrilateral in the diagonal AC, and reversing the triangle ABC, place it as in (*c*), and draw DB. Since AB = CD, and CB = AD, DB is perpendicular to CA (***130***). Now, revolving ABC upon CA, it will coincide with ADC. Hence, angle DCA = angle CAB, and AB is parallel to DC. Also, angle DAC = angle BCA, and AD is parallel to BC. Therefore, ABCD is a parallelogram.

Q. E. D.

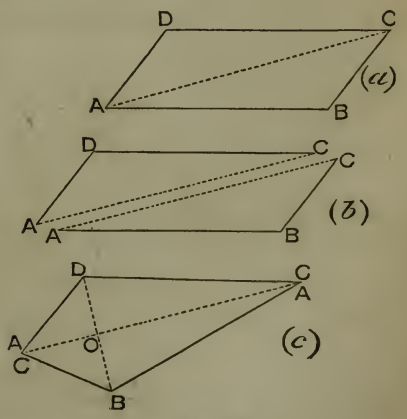

FIG. 181.

PROPOSITION XI.

241. Theorem.—Conversely to the last, *The opposite sides of a parallelogram are equal.*

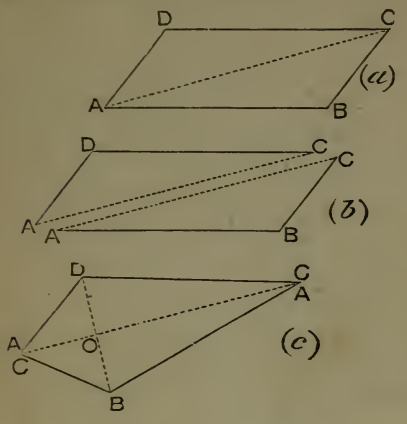

FIG. 182.

DEM.—Let ABCD be a parallelogram; then AB = DC, and AD = CB.

Since DC is parallel to AB, angle DCA = angle CAB. Also, since AD is parallel to BC, angle DAC = angle ACB (*152*). Now, divide the parallelogram (*a*) in the diagonal, and place ABC as in (*c*). Revolve ABC on AC, until it falls in the plane on the other side of AC. Since angle BAC = angle ACD, AB will take the direction CD, and B will fall in CD, or CD produced. Since angle BCA = angle DAC, CB will take the direction AD, and B will fall in AD, or in AD produced. Therefore, as B falls at the same time in AD and CD, it falls at the intersection D, and the triangles coincide. Hence, AB = CD, and AD = CB. Q. E. D.

242. COR. 1.—*Parallels intercepted between parallels are equal.*

243. COR. 2.—*A diagonal of a parallelogram divides it into two equal triangles.*

PROPOSITION XII.

244. Theorem.—*The diagonals of a parallelogram mutually bisect each other.*

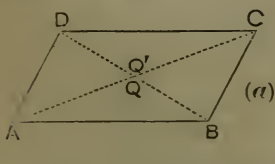

FIG. 183.

DEM.—Let AC and DB be the diagonals of the parallelogram ABCD (*a*), and Q their intersection; then, DQ = QB, and AQ = QC.

For, take the triangle AQB, and apply it to DQ'C, by placing BA in its equal DC, B falling at D, and A at C, with the vertices Q and Q' on the same side of this common line, as in (*b*). Now, since angle QBA = Q'DC (*152*), BQ will take the direction DQ', and Q will fall in DQ', or in DQ' produced. For a like reason AQ will take the direction CQ', and Q will fall in CQ', or in CQ' produced. Hence, as Q falls at the same time in DQ' and CQ', it falls at their intersection Q'; whence BQ = DQ', and AQ = CQ'. Q. E. D.

PROPOSITION XIII.

245. Theorem.—*The diagonals of a rhombus bisect each other at right angles.*

DEM.—Let AC and DB be the diagonals of the rhombus ABCD; then are they at right angles to each other, and bisected at Q.

For, since AB = AD, and DC = CB, AC has two of its points equally distant from D and B, and is, therefore, perpendicular to DB, at its middle point (***130***). In like manner D and B are each equidistant from A and C, whence Q is the middle point of AC.

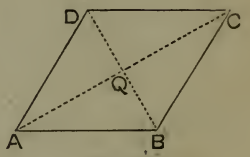

FIG. 184.

246. COR.—*The diagonals of a rhombus bisect its angles.*

For, revolve ABC upon AC as an axis, and it will coincide with ADC. Hence angles A and C are bisected. In like manner revolve DAB upon DB, and it will coincide with DCB. Hence D and B are bisected.

PROPOSITION XIV.

247. Theorem.—*The diagonals of a rectangle are equal.*

DEM.—Let AC and DB be the diagonals of the rectangle ABCD; then AC = DB.

For, upon AC as a diameter describe a circle. Since D and B are right angles, they are inscribed in semicircles (***211***), and DB is a diameter. Therefore, AC = DB.

Q. E. D.

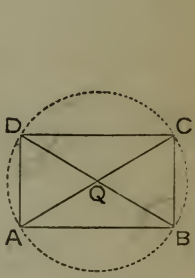

FIG. 185.

248. COR.—Conversely, *If the diagonals of a parallelogram are equal, the figure is a rectangle.*

DEM.—Since the diagonals of a parallelogram bisect each other, if they are equal, a circumference described from their intersection as a centre, with a radius equal to half of a diagonal, will pass through the vertices of the parallelogram. Hence the diagonals will be diameters, and the angles will be inscribed in semicircles, and consequently will be right angles.

OF POLYGONS OF MORE THAN FOUR SIDES.

249. A Salient Angle of a polygon is one whose sides, when produced, can only extend *without* the polygon.

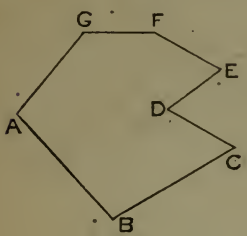

Fig. 186.

250. A Re-entrant Angle of a polygon is one whose sides, when produced, can extend *within* the polygon.

Ill.—In the polygon ABCDEFG, all the angles are salient except D, which is re-entrant.

251. A Convex Polygon is a polygon which has only salient angles. A polygon is always supposed to be convex, unless the contrary is stated.

252. A Concave or Re-entrant Polygon is a polygon with at least one re-entrant angle.

PROPOSITION XV.

253. Theorem.—*The sum of the interior angles of a polygon is equal to twice as many right angles as the polygon has sides, less four right angles.*

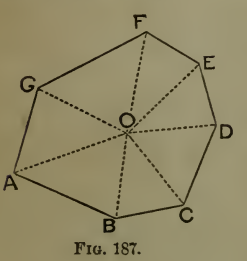

Fig. 187.

Dem.—Let n be the number of sides of any polygon; then the sum of its angles is

n times *two right angles* — 4 *right angles.*

For, from any point O, within, draw lines to the vertices of the angles. As many triangles will thus be formed as the polygon has sides, that is, n. The sum of the angles of these triangles is

n times *two right angles* (***219***).

But this exceeds the sum of the angles of the polygon by the sum of the angles at the common vertex O, that is, by 4 *right angles.* Hence the sum of the angles of the polygon is

n times *two right angles* — 4 *right angles.* Q. E. D.

254. Sch. 1.—The sum of the angles of a pentagon is 5 times *two right angles* — 4 *right angles*, or 6 *right angles.* The sum of the angles of a hexagon is 8 *right angles;* of a heptagon, 10; of an octagon, 12, etc.

255. Sch. 2.—This proposition is equally applicable to triangles and to quadrilaterals. Thus the sum of the angles of a triangle is 3 times *two right angles* − 4 *right angles* (or 6 − 4) = 2 right angles. So also the sum of the angles of a quadrilateral is 4 times *two right angles* − 4 *right angles*, or 4 right angles.

256. Sch. 3.—To find the value of an angle of an equiangular polygon, that is, one whose angles are equal each to each, divide the sum of all the angles by the number of angles.

PROPOSITION XVI.

257. Theorem.—*If the sides of a polygon be produced so as to form one exterior angle (and only one) at each vertex, the sum of these exterior angles is four right angles.*

Dem.—Let n be the number of sides of any polygon. At each of the n angles, there is an interior and an exterior angle, whose sum, as A + a, is two right angles. Hence the sum of all the exterior and interior angles is n times *two right angles*. Now, from this sum subtracting the sum of the exterior angles, the remainder is the sum of the interior angles. But, by the preceding proposition, 4 *right angles* subtracted from n times *two right angles*, leaves the sum of the interior angles. Therefore the sum of the exterior angles is 4 *right angles*. Q. E. D.

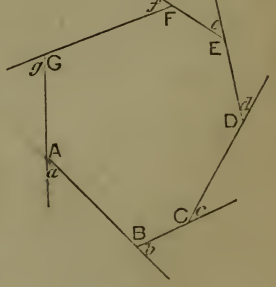

Fig. 183.

OF REGULAR POLYGONS.

PROPOSITION XVII.

258. Theorem.—*The angles of an inscribed equilateral polygon are equal; and the polygon is regular.*

Dem.—Let ABCDEF be an inscribed polygon, with AB = BC = CD, etc.; then is angle A = B = C = D, etc., and the polygon is regular.

For, from the centre of the circle draw OF, OA, and OB, and also the perpendiculars Oa and Ob. Revolve OFA upon OA as an axis, until it falls in the

plane of OAB. Since the chords FA and AB are equal, the *arc* FA = *arc* AB, and F falls at B. Hence the triangles OFA and OAB coincide. The angle A of the polygon is therefore bisected by OA; that is, OAF = OAB. In the same manner OBA can be shown equal to OBC. Moreover, since OA and OB are equal oblique lines drawn from a point in the perpendicular, *angle* OAB = *angle* OBA. Hence, as the half of A equals the half of B, A = B. In like manner, B can be shown equal to C, C to D, D to E, etc. Therefore the polygon is equiangular, as well as equilateral, and consequently regular (***117***). Q. E. D.

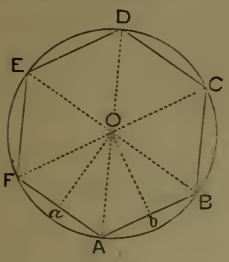

FIG. 189.

PROPOSITION XVIII.

259. Theorem.—*The sides of an inscribed equiangular polygon are equal when their number is* ODD; *and the polygon is regular.*

DEM.—Let ABCDEFG be an inscribed equiangular polygon of an *odd* number of sides; then is side AB = BC = CD, etc., and the polygon is regular.

For, from the centre of the circle draw the radii OA, OB, etc., to the vertices of the polygon, and O*a*, O*b*, etc., perpendicular to the sides. Revolve the quadrilateral OCA*a*, upon O*a* as an axis until it falls in the plane of OCB*a*. Since O*a* is perpendicular to the chord AB, A*a* = *a*B, and A will fall at B. Also, as the angle A of the polygon = B, AC will fall in BC. Now C falls at the same time in the arc BCD (***158***) and in BC, and hence falls at their intersection C. Therefore AC = BC. In like manner revolving OBC*c* upon O*c* as an axis, BC is found equal to ED. So also we can show that ED = FC; then that FC = AB; then that AB = DC; and finally, that DC = EF. Hence we have CA = BC = ED = FC = AB = DC = EF; and as the polygon is equiangular by hypothesis, it is regular (***117***). Q. E. D.

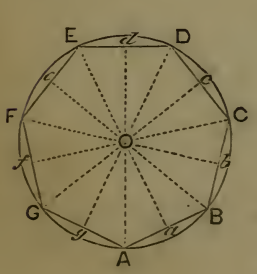

FIG. 190.

260. SCH.—It is easy to see that the above argument would fail in the case of a polygon of an even number of sides, because, in going around the second time the *same* sides would coincide as in going around the first time. Moreover, we can readily inscribe an equiangular polygon of an *even* number of sides which shall not be regular.

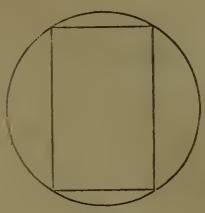
FIG. 191.

PROPOSITION XIX.

261. Theorem.—*The sides of a circumscribed equiangular polygon are equal; and the polygon is regular.*

DEM.—Let ABCDEF be a circumscribed polygon, with angle A = B = C, etc.; then is AB = BC = CD, etc., and the polygon is regular.

For, from the centre of the circle, draw OA, OB, etc., to the vertices of the polygon, and O*a*, O*b*, etc., to the points of tangency. The latter will be perpendicular to the sides by (***173***). Now reverse the triangle A*a*O, and apply it to A*b*O, placing O*a* in its equal O*b*; *a*A will take the direction *b*A. Then will OA of the triangle A*a*O, fall in OA of the triangle A*b*O, since there cannot be two equal oblique lines on the same side of O*b* (***140***). Hence *angle b*AO = *angle a*AO, and *b*A = *a*A. In the same way it can be shown that OB, OF, etc., bisect the other angles, and that *b*B = B*c*, etc. Whence, as the polygon is equiangular, these halves are equal, that is, OA*a* = OF*a*, etc. Then, as OA and OF make equal angles with AF, they cut off equal distances from *a*, and A*a* = *a*F. So, likewise, we can show that A*b* = *b*B, and that each side is bisected at the point of tangency. Therefore, as the halves of the sides are equal, the polygon is equilateral, as well as equiangular, and consequently regular (***117***). Q. E. D.

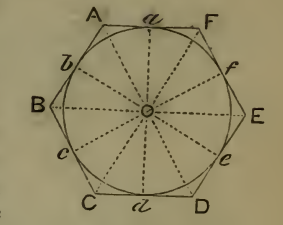

FIG. 192.

PROPOSITION XX.

262. Theorem.—*The angles of a circumscribed equilateral polygon are equal when their number is* ODD; *and the polygon is regular.*

DEM.—Let ABCDE be a circumscribed polygon with AB = BC = CD, etc.; then is *angle* A = B = C = D, etc., and the polygon is regular.

In the same manner as in the preceding demonstration, we may show that OA, OB, etc., bisect the angles of the polygon. [The student should go through the process.] Then revolve the triangle AOE upon AO as an axis till it falls in the plane of AOB; and as angle OAE = angle OAB, and AE = AB, the triangles will coincide. Hence angle OEA, the half of angle E of the polygon, equals angle OBA the half of B, and E = B. In like manner revolving AOB upon OB, we can show that A = C. So also we find B = D, and D = A. Therefore the polygon is equiangular as well as equilateral, and consequently regular. Q. E. D.

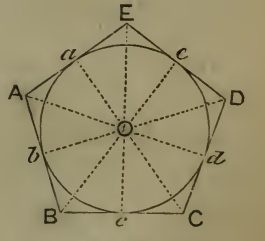

FIG. 193.

263. Sch.—That the above style of argument fails in the case of a polygon of an *even* number of sides, may be observed by attempting to apply it. Thus, from *Fig.* 192, we would have A = C, B = D, C = E, D = F, E = A, and F = B. From these we have A = C = E, and B = D = F. But the process will not give any one of the first three angles equal to any one of the second set. That is, it does not follow that two *adjacent* angles are equal in case the number of sides is *even*. We can readily construct a circumscribed equilateral polygon which shall not be equiangular.

Fig. 194.

PROPOSITION XXI.

264. *Theorem.*—*A circumference may be circumscribed about any regular polygon.***

Dem.—Let ABCDEF be a regular polygon. Bisect AF with a perpendicular O*a*. Any point in this perpendicular is equidistant from A and F. Bisect AB, adjacent to AF, with a perpendicular, as O*b*. Any point in this perpendicular is equidistant from A and B. Hence the intersection of these perpendiculars, O, is equidistant from A, F, and B, and a circumference described from O as a centre, with a radius OA, will pass through F and B. Now revolve the quadrilateral FO*b*A upon O*b* as an axis until it falls in the plane of CO*b*B, *b*A will fall in its equal *b*B; and since angle A = angle B, and side AF = side BC, F will fall at C. Thus it appears that the circumference described from O, and passing through F, A, and B, also passes through C. In a similar manner it can be shown that the same circumference passes through all the vertices, and hence is circumscribed. Q. E. D.

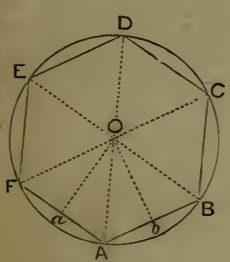

Fig. 195.

265. Cor. 1.—***A circumference may be inscribed in any regular polygon.***

Dem.—For, having circumscribed one about it, the equal sides become equal chords, and hence are equally distant from the centre. If, therefore, a circle be drawn from O as a centre, with O*a* as a radius, it will touch every side of the polygon at its middle point.

266. Cor. 2.—*The centres of the inscribed and circumscribed circles coincide.*

267. *The Centre* of a regular polygon is the common centre of its inscribed and circumscribed circles.

268. An Angle at the Centre of a regular polygon is the angle included by two lines drawn from the centre to the extremities of a side, as FOA, AOB.

269. Cor. 3.—*The angles at the centre of a regular polygon are equal each to each; and any one is equal to four right angles divided by the number of sides of the polygon.*

270. The Apothem of a regular polygon is the distance from the centre to any side, and is the radius of the inscribed circle.

PROPOSITION XXII.

271. Theorem.—*The side of a regular inscribed hexagon is equal to the radius.*

Dem.—Let ABCDEF be a regular inscribed hexagon; then is any side, as BC, equal to OB, the radius.

In the triangle BOC the angle O is measured by the arc BC, or $\frac{1}{6}$ of a circumference, and hence is $\frac{1}{6}$ of 4 right angles, or $\frac{2}{3}$ of a right angle. Angle ABC is measured by $\frac{1}{2}$ arc CDEFA, or $\frac{5}{6}$ of a circumference. Hence angle OBC, which is $\frac{1}{2}$ of ABC, is measured by $\frac{1}{2}$ of $\frac{5}{6}$, or $\frac{5}{12}$ of a circumference, and is, consequently, equal to BOC. So also OCB, the half of DCB, is measured by $\frac{5}{12}$ of a circumference. Hence OCB is equiangular, and consequently equilateral (*226*), and BC = OB. Q. E. D.

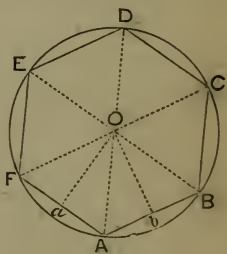

Fig. 196.

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272. A *Broken Line* is said to be *Convex* when no one of its parts will, when produced, enter the space included between it and a line joining its extremities.

PROPOSITION XXIII.

273. Theorem.—*A Convex broken line is less than any broken line which envelops it and has the same extremities.*

Dem.—Let A*bcd*B be a broken line enveloped by the broken line ACDEFB, and having the same extremities A and B; then is A*bcd*B < ACDEFB.

For, produce the parts of A*bcd*B till they meet the enveloping line, as A*b* to *e*, *bc* to *f*, and *cd* to *g*. Now, since a straight line is the shortest path between two points, A*e* < AC*e*, *bf* < *be*DE*f*,

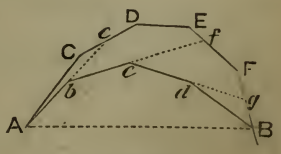

Fig. 197.

$cg < cf$Fg, and dB $< dg$B. Hence, if a point starts from A to move to B, AeDEFB will be a shorter path than ACDEFB, AbfFB shorter than AeDEFB, AbcgB shorter than AbfFB, and AbcdB shorter than AbcgB. Therefore, AbcdB is shorter than ACDEFB. Q. E. D.

274. Cor. 1.—*The sum of any two sides of a triangle is greater than the third side.*

This is the same as the axiom that the shortest distance between two points is a straight line.

275. Cor. 2.—*The difference between any two sides of a triangle is less than the third side.*

Dem.—Let a, b, and c be the sides. By Corollary 1st, $a + b > c$. Therefore, transposing, $a > c - b$.

276. Cor. 3.—*If from any point within a triangle lines be drawn to the extremities of any side, the sum of these lines is less than the sum of the other two sides of the triangle.*

EXERCISES.

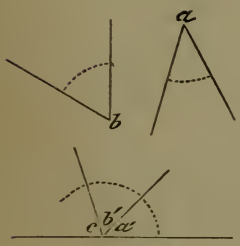

Fig. 198.

1. Given two angles of a triangle, to find the third.

Sug's.—The student should draw two angles on the blackboard, as a and b, and then proceed to find the third. The figure will suggest the method. The third angle is c.

The solution is effected also by constructing the two given angles at the extremities of any line, and producing the sides till they meet.

2. Two angles of a triangle are respectively $\frac{2}{3}$ and $\frac{1}{2}$ of a right angle. What is the third angle?

3. The angles of a triangle are respectively $\frac{2}{3}$, $\frac{1}{2}$, and $\frac{5}{6}$ of a right angle. Which is the greatest side? Which the least? Can you tell the ratio of the sides?

4. What is the value of one of the equal angles of an isosceles triangle whose third angle is $\frac{1}{3}$ of a right angle?

5. Two consecutive angles of a quadrilateral are respectively $\frac{4}{3}$ and $\frac{2}{3}$ of a right angle, and the other two angles are mutually equal to

each other. What is the form of the quadrilateral? What the value of each of the two latter angles?

6. One of the angles of a parallelogram is $\frac{5}{7}$ of a right angle. What are the values of the other angles?

7. The two opposite angles of a quadrilateral are respectively $\frac{2}{3}$ and $\frac{4}{3}$ of a right angle. Can a circumference be circumscribed? If so, do it.

8. Two of the opposite sides of a quadrilateral are parallel, and each is 15 in length. What is the figure? Do these facts determine the angles?

9. Two of the opposite sides of a quadrilateral are 12 each, and the other two 7 each. What do these facts determine with reference to the form of the figure?

10. What is the value of an angle of a regular dodecagon?

11. What is the sum of the angles of a nonagon? What is the value of one angle of a regular nonagon? Of one exterior angle?

12. What is the regular polygon, one of whose angles is $1\frac{3}{17}$ right angles? 17 gon

13. What is the regular polygon, one of whose exterior angles is $\frac{2}{5}$ of a right angle?

14. Can you cover a plane surface with equilateral triangles without overlapping them or leaving vacant spaces? With quadrilaterals? Of what form? With pentagons? Why? With hexagons? Why? What insect puts the latter fact to practical use? Can you cover a plane surface thus with regular polygons of more than 6 sides? Why?

15. Is an equilateral hexagon circumscribed about a circle necessarily regular? A heptagon? An octagon? A nonagon?

16. Is an equiangular circumscribed quadrilateral necessarily regular? A pentagon? A hexagon? A heptagon?

17. Is an equilateral inscribed pentagon necessarily regular? An octagon? How is it if they are equiangular; are they necessarily equilateral and regular?

SYNOPSIS.

- **Angles and Sides of Polygons.**
  - **Triangles.**
    - Prop. I. Sum of angles.
      - *Cor.* 1. Only one right or obtuse.
      - *Cor.* 2. Two angles given.
      - *Cor.* 3. Acute angles if right angled
      - *Cor.* 4. One angle if equiangular.
    - Prop. II. Sides and opp. angles.
      - *Cor.* 1. Converse.
      - *Cor.* 2. Equiangular, equilateral, and converse.
      - *Cor.* 3. Isosceles, equiangular, and converse.
      - *Sch.* These only general relations.
    - Prop. III. Angle within a triangle.
    - Def. Exterior angle.
    - Prop. IV. Exterior angle.—*Cor.* Non-adjacent interior.
  - **Quadrilaterals.**
    - Prop. V. Sum of angles.
    - Prop. VI. Angles of inscribed.
    - **Parallelograms.**
      - Prop. VII. Angles of.
        - *Cor.* 1. Of a trapezoid.
        - *Cor.* 2. Of a rectangle.
      - Prop. VIII. Converse to last.
      - Prop. IX. Two op. sides of a quadrilat'l equal and parallel.
      - Prop. X. Opposite sides of a quadrilateral equal. [parallels.
      - Prop. XI. Converse to last.
        - *Cor.* 1. Parallels intercepted bet.
        - *Cor.* 2. Diagonal of a parallelogram.
      - **Diagonals.**
        - Prop. XII. Bisect.
        - Prop. XIII. Of a rhombus.—*Cor.* Bisect angles.
        - Prop. XIV. Of a rectangle.—*Cor.* Converse.
  - **Polygons of more than 4 Sides.**
    - Def's.—Salient angle.—Re-entrant.—Convex polygon.—Concave.
    - Prop. XV. Sum of angles.
      - *Sch.* 1. Application.
      - *Sch.* 2. Applied to triangles.
      - *Sch.* 3. Angle of equiangular polygon.
    - Prop. XVI. Sum of exterior angles.
    - **Regular.**
      - Prop. XVII. Equilateral inscribed, regular.
      - Prop. XVIII. Equiangular inscribed if odd No. of sides. *Sch.* Fails for even No.
      - Prop. XIX. Equiangular circumscribed, regular.
      - Prop. XX. Equilateral circbd. if odd No. of sides. *Sch.* Fails for even No.
      - Prop. XXI. Circf. can be circumscribed.
        - *Cor.* 1. Inscribed.
        - *Cor.* 2. Centres.
        - *Def.* Angle at cntr.
        - *Cor.* 3. Value of angle at centre.
        - *Def.* Apothem.
      - Prop. XXII. Side of inscribed hexagon.
    - Def. Convex Broken Line.
    - Prop. XXIII. Convex broken line < than —.
      - *Cor.* 1. Sum of two sides of triangle.
      - *Cor.* 2. Diff. of two sides of triangle.
      - *Cor.* 3. Lines from point within triangle.
    - Exercises.

SECTION VIII.

OF EQUALITY.

277. Equality signifies likeness in every respect.

278. The equality of magnitudes is usually shown by applying one to the other, and observing that they coincide.

PROPOSITION I.

279. Theorem.—*Two straight lines of the same length are equal magnitudes.**

DEM.—Let AB and CD be two straight lines of the same length; then are they equal.

For, conceive the extremity C of CD placed at A, and the other extremity somewhere in AB, or in AB produced, as the case may be. Now, the point which traces AB passes through all points in the direction of B from A; and hence, if CD is traced from A towards B, it will pass through the same points as far as they mutually extend. The lines therefore coincide, as far as they both extend; and, being of the same length, D falls at B, and they coincide throughout; they are, therefore, equal. Q. E. D.

FIG. 198.*

ILL.—The truth of this theorem is so evident, that the student may fail to see the point of the demonstration. Let him see if he can say the same things of two curved lines A*m*B, and C*n*D, which are of the same length.

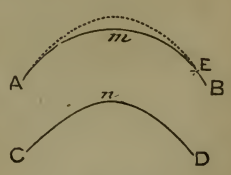

FIG. 199.

The substance of the demonstration is as follows: A line has two properties, and *only* two, *form* and *magnitude*. *Straight* lines, being of the same form, if they are of the same magnitude, are alike in all respects; *i. e.*, they are equal. Now, a line, as a magnitude, has only one dimension, viz., length. If, therefore, two lines have the same length, they have the same magnitude.

* See Preface.

PROPOSITION II.

280. Theorem.—*Two circles whose radii are of the same length are equal;* i. e., *the circumferences are equal, and the circles equal.*

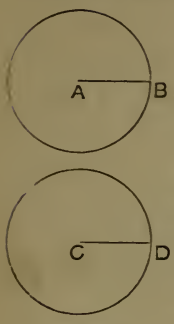

Fig. 200.

Dem.—Let there be two circles whose radii AB and CD are of the same length; then are the circles equal.

For, place the second circle on the first, with the centre C at A, and CD in AB. As CD = AB, D will fall at B. Now, every point in the plane at a distance AB from A is in the circumference of circle A. But every point at a distance CD from the common centre is in the circumference of circle C. Hence, the two figures coincide, and the circles are alike in all respects, *i. e.*, are equal. Q. E. D.

OF ANGLES.

PROPOSITION III.

281. Theorem.—*Two angles whose sides are parallel, two and two, and lie in the same or in opposite directions from their vertices, are equal.*

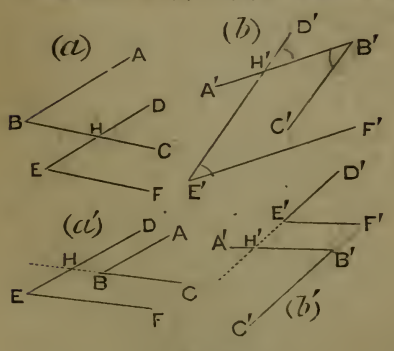

Fig. 201.

Dem.—1st. In (*a*) or (*a'*) let B and E have BA and ED parallel, and extending in the same direction from the vertices, and also BC and EF; then are B and E equal. For, produce (if necessary) either two sides which are not parallel, till they intersect, as at H; then are the corresponding angles DHC and DEF, and DHC and ABC equal (***152***). Hence, ABC = DEF.

2nd. In (*b*) and (*b'*) let B' and E' have B'A' parallel with E'F', but extending in an opposite direction from its vertex; and in like manner B'C' parallel with, but extending in an opposite direction from E'D'; then are B' and E' equal. For, produce (if necessary) two of the sides which are not parallel till they intersect, as at H'; then D'H'B' = the corresponding angle D'E'F', and also = the alternate interior angle A'B'C'; whence A'B'C' = D'E'F'. Q. E. D.

PROPOSITION IV.

282. Theorem.—*If two angles have two sides parallel and extending in the same direction with each other, while the other two sides are parallel and extend in opposite directions from each other, the angles are supplemental.*

DEM.—Let ABC and DEF be two angles, having BC and ED parallel, and extending in the same direction from the vertices, and AB and EF parallel, and extending in opposite directions from the vertices; then are ABC and DEF supplements of each other.

For, produce the two sides not parallel, if necessary, till they meet. Now, BHD is the supplement of BHE by (***131***), BHE = the alternate interior angle DEF, and BHD = the corresponding angle ABC. Therefore, ABC is the supplement of DEF. Q. E. D.

[This demonstration is adapted to the upper cut; let the student adapt it to the lower.]

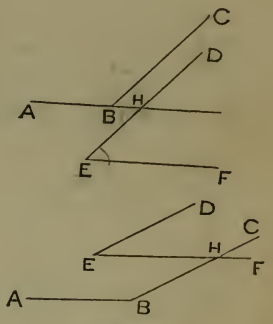

FIG. 202.

PROPOSITION V.

283. Theorem.—*If two angles have their sides respectively perpendicular to each other, the angles are either equal or supplementary.*

DEM.—Let BA be perpendicular to EF or to E′F′, and BC to ED; then is ABC = DEF. For, through B draw BO and BN, respectively parallel to ED and EF; then by the preceding propositions NBO = DEF, and is the supplement of F′E′D. But NBA = OBC, since both are right angles. Take away OBA from each, and we have NBO = ABC; and as NBO is the supplement of F′E′D, ABC is also the supplement of F′E′D. Q. E. D.

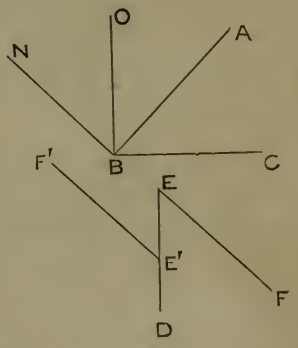

FIG. 203.

OF TRIANGLES.

PROPOSITION VI.

284. Theorem.—*Two triangles which have two sides and the included angle of one equal to two sides and the included angle of the other, each to each, are equal.*

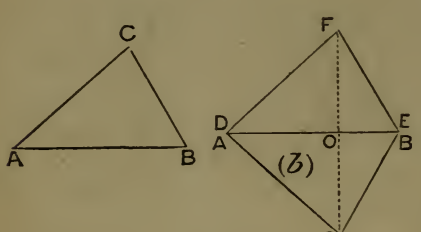

FIG. 204.

DEM.—Let ABC and DEF be two triangles, having AC = DF, AB = DE, and angle A = angle D; then are the triangles equal.

For, place the triangle ABC in the position (*b*), the side AB in its equal DE, and the angle A adjacent to its equal angle D. Then revolving ABC upon DB, until it falls in the plane on the opposite side of DB, since angle A = angle D, AC will take the direction DF; and as AC = DF, C will fall at F. Hence BC will fall in EF, and the triangles will coincide. Therefore the two triangles are equal. Q. E. D.

We may also make the application of ABC to DEF directly, as in (*85*). The method here given is used for the purpose of uniformity in this and the following. We may observe that in this, as in the other cases, DB is perpendicular to FC, and bisects it at O. This fact might easily be shown, and the demonstration be based upon it.

285. SCH.—This proposition signifies that the two triangles *are equal in all respects, i. e.*, that the two remaining sides are equal, as CB = FE; that angle C = angle F, angle B = angle E, and that the areas are equal.

PROPOSITION VII.

286. Theorem.—*Two triangles which have two angles and the included side of the one equal to two angles and the included side of the other, each to each, are equal.*

DEM.—Let ABC and DEF be two triangles, having angle A = angle D, angle B = angle E, and side AB = side DE; then are the triangles equal.

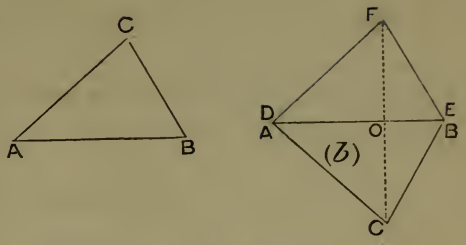

FIG. 205.

For, place ABC in the position (*b*), the side AB in its equal DE, the angle A adjacent to its equal angle D, and B adjacent to its equal angle E. Then revolving ABC upon DB till it falls in the plane on the same side as DFE, since angle A = angle D, AC will take the direction DF, and C will fall somewhere in DF or DF produced. Also, since angle B = angle E, BC will take the direction EF, and C will fall somewhere in EF, or EF produced. Hence, as C falls at the same time in DF and EF, it falls at their intersection F. Therefore the two triangles coincide, and are consequently equal. Q. E. D.

287. COR.—*If one triangle has a side, its opposite angle, and one adjacent angle, equal to the corresponding parts in another triangle, each to each, the triangles are equal.*

For the third angle in each is the supplement of the sum of the given angles, and they are consequently equal. Whence the case is included in the proposition.

288. SCH.—A triangle may have a side and one adjacent angle equal to a side and an adjacent angle in another, and the second adjacent angle of the first equal to the angle *opposite* the equal side in the second, and the triangles *not* be equal. Thus, in the figure, AB = C′A′, A = A′, and B = B′; but the triangles are evidently not equal. [Such triangles are, however, *similar*, as will be shown hereafter.]

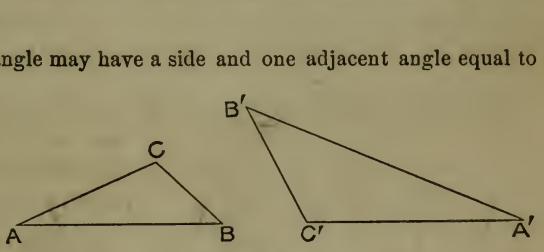

FIG. 206.

PROPOSITION VIII.

289. Theorem.—*Two triangles which have two sides and an angle opposite one of these sides, in the one, equal to the corresponding*

parts in the other, are equal, if of these two sides the one opposite the given angle is equal to or greater than the one adjacent.

DEM.—In the triangles ABC and DEF, let AC = DF, CB = FE, A = D, and CB (= FE) $\gtreqless$ AC (= DF); then are the triangles equal.

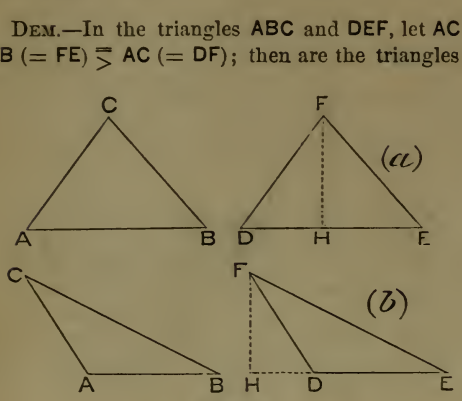

FIG. 207.

For, apply AC to its equal DF, the point A falling at D and C at F. Since A = D, AB will take the direction DE. Let fall the perpendicular FH upon DE, or DE produced. Now, CB being $\gtreqless$ DF, cannot fall between it and the perpendicular, but must fall in FD or beyond both. As there can be but one line on the same side of the perpendicular equal to CB, and as FE = CB, CB must fall in FE. Hence, the two triangles coincide, and are consequently equal. Q. E. D.

290. SCH. 1.—If A and D are acute and CB (= FE) = AC (= DF), the triangles are isosceles. If A and D are right or obtuse, CB (=FE) must be *greater* than AC (= DF), in order that there may be a triangle, since the right or obtuse angle is the greatest angle in a triangle, and the greatest side is opposite the greatest angle. This impossibility appears also from the demonstration above.

291. SCH. 2.—If A and D are *acute*, and the side opposite A, *i. e.*, CB, is less than AC, it must be equal to or greater than the perpendicular CI (= FH) in order to have a triangle. Then, applying AC to DF, and observing that AB takes the direction DE, and that EF, which = CB, being intermediate in length between DF and FH, may lie on either side of FH, we see that ABC may or may not coincide with DEF. Whether it does or not will depend upon whether angle C = angle F, or whether AB = DE. This is the AMBIGUOUS CASE in the solution of triangles, and should receive special attention.

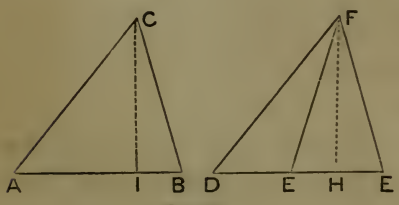

FIG. 208.

PROPOSITION IX.

292. Theorem.—*Two triangles which have the three sides of the one equal to the three sides of the other, each to each, are equal.*

DEM.—Let ABC and DEF be two triangles, in which AB = DE, AC = DF, and BC = EF; then are the triangles equal.

For, place the triangle ABC in the position (*b*), and the side AB in its equal DE, so that the other equal sides shall be adjacent, as AC adjacent to DF, and BC to EF. Draw FC. Now, since DC = DF, and EC = EF, DB is perpendicular to FC at its middle point (***130***). Hence, revolving ABC upon DB, it will coincide with DEF when brought into the plane of the latter Therefore the two triangles are equal. Q. E. D.

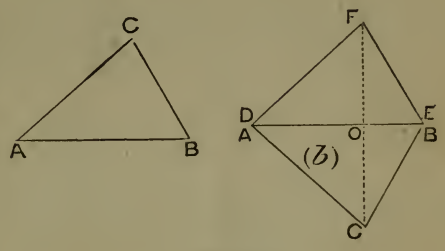

FIG. 209.

293. COR.—*In two equal triangles, the equal angles lie opposite the equal sides.*

294. SCH.—If the triangles compared, as in the three preceding propositions, have an obtuse angle, and the two sides first brought together are sides about the obtuse angle, the figure will take the form in the margin; but the demonstration will be the same. When the three sides are the given equal parts, the form of figure given in the demonstration above can always be secured by bringing together the two *greatest* sides.

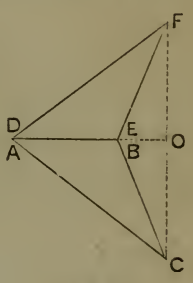

FIG. 210.

PROPOSITION X.

295. Theorem.—*If two triangles have two sides of the one respectively equal to two sides of the other, and the included angles unequal, the third sides are unequal, and the greater third side belongs to the triangle having the greater included angle.*

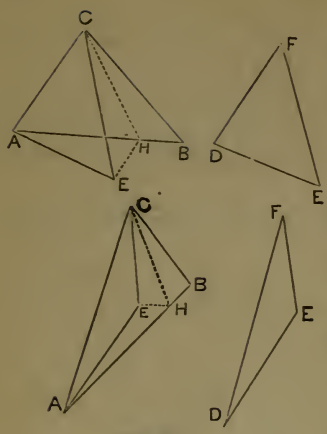

Fig. 211.

Dem.—Let ACB and DEF be two triangles having AC = DF, CB = FE, and C > F; then is AB > DE.

For, placing the side DF in its equal AC, since angle F < angle C, FE will fall within the angle ACB, as in CE. Then let the triangle ACE = the triangle DFE. Bisect ECB with CH, and draw HE. The triangles HCB and HCE have two sides and the included angle of the one, respectively equal to the corresponding parts of the other, whence HE = HB. Now AH + HE > AE; but AH + HE = AH + HB = AB. Therefore, AB > AE. Q. E. D.

296. Cor. — Conversely, *If two sides of one triangle are respectively equal to two sides of another, and the third sides unequal, the angle opposite this third side is the greater in the triangle which has the greater third side.*

Dem.—If AC = DF, CB = FE, and AB > DE, angle C > angle F. For, if C = F, the triangles would be equal, and AB = DE (***284***); and, if C were less than F, AB would be less than DE, by the proposition. But both these conclusions are contrary to the hypothesis. Hence, as C cannot be equal to F, nor less than F, it must be greater.

PROPOSITION XI.

297. Theorem.—*Two right angled triangles which have the hypotenuse and one side of the one equal to the hypotenuse and one side of the other, each to each, are equal.*

Dem.—In the two triangles ABC and DEF, right angled at B and E, let AC = DF, and BC = EF; then are the triangles equal.

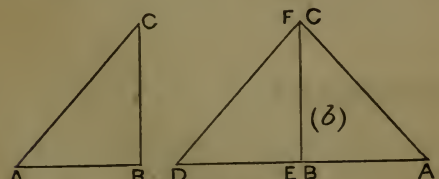

Fig. 212.

For, place BC in its equal EF, so that the right angles shall be adjacent, the angles A and D lying on opposite sides of EF, as in (*b*). Since E and B are right angles, DA is a straight line. Now, since equal oblique lines, as FD and CA, cut off equal distances from the foot of the perpendicular (***141***), DE = BA; and revolving CAB upon FB, the two

triangles will coincide when CAB falls in the plane on the side D. Therefore, the triangles are equal. Q. E. D.

PROPOSITION XII.

298. Theorem.—*Two right angled triangles having the hypotenuse and one acute angle of the one equal to the hypotenuse and an acute angle of the other, are equal.*

DEM.—One acute angle in each being equal, the other acute angles are equal, since they are complements of the same angle (*222*). The case is, then, that of two angles (the acute angles in each), and their included side (the hypotenuse), and falls under (*286*).

PROPOSITION XIII.

299. Theorem.—*Two right angled triangles having a side and a corresponding acute angle in each equal, are equal.*

This also falls under (*286*). Let the student show why.

OF QUADRILATERALS.

PROPOSITION XIV.

300. Theorem.—*Two quadrilaterals are equal when the following parts are equal, each to each, in both quadrilaterals, and similarly arranged:*

1. *The triangles into which either diagonal divides the quadrilaterals.*
2. *The four sides and either diagonal.*
3. *The four sides and one angle.*
4. *Three sides and the two included angles.*
5. *Three angles and any two sides, if the other two sides are non-parallel.*

DEM.—1. This case is demonstrated by applying one quadrilateral to the other.

2. This case is reduced to the former by (*292*).

3. Drawing the diagonal opposite the known angle, this case is reduced to the first by (*284*), and (*292*).

4. This is demonstrated by applying one quadrilateral to the other; or, drawing a diagonal, it may be reduced to case first by (*284*).

5. In this case the quadrilaterals are mutually equiangular by (*255*). If, then, the two sides are adjacent, by drawing the diagonal joining their extremi-

ties the case is brought under the first by (*284*), and (*286*). If, however, the two known sides are non-adjacent, by producing the unknown sides two triangles are formed in each figure, which are mutually equal by (*286*), and the case comes under the axiom concerning equals subtracted from equals.

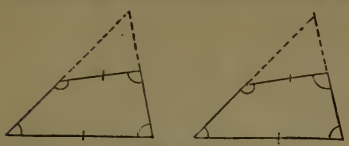

Fig. 213.

[Let the pupil draw the figures and give the demonstrations in full form. Trapeziums should be used; although it should be seen in each case—except the fifth—that the truth applies to any other form of quadrilateral.]

PROPOSITION XV.

301. Theorem.—*Two parallelograms having two sides and the included angle of the one equal to two sides and the included angle of the other, each to each, are equal.*

Dem.—Let AC and EG be two parallelograms, with AD = EH, AB = EF, and A = E; then are they equal.

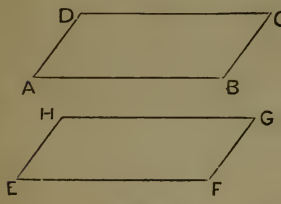

Fig. 214.

For, applying the angle E to A, since EH = AD, H will fall at D; and since EF = AB, F will fall at B. Now, through D but one line can be drawn parallel to AB; hence HC will fall in DC, and G will be found in DC, or in DC produced. In like manner, since but one parallel to AD can be drawn through B, FG must fall in BC, and G be found in BC, or in BC produced. Therefore, as G falls at the same time in DC and BC, it falls at C, and the parallelograms coincide.

302. Cor.—*Two rectangles of the same base and altitude are equal.*

OF POLYGONS.

PROPOSITION XVI.

303. Theorem.—*Two polygons of the same number of sides, having all the parts of the one except three angles, known to be respectively equal to the corresponding parts of the other, are equal.*

Dem.—If the two polygons AE* and A′E′, have all the parts of the one equal to the corresponding parts of the other, each to each, except three angles; then are the polygons equal.

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* It is often more convenient to read a polygon by two letters, instead of all those at the vertices.

1st. When the three unknown angles are consecutive, as G, F, E, and G′, F′, E′. Draw GE, and G′E′. Apply polygon A′E′ to AE, beginning with g' in its equal g: A′ = A, and $a' = a$; hence, B′ falls at B; B′ = B, and $b' = b$, hence, C′ falls at C; etc. Thus, we may show that the perimeters coincide till we reach E′ and E. Then will G′E′ = GE, and the triangles GFE and G′F′E′, having their corresponding sides respectively equal, are themselves equal, and the polygons coincide throughout.

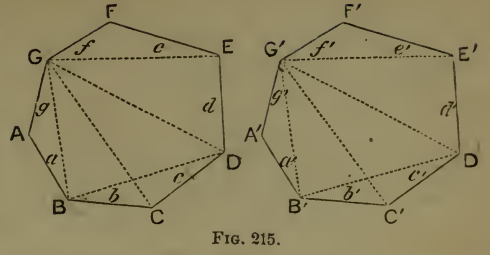


Fig. 215.

2d. When two of the unknown angles are consecutive, and the third not consecutive with these, as G, E, D, and G′, E′, D′. From the angle which is not consecutive with the other two, draw diagonals to the other angles, as GE, GD, and G′E′, G′D′. Now, G′A′B′C′D′ can be applied to GABCD, and G′F′E′ to GFE, in the ordinary way. Hence, the triangles G′E′D′ and GED are mutually equilateral, and consequently equal. Therefore the polygons are equal.

3d. When no two of the three unknown angles are consecutive, as G, B, D, and G′, B′, D′. Join the unknown angles by diagonals, as GB, GD, BD, and G′B′, G′D′, B′D′. Now, polygon G′F′E′D′ can be applied to GFED, D′C′B′ to DCB, and G′A′B′ to GAB in the ordinary way. Hence, the triangles G′D′B′ and GDB are mutually equilateral, and consequently equal. Therefore the polygons are equal.*

304. Cor.—*Two quadrilaterals having their corresponding sides equal, and an angle in one equal to the corresponding angle in the other, are equal.*

PROPOSITION XVII.

305. Theorem.—*Two polygons of the same number of sides, having all the parts of the one except two angles and the included side, known to be respectively equal to the corresponding parts of the other, are equal.*

Dem.—Let the unknown parts be C, c, D, and C′, c', D′. From any other two of the mutually equal angles, as G and G′, draw the diagonals GC, GD, G′C′, G′D′,

---

* Notice that in each case the unknown angles are to form the vertices of triangles, which the argument shows to be equilateral, and therefore equal. In Case 1st, we have to draw only one line in order to give the triangles, as two sides are sides of the polygon; in Case 2d, we have to draw two sides, and in Case 3d, three sides, for analogous reasons.

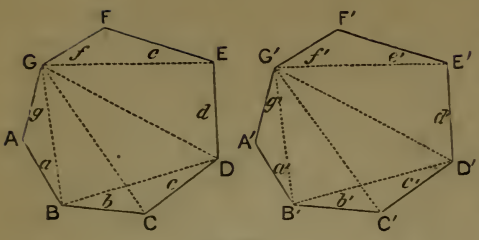


Fig. 216.

to the unknown angles. Then the polygon G′F′E′D′ can be applied in the ordinary way to GFED, f' being placed in f, etc. So also G′A′B′C′ can be applied to GABC, beginning with g' in its equal g. Hence, angle F′G′D′ = FGD, A′G′C′ = AGC; and, adding, F′G′D′ + A′G′C′ = FGD + AGC. Subtracting these equals from G′ = G, we have C′G′D′ = CGD. Whence the triangles C′G′D′ and CGD have two sides and their included angle equal in each, and are equal; therefore the polygons are equal in all their parts.

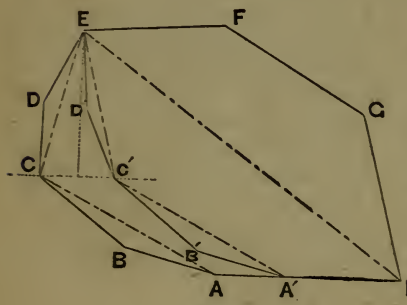


Fig. 217.

306. Sch.—When the unknown angles are both separated from the unknown side, the polygons may or may not be equal—the case is ambiguous. Thus, if C and E are the unknown angles and AH the unknown side, the polygons ABCDEFG, and A′B′C′D′EFG fulfill the conditions, but are not equal. By drawing CE, CA, and EH, the case is reduced to that of two quadrilaterals having all the parts equal, each to each, except two angles and their non adjacent side; in which case the quadrilaterals are not necessarily equal.

So, also, when one of the unknown angles is adjacent to the unknown side and the other separated, the polygons may or may not be equal. Thus, let

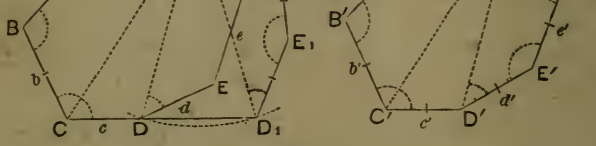


Fig. 218.

the unknown parts be D, c, G, and D′, c', G′. From the separated angle draw the diagonals to the extremities of the unknown side, as GC, GD (or GD₁), and G′C′, G′D′. In the usual way G′A′B′C′ can be applied to GABC, and G′F′E′D′ to

GFED. Whence G′C′ = GC, G′D′ = GD, and angle G′C′D′ = GCD. Thus the case is reduced to that of two triangles having two sides and an angle opposite one of them mutually equal, and is, therefore, ambiguous. The polygon (*a*) may have the part corresponding to G′F′E′D′ situated as GFED, or as $GF_1E_1D_1$. In the former case the polygons are equal, in the latter not.

307. Cor.—*Two quadrilaterals having three sides and the corresponding angles included by these sides equal, are equal.*

This falls under the 1st case.

308. Sch.—If the three unknown or excepted parts are all sides, the polygons are not necessarily equal, as will appear by an inspection of the figure. The

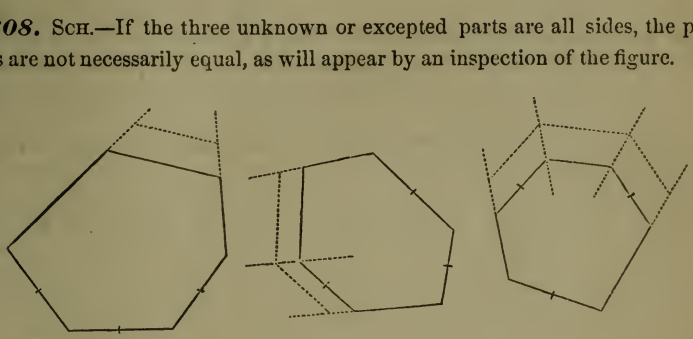


Fig. 219.

unmarked sides being the excepted ones, the polygons may be those included by the continuous lines, or those included in part by the broken lines, all the parts being equal in each two, except the three unknown ones.

PROPOSITION XVIII.

309. Theorem.—*Two polygons of the same number of sides, having two adjacent sides and the diagonals drawn from the included angle, in the one, respectively equal to the corresponding parts in the other, and their corresponding included angles equal, are equal figures.*

Dem.—The demonstration is based upon (***284***). Let the student draw the figures, and make the applications.

PROPOSITION XIX.

310. Theorem.—*Two polygons of the same number of sides, having all the parts (sides and angles) of the one respectively equal to the corresponding parts of the other, except two parts, are equal, unless the excepted parts are parallel sides.*

DEM.—The demonstration can be supplied by the pupil, as it is similar to the several preceding. The cases will be, 1st, When two angles are excepted, (*a*) they being consecutive, (*b*) they not being consecutive;—2d, An angle and a side, (*a*) consecutive, (*b*) not consecutive;—3d, Two sides, (*a*) consecutive, (*b*) not consecutive.

EXERCISES.

1. ***Prob.***—*Having two sides and their included angle given, to construct a triangle.*

SUG'S.—The student should draw two lines on the blackboard, and a detached angle, as the given parts. Then, making an angle equal to the given angle (***200***), he should lay off the given sides from the vertex on the sides of the angle, and join their extremities. The triangle thus formed is the one required, for any other triangle formed with these two sides and this angle will be just like this by (***284***).

2. ***Prob.***—*Having two angles and their included side given, to construct a triangle.*

3. ***Prob.***—*Having the three sides of a triangle given, to construct the triangle.*

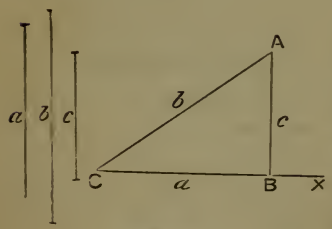
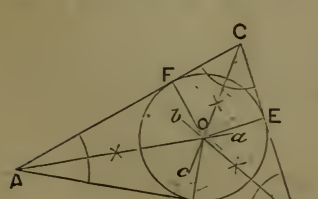


FIG. 220.

SOLUTION.—Let *a*, *b*, and *c*, be the given sides. Draw an indefinite line CX, and on it take CB = *a*. From C as a centre with *b* as a radius, describe an arc as near as can be discerned where the angle A will fall. From B, with a radius *c*, describe an arc intersecting the former. Then is ABC the triangle required, since any other triangle having the same sides would be equal to ABC (***292***).

4. ***Prob.***—*To inscribe a circle in a given triangle.*

SOLUTION.—For the method of doing it see PART I. (***79***). To prove the method correct, we observe that the triangles ODB and OBE have OB common, and are mutually equiangular; hence they are equal, and OD = OE. In like manner triangle OEC = OFC, and OE = OF. [Triangle OFA = ODA; but we do not need the fact in the demonstration.] Since OD = OE = OF, the circumference struck from O as a centre with a radius OD, passes through E and F. Moreover, since each side of the triangle is perpendicular to a radius at its extremity, it is tangent to the circle (***172***); and the circle is inscribed.



5. ***Prob.***—*Having two sides and an angle opposite one of them given, to construct the triangle.*

Solution.—1st. When the given angle is right or obtuse, the side opposite must be greater than the side adjacent, as the greatest side is opposite the greatest angle (***224***), and the greatest angle in such a triangle is the right or

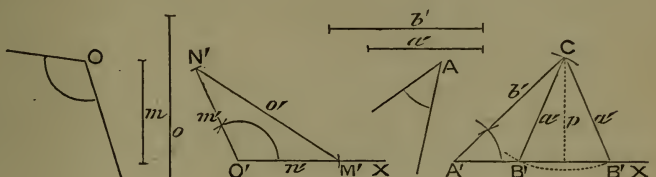


Fig. 221.

obtuse angle. In this case let m and o be the given sides, and O the angle opposite o. Draw an indefinite line O'X, construct O' equal to O, and take O'N' equal to m. From N' as a centre, with a radius equal to O, describe an arc cutting O'X, as at M'. Draw N'M'. Then is N'M'O' the triangle required, since all triangles having their corresponding parts equal to m', o', and O' are equal.

2d. When the given angle is acute, as A, there will be *no solution* if the given side, a, opposite A, is less than the perpendicular; *one solution* if $a = p$, or if $a >$ than both p and b, and *two solutions* if $a > p$, and less than b. This will appear from the construction, which is the same as in Case 1st.

6. If a perpendicular be let fall from the right angle C of the triangle ACB upon the hypotenuse, as CD, show from (***222***) that the three triangles in the figure are mutually equiangular.

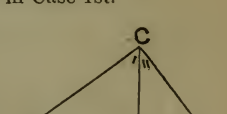


Fig. 222.

7. Given the sides of a triangle, as 15, 8, and 5, to construct the triangle.

8. Given two sides of a triangle $a = 20$, $b = 8$, and the angle B opposite the side b equal $\frac{1}{3}$ of a right angle,* to construct the triangle.

9. Same as in the 8th, except $b = 12$. Same, except that $b = 25$.

10. Construct a triangle with angle A $= \frac{2}{3}$ of a right angle, angle B $= \frac{1}{2}$ of a right angle, and side a opposite angle A, 15.

11. Construct a right angled triangle whose hypotenuse is 16, and

* To construct this angle, bisect an angle of an equilateral triangle.

one of the other sides 7. The same with one acute angle $\frac{2}{3}$ of a right angle, and a side about the right angle 12. Will there be any difference in the *shape* of the triangles if one is constructed with the given angle adjacent to the given side, and the other with it opposite? Will there be any difference in the *size?*

12. Construct a right angled triangle having its hypotenuse 20, and one acute angle $\frac{1}{3}$ of a right angle.

13. Construct a quadrilateral three of whose sides are 20, 12, and 15, and the angle included between 20 and the unknown side $\frac{2}{3}$ of a right angle, and that between 15 and the unknown side $\frac{1}{2}$ a right angle.

Sug's.—Make A = $\frac{2}{3}$ of a right angle, and $b = 20$. From D as a centre, with a radius 12, strike the arc *on*. At any point on side *a*, make an angle B′ = $\frac{1}{2}$ a right angle. Take B′*m* = 15, and draw C*m* parallel to AB′. From the intersection C draw CB parallel to *m*B′. Draw CD. Then is ABCD the quadrilateral required.

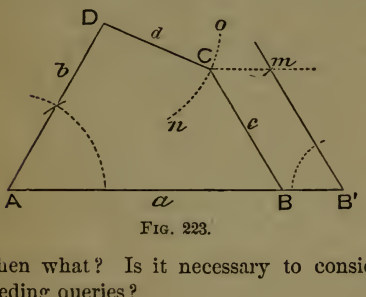


Fig. 223.

Queries.—If $d + c$ is less than the perpendicular from D upon AB, then what? If equal to the perpendicular, then what? Is it necessary to consider angle B in answering the two preceding queries?

14. Construct a parallelogram whose two adjacent sides are 6 and 8, and whose included angle equals $1\frac{1}{3}$ right angles.

15. Construct a heptagon whose sides in order are $a = 4$, $b = 5$, $c = 5$, $d = 6$, $e = 6$, $f = 3$, $g = 4$; and the angle included between a and b, $1\frac{1}{2}$ right angles; between b and c, $1\frac{2}{3}$; c and d, $1\frac{1}{3}$; d and e, $1\frac{1}{4}$.

Sug's.—See *Fig.* 187. Proceed in order, laying off the parts as given, from A to F. Draw AF. From F as a centre, with a radius $f = 3$, strike an arc, and also from A, with a radius $g = 4$. The intersection of these arcs will determine G.

Queries.—What is the limit of the sum of the possible values of the given angles? What the limit of the sum of the sides included between the unknown angles?

SYNOPSIS.

OF EQUALITY.

What? How shown?

PROP. I. Of straight lines.

PROP. II. Of circles.

ANGLES.

PROP. III. Sides parallel. Direction same or opposite.

PROP. IV. " " " one same, other opposite.

PROP. V. " perpendicular.

TRIANGLES.

PROP. VI. Two sides and included angle. { *Sch.* All parts equal.

PROP. VII. Two angles and included side. { *Cor.* Side, one adjacent and one opposite angle equal. *Sch.* Exception.

PROP. VIII. Two sides and angle opposite one. { *Sch.* 1. When isosceles. *Sch.* 2. When ambiguous.

PROP. IX. Three sides. { *Cor.* Equal angles opposite equal sides. *Sch.* Case of obtuse angle. Form of *Fig.*

PROP. X. Two sides equal. included angles unequal. { *Cor.* Converse.

RIGHT ANGLED.

PROP. XI. Hypotenuse and one side.

PROP. XII. Hypotenuse and one acute angle.

PROP. XIII. Side and one acute angle.

QUADRILATERALS.

PROP. XIV. Three sides and non-included angles equal.

PROP. XV. Two parallelograms having two sides and the included angles equal. { *Cor.* Rectangles of same base and altitude.

POLYGONS OF MORE THAN 4 SIDES.

PROP. XVI. Three angles excepted. { *Cor.* Quadrilaterals.

PROP. XVII. Two angles and one side excepted. { *Sch.* 1. The ambiguous case. *Cor.* Quadrilaterals. *Sch.* 2. Three sides excepted.

PROP. XVIII. Two sides and included diagonals.

PROP. XIX. Any two parts excepted.

EXERCISES.

Prob. In a triangle, given two sides and included angle.

Prob. " " " angles " side.

Prob. " " " sides and angle opposite one.

Prob. " " " three sides.

Prob. To inscribe a circle in a triangle.

SECTION IX.

OF EQUIVALENCY AND AREA.

311. Equivalent Figures are such as are equal in magnitude.

PROPOSITION I.

312. Theorem.—*Parallelograms having equal bases and equal altitudes are equivalent.*

DEM.—Let ABCD and EFGH be two parallelograms having equal bases, BC and FG, and equal altitudes; then are they equivalent.

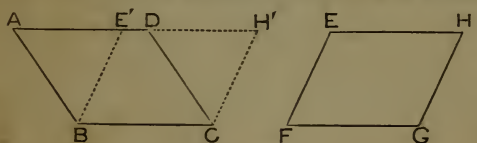


FIG. 223.*

For, place FG in its equal BC; and, since the altitudes are equal, the upper base EH will fall in AD or AD produced, as E′H′. Now, the two triangles AE′B and DH′C are equal, because the three sides of the one are respectively equal to the three sides of the other. Thus AB = DC, being opposite sides of the same parallelogram. For a like reason, E′B = H′C. Also, E′H′ = BC = AD. From AH′ taking E′H′, AE′ remains, and taking AD, DH′ remains. Therefore AE′ = DH′. These triangles being equal, the quadrilateral ABCH′ − the triangle AE′B = ABCH′ − DH′C. But ABCH′ − AE′B = E′BCH′ = EFGH; and ABCH′ − DH′C = ABCD. Hence, ABCD = EFGH. Q. E. D.

313. COR.—*Any parallelogram is equivalent to a rectangle having the same base and altitude.*

PROPOSITION II.

314. Theorem.—*A triangle is equivalent to one-half of any parallelogram having an equal base and an equal altitude with the triangle.*

DEM.—Let ABC be a triangle. Through C draw CD parallel to AB; and through A draw AD parallel to BC. Then is ABCD a parallelogram, of which ABC is one-half (*243*). Now, as any other parallelogram having an equal base and altitude with ABCD is equivalent to ABCD (*312*), ABC is equivalent to one-half of *any* parallelogram having an equal base and altitude with ABC. Q. E. D.

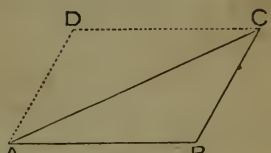


FIG. 224.

315. COR. 1.—*A triangle is equivalent to one-half of a rectangle having an equal base and an equal altitude with the triangle.*

316. COR. 2.—*Triangles of equal bases and equal altitudes are equivalent,* for they are halves of equivalent parallelograms.

PROPOSITION III.

317. Theorem.—*The square described on a line is equivalent to four times the square described on half the line, nine times the square described on one-third the line, sixteen times the square on one-fourth the line,* etc.

DEM.—Let AB be any line. Upon it describe the square ABCD. Bisect AB, as at *d*, and AD, as at *a*. Draw *dc* parallel to AD, and *ab* parallel to AB. Now, the four quadrilaterals thus formed are parallelograms by construction, hence their opposite sides and angles are equal; and as A, B, C, and D are right angles, and $Aa = Ad = dB = bB =$ etc., the four figures 1, 2, 3, 4, are equal squares. Hence $Adoa = \frac{1}{4}$ ABCD. In like manner it can be shown that the nine figures into which the square on A′B′ is divided by drawing through the points of trisection of the sides, lines parallel to the other sides, are equal squares. Hence A′*o*′, the square on $\frac{1}{3}$ of A′B′, is $\frac{1}{9}$ of the square A′B′C′D′. The same process of reasoning can be extended at pleasure, showing that the square on $\frac{1}{4}$ a line is $\frac{1}{16}$ the square of the whole, etc.

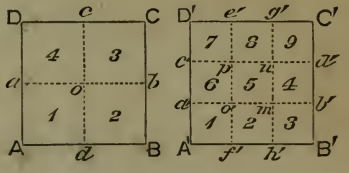


FIG. 225.

PROPOSITION IV.

318. Theorem.—*A trapezoid is equivalent to two triangles having for their bases the upper and lower bases of the trapezoid, and for their common altitude the altitude of the trapezoid.*

DEM.—By constructing any trapezoid, and drawing either diagonal, the student can show the truth of this theorem.

PROPOSITION V.

319. Prob.—*To reduce any polygon to an equivalent triangle.*

SOLUTION.—Let ABCDEF be a polygon which it is proposed to reduce to an equivalent triangle. Produce any side, as BC, indefinitely. Draw the diagonal EC and DH parallel to it. Draw EH. Now, consider the triangle CDE as cut off from the polygon and replaced by CHE. The magnitude of the polygon will not be changed, since CDE and CHE have the same base CE, and the same altitude, as their vertices lie in DH parallel to EC. From the polygon thus reduced we cut the triangle FHE, and replace it by its equivalent FHI, by drawing the diagonal FH, and the parallel EI. In like manner, by drawing FB and the parallel AG, we can replace FBA by its equivalent FGB. Hence, GFI is equivalent to ABCDEF. It is evident that a similar process would reduce a polygon of any number of sides to an equivalent triangle.

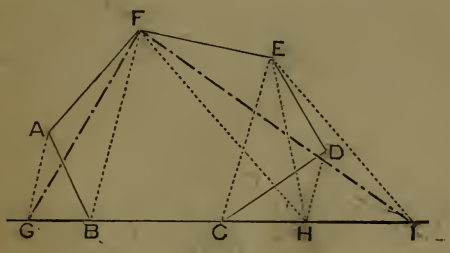


FIG. 226.

AREA.

PROPOSITION VI.

320. Theorem.—*The area of a rectangle is equal to the product of its base and altitude.*

DEM.—Let ABCD be a rectangle, then is its area equal to the base AB multiplied by the altitude AC.

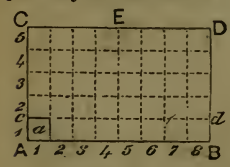


FIG. 227.

If the sides AB and AC are commensurable, take some unit of length, as E, which is contained a whole number of times in each, as five times in AC, and eight times in AB, and apply it to the lines, dividing them respectively into five and eight equal parts. From the several points of division draw lines through the rectangle perpendicular to its sides. The rectangle will be divided into small parallelograms, which are all equal squares, as the angles are all right angles, and the sides all

equal to each other. Each square is a unit of surface, and the area of the rectangle is expressed by the number of these squares, which is evidently equal to the number in the row on AB, multiplied by the number of such rows, or the number of linear units in AB multiplied by the number in AC.

If the two sides of the rectangle are not commensurable, take some very small unit of length which will divide one of the sides, as AC, and divide the rectangle into squares as before; the number of these squares will be the measure of the rectangle, except a small part along one side, not covered by the squares. By taking a still smaller unit, the part left unmeasured by the squares will be still less, and by diminishing the unit of length E, we can make the part unmeasured as small as we choose. It may, therefore, be made infinitely small by regarding the unit of measure as infinitesimal, and consequently is to be neglected.* Hence, in any case, the area of a rectangle is equal to the product of its base into its altitude. Q. E. D.

321. Cor. 1.—*The area of a square is equal to the second power of one of its sides,* as in this case the base and altitude are equal.

322. Cor. 2.—*The area of any parallelogram is equal to the product of its base into its altitude;* for any parallelogram is equivalent to a rectangle of the same base and altitude (*313*).

323. Cor. 3.—*The area of a triangle is equal to one-half the product of its base and altitude;* for a triangle is one-half of a parallelogram of the same base and altitude (*314*).

324. Cor. 4.—*Parallelograms or triangles*† *of equal bases are to each other as their altitudes; of equal altitudes, as their bases; and in general they are to each other as the products of their bases by their altitudes.*

PROPOSITION VII.

325. Theorem.—*The area of a trapezoid is equal to the product of its altitude into one-half the sum of its parallel sides, or, what is the same thing, the product of its altitude and a line joining the middle points of its inclined sides.*

---

* This principle may be thus stated: An infinitesimal is a quantity conceived, *and to be treated*, as less than any assignable quantity; hence, as added to or subtracted from finite quantities, it has no value. Thus, suppose $\frac{m}{n} = a$, m, n, and a being finite quantities. Let c represent an infinitesimal; then $\frac{m \pm c}{n}$, or $\frac{m}{n \pm c}$, or $\frac{m \pm c}{n \pm c}$, is to be considered as still equal to a, for to consider it to differ from a by any amount we might name, would be to assign *some* value to c.

† By this is meant the *areas* of the figures.

DEM.—In the trapezoid ABCD draw either diagonal, as AC. It is thus divided into two triangles, whose areas are together equal to one-half the product of their common altitude (the altitude of the trapezoid), into their bases DC and AB, or this altitude into $\frac{1}{2}$ (AB + DC).

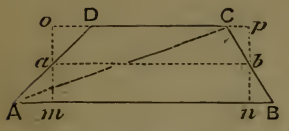


FIG. 228.

Secondly, if *ab* be drawn bisecting AD and CB, then is $ab = \frac{1}{2}$ (AB + CD). For, through *a* and *b* draw the perpendiculars *om* and *pn*, meeting DC produced when necessary. Now, the triangles *a*oD and A*am* are equal, since A*a* = *a*D, angle *o* = *m*, both being right, and angle *oa*D = A*am* being opposite. Whence A*m* = *o*D. In like manner we may show that C*p* = *n*B. Hence, $ab = \frac{1}{2}(op + mn) = \frac{1}{2}$(AB + DC); and area ABCD, which equals *altitude* into $\frac{1}{2}$(AB + DC), = *altitude* into *ab*. Q. E. D.

PROPOSITION VIII.

326. Theorem.—*The area of a regular polygon is equal to one-half the product of its apothem into its perimeter.*

DEM.—Let ABCDEFG be a regular polygon whose apothem is O*a*; then is its area equal to $\frac{1}{2}$ O*a* (AB + BC + CD + DE + EF + FG + GA).

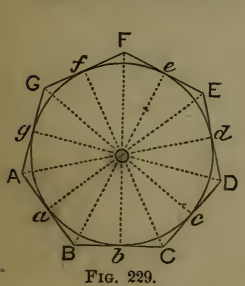


FIG. 229.

Drawing the inscribed circle, the radii O*a*, O*b*, etc., to the points of tangency, and the radii of the circumscribed circle OA, OB, etc. (***264***, ***265***), the polygon is divided into as many equal triangles as it has sides. Now, the apothem (or radius of the inscribed circle) is the common altitude of these triangles, and their bases make up the perimeter of the polygon. Hence, the area = $\frac{1}{2}$ O*a* (AB + BC + CD + DE + EF + FG + GA). Q. E. D.

327. COR.—*The area of any polygon in which a circle can be inscribed is equal to one-half the product of the radius of the inscribed circle into the perimeter.*

The student should draw a figure and observe the fact. It is especially worthy of note in the case of a triangle. See *Fig.* 60.

PROPOSITION IX.

328. Theorem.—*The area of a circle is equal to one-half the product of its radius into its circumference.*

DEM.—Let O*a* be the radius of a circle. Circumscribe any regular polygon. Now the area of this polygon is one-half the product of its apothem and perimeter. Conceive the number of sides of the polygon, indefinitely increased, the polygon still continuing to be circumscribed. The apothem continues to be the radius of the circle, and the perimeter approaches the circumference. When, therefore, the number of sides of the polygon becomes infinite, it is to be considered as coinciding with the circle, and its perimeter with the circumference. Hence the area of the circle is equal to one-half the product of its radius into its circumference. Q. E. D.

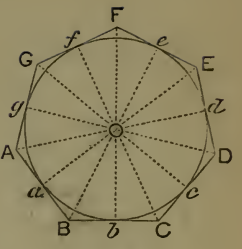


FIG. 230.

329. DEF.—***A Sector*** is a part of a circle included between two radii and their intercepted arc. *Similar Sectors* are sectors in different circles, which have equal angles at the centre.

330. COR. 1.—*The area of a sector is equal to one-half the product of the radius into the arc of the sector.*

331. COR. 2.—*The area of a sector is to the area of the circle as the arc of the sector is to the circumference, or as the angle of the sector is to four right angles.*

EXERCISES.

1. What is the area in acres of a triangle whose base is 75 rods and altitude 110 rods?

2. What is the area of a right angled triangle whose sides about the right angle are 126 feet and 72 feet?

3. If 2 lines be drawn from the vertex of a triangle to the base, dividing the base into parts which are to each other as 2, 3, and 5, how is the triangle divided? How does a line drawn from an angle to the middle of the opposite side divide a triangle?

4. Review the exercises on pages 49 and 50, giving the reasons, in each case.

SYNOPSIS.

- EQUIVALENCY AND AREA.
  - EQUIVALENCY.
    - Definition.
    - PROP. I. Of parallelograms. — *Cor.* Paral. and rectangle.
    - PROP. II. Of triangles. — *Cor.* 1. Triangle and rectangle. *Cor.* 2. Of equal bases and equal altitudes.
    - PROP. III. Square on $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ a line, etc.
    - PROP. IV. Trapezoid.
    - PROP. V. To reduce a polygon to a triangle.
  - AREA.
    - PROP. VI. Of rectangle. — *Cor.* 1. Of square. *Cor.* 2. Any parallelogram. *Cor.* 3. Of triangle. *Cor.* 4. Relation of parallelograms and of triangles.
    - PROP. VII. Of trapezoid.
    - PROP. VIII. Of regular polygons. — *Cor.* Of any circumscribed polygon.
    - PROP. IX. Of a circle. — *Def.* Of sector. *Cor.* 1. Area of sector. *Cor.* 2. Relation of sector to circle.
  - EXERCISES.

SECTION X.

OF SIMILARITY.

332. The primary notion of similarity is *likeness of form.* Two figures are said to be similar which have the same shape, although they may differ in magnitude.* A more scientific definition is as follows:

333. Similar Figures are such as have their angles respectively equal, and their homologous sides proportional.

334. Homologous Sides of similar figures are those which are included between equal angles in the respective figures.

* The student should be careful, at the outset, to mark the fact that *similarity involves two things*, EQUALITY OF ANGLES and PROPORTIONALITY OF SIDES. It will appear that, in the case of triangles, if *one* of these facts exists, the other does also; but this is not so in other polygons, as is illustrated in PART I.

IN SIMILAR TRIANGLES, THE HOMOLOGOUS SIDES ARE THOSE OPPOSITE THE EQUAL ANGLES.

PROPOSITION I.

335. Theorem.—*Triangles which are mutually equiangular are similar.*

DEM.—Let ABC and DEF be two mutually equiangular triangles, in which A=D, B=E, and C=F; then are the sides opposite these equal angles proportional, and the triangles possess both requisites of similar figures; *i. e.*, they are mutually equiangular and have their homologous sides proportional, and are consequently similar.

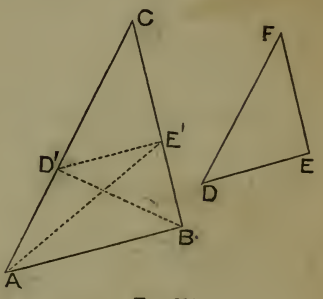


FIG. 231.

To prove that the sides opposite the equal angles are proportional, place the triangle DEF upon ABC, so that F shall coincide with its equal C, CE'=FE, and CD'=FD. Draw AE', and D'B. Since angle CE'D'=CBA, D'E' is parallel to AB, and as the triangles D'E'A and D'E'B have a common base D'E' and the same altitude, their vertices lying in a line parallel to their base, they are equivalent (***316***). Now, the triangles CD'E' and D'E'A, having a common altitude, are to each other as their bases (***324***). Hence,

$$CD'E' : D'E'A :: CD' : D'A.$$

For like reason $$CD'E' : D'E'B :: CE' : E'B.$$

Then, since D'E'A and D'E'B are equivalent, the two proportions have a common ratio, and we may write CD' : D'A :: CE' : E'B.

By composition $CD' : CD'+D'A :: CE' : CE'+E'B,$

or $CD' : CA :: CE' : CB,$ or $FD : CA :: FE : CB.$

In a similar manner, by applying angle E to B, we can show that FE : CB :: ED : BA. Therefore, FD : CA :: FE : CB :: ED : BA. Q. E. D.

336. COR. 1.—*If two triangles have two angles of the one respectively equal to two angles of the other, the third angles being equal* (***221***), *the triangles are similar.*

337. COR. 2.—*A line drawn through a triangle parallel to any side divides the other sides proportionally.*

Thus D'E' being parallel to AB, it is shown in the proposition that CD' : D'A :: CE' : E'B.

338. Cor. 3.—*If any two lines cut a series of parallels, they are divided proportionally.*

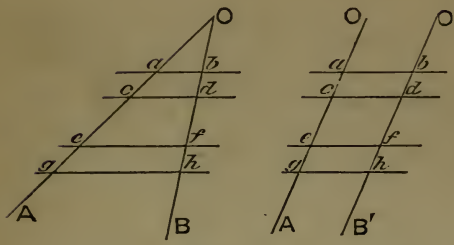


Fig. 232.

Dem.—If the two secant lines are parallel, as OA and O′B′, the intercepted parts are equal, *i. e.*, $ac = bd$, $ce = df$, $eg = fh$, etc. (***242***). Hence, $ac : bd :: ce : df :: eg : fh$. Secondly, if the secant lines are not parallel, let them meet in some point, as O. Then, by the proposition, we have

$$Oa : ac :: Ob : bd \quad (1), \qquad \text{and also } Oc : ce :: Od : df \quad (2).$$

Taking the first by composition, it becomes

$$Oa + ac : ac :: Ob + bd : bd, \text{ or } Oc : ac :: Od : bd \quad (3).$$

Now, as the antecedents in (2) and (3) are the same, we have

$$ac : bd :: ce : df, \text{ or } ac : ce :: bd : df.$$

In like manner, we may show that

$$ce : df :: eg : fh, \text{ or } ce : eg :: df : fh.$$

PROPOSITION II.

339. Theorem.—Conversely, *If two triangles have their corresponding sides proportional, they are similar.*

Dem.—In the triangles ABC and DFE, let FD : CA :: FE : CB :: DE : AB; then are the triangles similar.

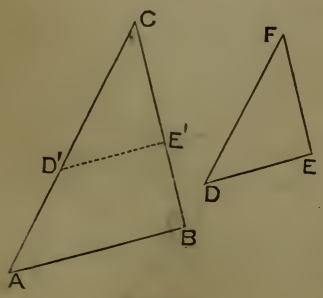


Fig. 233.

As one of the characteristics of similarity, viz., proportionality of sides, exists by hypothesis, we have only to prove the other, *i. e.*, that the triangles are mutually equiangular. Make CD′ equal to FD, and draw D′E′ parallel to AB. By the preceding proposition CD′ (= FD) : CA :: D′E′ : AB. But, by hypothesis, FD : CA :: DE : AB. Whence, D′E′ = DE. In like manner CE′ : CB :: CD′ (=FD) : CA. But, by hypothesis, FE : CB :: FD : CA. Whence CE′ = FE; and the triangle CD′E′ is equal to the triangle FDE (***292***). Now, CD′E′ and CAB are mutually equiangular, since D′E′ is parallel to AB (***152***), and C is common. Hence.

the triangles ABC and DEF are mutually equiangular, and consequently similar. Q. E. D.

340. SCH.—As we now know that if two triangles are mutually equiangular, they are similar; or, if they have their sides proportional, they are similar, it will be sufficient hereafter, in any given case, to prove *either one* of these facts, in order to establish the similarity of two triangles. For, either fact being proved, the other follows as a consequence. See Section VI., PART I., for familiar illustrations of this most important subject.

PROPOSITION III.

341. Theorem.—*Two triangles which have the sides of the one respectively parallel or perpendicular to the sides of the other, are similar.*

DEM.—Let ABC and A'B'C' be two triangles whose sides are respectively parallel or perpendicular to each other, then are the triangles similar.

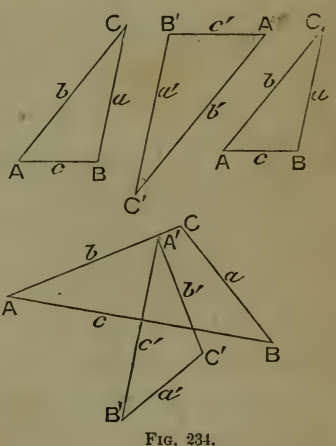


FIG. 234.

For, any angle in one triangle is either equal or supplementary to the angle in the other which is included between the sides which are parallel or perpendicular to its own sides. Thus A either equals A', or A + A' = 2 right angles (***281, 282, 283***). Now, if the corresponding angles are all supplementary, that is, if A + A' = 2 R.A., B + B' = 2 R.A., and C + C' = 2 R.A., the sum of the angles of the two triangles is 6 right angles, which is impossible. Again, if one angle in one triangle equals the corresponding angle in the other, as A = A', and the other angles are supplementary, the sum is 4 right angles plus twice the equal angle, which is impossible. Hence, two of the angles of one triangle must be equal respectively to two angles of the other; and, if two are equal, the third angle in one is equal to the third in the other (***221***). Hence, the triangles are mutually equiangular, and therefore similar (***335***). Q. E. D.

PROPOSITION IV.

342. Theorem.—*Two triangles, which have an angle in each equal, and the sides about the equal angles proportional, are similar.*

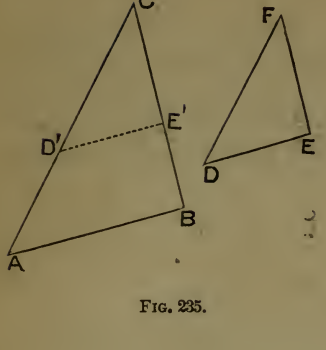


FIG. 235.

DEM.—In the triangles ABC and DEF let C = F, and AC : DF :: CB : FE; then are the triangles similar.

For, place F on its equal C, and let D fall at D'. Draw D'E' parallel to AB. Then AC : D'C (= DF) :: BC : CE' (337). But by hypothesis AC : DF :: BC : FE. ∴ CE' = FE, and the triangles D'CE' and DFE are equal (284). Therefore, D'CE' being equiangular with ACB, is similar to it (335); and as DFE is equal to D'CE', DFE is similar to ACB. Q. E. D.

PROPOSITION V.

343. Theorem.—*In any right angled triangle, if a line be drawn from the right angle perpendicular to the hypotenuse, it divides the triangle into two triangles, which are similar to the given triangle, and consequently similar to each other.*

DEM.—Let ACB be a triangle right-angled at C, and CD a perpendicular upon the hypotenuse AB; then are ACD and CDB similar to ACB, and consequently to each other.

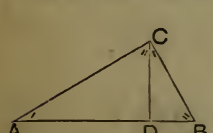


FIG. 236.

For, the triangles ACD and ACB have the angle A common, and a right angle in each; hence they are mutually equiangular, and consequently similar(***336***). For a like reason CDB and ACB are similar. Finally, as ACD and CDB are both similar to ACB, they are similar to each other. Q. E. D.

344. COR. 1.—*Either side about the right angle is a mean proportional between the whole hypotenuse and the adjacent segment.*

DEM.—This is a direct consequence of the similarity of the partial triangles with the whole triangle. Thus, comparing the homologous sides of ACD and ACB, we have AD : AC :: AC : AB;* and from CDB and ACB, we have DB : CB :: CB : AB.

345. COR. 2.—*The perpendicular is a mean proportional between the segments of the hypotenuse.*

DEM.—This is a consequence of the similarity of ACD and CDB. Thus, AD : CD : : CD : DB.

* Notice that AD of the triangle ACD is opposite angle ACD, and AC, its consequent, is of the triangle ACB, and opposite the angle B, which equals angle ACD. The student must be sure that he knows in what order to take the sides, and *why*.

Queries.—To which triangle does the first CD belong? To which the second? Why is CD made the consequent of AD? Why, in the second ratio, are CD and DB to be compared?

346. COR. 3.—*The square described on the hypotenuse of a right angled triangle is equivalent to the sum of the squares described on the other two sides.*

DEM.—From *Cor.* 1, $\overline{AC}^2 = AB \times AD$

and also $\overline{CB}^2 = AB \times DB.$

Therefore, adding, $\overline{AC}^2 + \overline{CB}^2 = AB\,(AD + DB) = \overline{AB}^2.$

347. COR. 4.—*If a perpendicular be let fall from any point in a circumference upon a diameter, this perpendicular is a mean proportional between the segments of the diameter.*

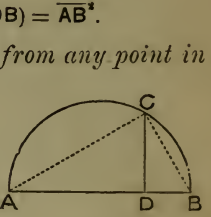


FIG. 237.

DEM.—Thus, AD : CD :: CD : DB, or $\overline{CD}^2 = AD \times DB.$

For, drawing AC and CB, ACB is a right angle, and the case falls under *Cor.* 2.

The chords AC and CB are mean proportionals between the whole diameter and their adjacent segments by *Cor.* 1.

348. SCH.—This proposition, with its corollaries, is perhaps the most fruitful in direct practical results of any in Geometry. *Cor.* 3 will be recognized as a demonstration of the Pythagorean proposition (***109***), PART I. There are many other demonstrations of exceeding beauty, some of which will be given in PART III. The one here given is the simplest, and shows best the way in which this truth grows out of the more general fact of similarity.

PROPOSITION VI.

349. Theorem.—*Regular polygons of the same number of sides are similar figures.*

DEM.—Let P and P′ be two regular polygons of the same number of sides,* a, b, c, d, etc., being the sides of the former, and a', b', c', d', etc., the sides of the latter. Now, by the definition of regular polygons, the sides a, b, c, d, etc., are equal each to each, and also a', b', c', d', etc. Hence, we have $a : a' :: b : b' :: c : c' :: d : d'$, etc. Again, the angles are equal, since n being the number of sides of each polygon, each angle is

$$\frac{n \times 2 \text{ right angles} - 4 \text{ right angles}}{n} \quad (\textit{256}).$$

Hence the polygons are mutually equiangular, and have their sides proportional; that is, they are similar. Q. E. D.

---

* The student may construct two regular hexagons, if thought desirable.

350. Cor. 1.—*The corresponding diagonals of regular polygons of the same number of sides are in the same ratio as the sides of the polygons.*

Let the student draw a figure and demonstrate the fact.

351. Cor. 2.—*The radii of the inscribed, and also of the circumscribed circles, of regular polygons of the same number of sides, are in the same ratio as the sides of the polygons.*

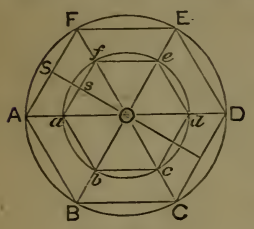


Fig. 238.

Dem.—Since the angles F and *f* are equal, and bisected by FO, the right angled triangles OSF, O*sf* are equiangular, and hence similar. Therefore FS : *fs* : : SO : *s*O or FO : *f*O. Whence, doubling both terms of the first couplet,

FA : *fa* : : SO : *s*O or FO : *f*O.

PROPOSITION VII.

352. ***Theorem.***—*Circles are similar figures.*

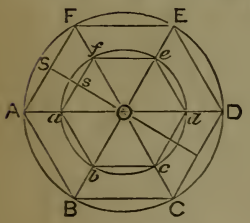


Fig. 239.

Dem.—Let O*a* and OA be the radii of any two circles. Place the circles so that they shall be concentric, as in the figure. Inscribe the regular hexagons, as *abcdef*, ABCDEF. Conceive the arcs AB, BC, etc., of the outer circumference, bisected, and the regular dodecagon inscribed, and also the corresponding regular dodecagon in the inner circumference. These are similar figures by (***349***). Now, as the process of bisecting the arcs of the exterior circumference can be conceived as indefinitely repeated, and the corresponding *regular* polygons as inscribed in each circle, the circles may be considered as regular polygons of the same number of sides, and hence similar. Q. E. D.

353. Cor.—Arcs of similar sectors are to each other as the radii of their circles; *i. e.*, arc *fe* : arc FE : : O*f* : OF.

Sch.—The circle is said to be the *Limit* of the inscribed polygon, and the circumference the *limit* of the perimeter. By this is meant that as the number of the sides of the inscribed polygon is increased it approaches nearer and nearer to equality with the circle. The apothem approaches equality with the radius, and hence has the radius for its limit. It is an axiom of great importance in mathematics that, *Whatever can be shown to be true of a magnitude as it approaches its limit indefinitely, is true of that limit.*

EXERCISES.

1. ***Prob.***—*To divide a given line into parts which shall be proportional to several given lines.*

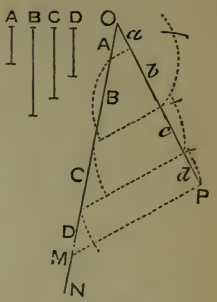


FIG. 240.

SOLUTION.—Let it be required to divide OP into parts proportional to the lines A, B, C, and D. Draw ON making any convenient angle with OP, and on it lay off A, B, C, and D, in succession, terminating at M. Join M with the extremity P, and draw parallels to MP through the other points of division. Then by reason of the parallels we shall have

A : B : C : D :: a : b : c : d, (***338***).

2. ***Prob.***—*To find a fourth proportional to three given lines.*

For the solution see (***89***). Repeat the process, and give the reasons.

3. ***Prob.***—*To find a third proportional to two given lines.*

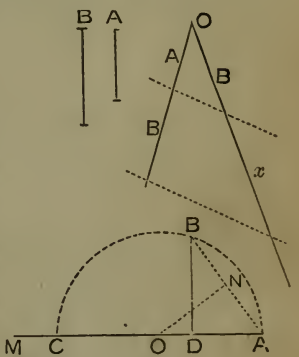


FIG. 241.

SOLUTION.—This may be solved as the two preceding. Thus, take any two lines, as A and B, for the given lines. We are to find a third line x, such that A : B :: B : x. The figure will suggest the details.

The following is a solution based on (***347***). Draw an indefinite line AM. Take AD = A, and erect BD = B. Join AB, and bisect it by the perpendicular ON. Then with O as a centre, and OA as a radius, describe a semi-circumference. This will pass through B. (Why?) Also AD : BD :: BD : CD (= x). (Why?)

4. Draw any straight line on the blackboard, and divide it into 5 equal parts, upon the principle used in the preceding solutions.

5. Review the exercises under (***89, 90***), and give the reasons.

6. ***Prob.***—*To find a mean proportional between two given lines.*

For the solution see (***110***). Repeat the process, and give the reasons for the method.

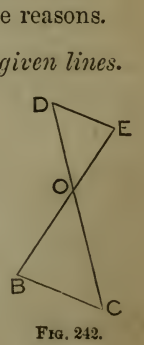


FIG. 242.

7. DE being parallel to BC, prove that the triangles DOE and BOC are similar, and hence that OD : OC :: OE : OB. Are the following proportions true?

OD : OC :: OE : OB, OD : DE :: OC : BC,
OD : OE :: OC : OB, OB : BC :: OE : DE.

8. Show that if ABCDEF is a regular polygon, A*bcdef* is also regular, *bc*, *cd*, etc., being parallel to BC, CD, etc. Show that any two similar polygons may be placed in similar relative positions, and hence show that the corresponding diagonals are in the same ratio as the homologous sides.

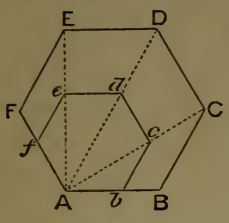


FIG. 243.

9. The sides of one triangle are 7, 9, and 11. The side of a second similar triangle, homologous with side 9, is $4\frac{1}{2}$. What are the other sides of the latter?

10. The diameter of a circle is 20. What is the perpendicular distance to the circumference from a point in the diameter 15 from one extremity? What are the distances from the point where this perpendicular meets the circumference to the extremities of the diameter?

SYNOPSIS.

OF SIMILARITY.

- Primary notion of similarity.
- Definition of similarity.
- Homogeneity of sides. In general. In triangles.
- **CONDITIONS OF SIMILARITY IN TRIANGLES.**
  - PROP. I. Mutually equiangular.
    - *Cor.* 1. Two angles equal.
    - *Cor.* 2. A parallel to a side.
    - *Cor.* 3. Lines cutting parallels.
  - PROP. II. Sides proportional.
    - *Sch.* Either of two facts sufficient.
  - PROP. III. Sides parallel or perpendicular.
  - PROP. IV. An angle equal in each, and sides proportional.
  - PROP. V. Perpendicular from right angle upon hypotenuse.
    - *Cor.* 1. Side about right angle.
    - *Cor.* 2. Perpendicular.
    - *Cor.* 3. Square on hypotenuse.
    - *Cor.* 4. Perpendicular on diameter.
    - *Sch.* Importance of this *Prop.* and *Cor's.*
- PROP. VI. Regular polygons similar.
  - *Cor.* 1. Corresponding diagonals.
  - *Cor.* 2. Radii of inscribed and circumscribed circles.
- PROP. VII. Circles similar.
  - *Sch.* Circle limit of polygon.
- EXERCISES.
  - *Prob.* To divide a line into proportional parts.
  - *Prob.* To find a fourth proportional.
  - *Prob.* To find a third proportional.
  - *Prob.* To find a mean proportional.

SECTION XI.

APPLICATIONS OF THE DOCTRINE OF SIMILARITY TO THE DEVELOPMENT OF GEOMETRICAL PROPERTIES OF FIGURES.

354. The doctrine of similarity, as presented in the preceding section, is the chief reliance for the development of the geometrical properties of figures. This section will be devoted to the investigation of a few of the more elementary properties of plane figures, which we are able to discover by means of this doctrine.

OF THE RELATIONS OF THE SEGMENTS OF TWO LINES INTERSECTING EACH OTHER, AND INTERSECTED BY A CIRCUMFERENCE.

PROPOSITION I.

355. Theorem.—*If two chords intersect each other in a circle, their segments are reciprocally proportional; whence the product of the segments of one chord equals the product of the segments of the other.*

DEM.—Let the chords AC and BD intersect at O; then is AO : BO :: DO : CO, whence AO × CO = BO × DO.

For, draw AD and BC. The two triangles AOD and BOC are equiangular, and hence similar; since the angles at O are vertical, and consequently equal (***134***), and D = C, because both are measured by ½ arc AB (***210***). (A = B because both are measured by ½ arc DC; but it is only necessary to show that *two* angles are equal in order to show that the triangles are equiangular, and hence similar.) Now, comparing the homologous sides (those opposite the equal angles), we have AO : BO :: DO : CO; whence, AO × CO = BO × DO. Q. E. D.

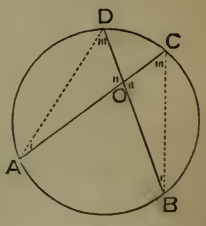


FIG. 244.

QUERIES.—Why is AO compared with BO? Why DO with CO? Would AO : CO :: BO : DO be true? Would AO : DO :: BO : CO? What is the force of the word "reciprocally," as used in the proposition?

PROPOSITION II.

356. Theorem.—*If from a point without a circle, two secants be drawn terminating in the concave arc, the whole secants are reciprocally proportional to their external segments; whence the product of one secant into its external segment equals the product of the other into its external segment.*

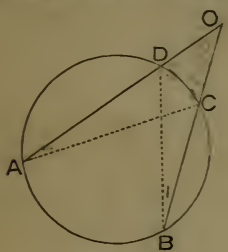


Fig. 245.

Dem.—OA and OB being secants, OA : OB :: OC : OD, and consequently OA × OD = OB × OC. For, drawing AC and DB, the two triangles ACO and BDO have angle O common, and A = B, since both are measured by ½ DC; hence the triangles are similar, and we have OA : OB :: OC : OD, and consequently OA × OD = OB × OC. Q. E. D.

Same queries as under the preceding demonstration.

PROPOSITION III.

357. Theorem.—*If from a point without a circle a tangent be drawn, and a secant terminating in the concave arc, the tangent is a mean proportional between the whole secant and its external segment; whence the square of the tangent equals the product of the secant into its external segment.*

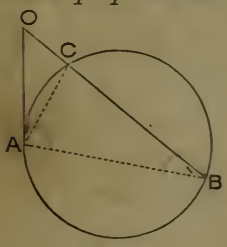


Fig. 246.

Dem.—OA being a tangent and OB a secant, OB : OA :: OA : OC, whence $\overline{OA}^2$ = OB × OC. For, drawing AB and AC, the triangles OAB and ACO have angle O common, and OAC = B, since each is measured by ½ arc AC; hence the triangles are similar, and OB : OA :: OA : OC, whence $\overline{OA}^2$ = OB × OC. Q. E. D.

OF THE BISECTOR OF AN ANGLE OF A TRIANGLE.

PROPOSITION IV.

358. Theorem.—*A line which bisects any angle of a triangle divides the opposite side into segments proportional to the adjacent sides.*

DEM.—Let CD bisect the angle ACB; then AD : DB :: AC : CB.

For, draw BE parallel to CD, and produce it till it meets AC produced, as at E. Now, by reason of the parallels CD and EB, angle ACD = AEB, and angle DCB = CBE (*152*). Whence, as ACD = DCB by hypothesis, E = CBE, and CE = CB (*227*). Also, since CD is parallel to EB, AD : DB :: AC : CE (*337*). Substituting for CE its equal CB, we have AD : DB :: AC : CB. Q. E. D.

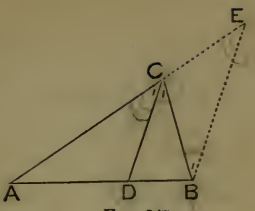


FIG. 247.

PROPOSITION V.

359. Theorem.—*If a line be drawn from any vertex of a triangle bisecting the exterior angle and intersecting the opposite side produced, the distances from the other vertices to this intersection are proportional to the adjacent sides.*

DEM.—Through the vertex C let CD be drawn, bisecting the exterior angle FCB, and intersecting AB produced in D; then AD : BD :: AC : CB.

For, draw BE parallel to CD. By reason of these parallels, angle FCD = CEB, and BCD = CBE (*152*). Hence CEB = CBE, and CB = CE. Also, by reason of the parallels, AD : BD :: AC : CE, or its equal CB (*335*). Q. E. D.

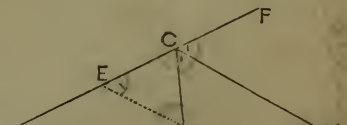


FIG. 248.

PROPOSITION VI.

360. Theorem.—*If a line be drawn bisecting any angle of a triangle and intersecting the opposite side, the rectangle of the sides about the bisected angle equals the rectangle of the segments of the third side, plus the square of the bisector.*

DEM.—Let CD bisect the angle ACB; then AC × CB = AD × DB + $\overline{CD}^2$.

For, circumscribe the circle about the triangle, produce the bisector till it meets the circumference at E, and draw EB. The triangles ADC and CBE are similar, since angle ACD = ECB, by hypothesis, and A = E, because each is measured by ½ arc CB. Therefore, AC : CE :: CD : CB, whence AC × CB = CE × CD = (DE + CD) CD = DE × CD + $\overline{CD}^2$. For DE × CD, substituting its equivalent AD × DB (*355*), we have AC × CB = AD × DB + $\overline{CD}^2$. Q. E. D.

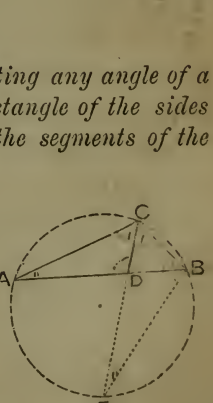


FIG. 249.

PROPOSITION VII.

361. Theorem.—*The bisectors of the angles of a triangle all pass through the same point, which is the centre of the inscribed circle.*

DEM.—Draw two lines bisecting two of the angles, and from their intersection draw a line to the other angle. Then show that the latter angle is bisected. By (Ex. 4, page 134) this point is shown to be the centre of the inscribed circle. [The student should fill out the demonstration.]

AREAS OF SIMILAR FIGURES.

PROPOSITION VIII.

362. Theorem.—*The areas of similar triangles are to each other as the squares described on their homologous sides.*

DEM.—Let ABC and DEF be any two similar triangles; then is

area ABC : *area* DEF :: $\overline{CB}^2 : \overline{FE}^2$ :: $\overline{AC}^2 : \overline{DF}^2$ :: $\overline{AB}^2 : \overline{DE}^2$

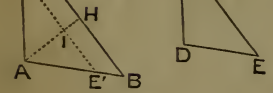


FIG. 250.

For, place the largest angle of the triangle DEF, as D, on its equal angle A, of the triangle ABC*; let E fall at E', F at F', and draw E'F'; then is triangle AE'F' = DEF (*284*), and E'F' is parallel to BC. Let fall a perpendicular from A to CB. Then AI is the altitude of AE'F', and AH of ABC. Now, by similar triangles we have CB : F'E' :: AH : AI.

But $\frac{1}{2}$AH : $\frac{1}{2}$AI :: AH : AI; and, multiplying corresponding terms, $\frac{1}{2}$AH × CB : $\frac{1}{2}$AI × F'E' :: $\overline{AH}^2 : \overline{AI}^2$. Whence, since

$\frac{1}{2}$AH × CB = *area* ABC, and $\frac{1}{2}$AI × F'E' = *area* AE'F' = *area* DEF, and

AH : AI :: CB : FE :: AC : DF :: AB : DE, or $\overline{AH}^2 : \overline{AI}^2$ :: $\overline{CB}^2 : \overline{FE}^2$ :: $\overline{AC}^2 : \overline{DF}^2$ :: $\overline{AB}^2$ :: $\overline{DE}^2$; we have

area ABC : *area* DEF :: $\overline{CB}^2 : \overline{FE}^2$ :: $\overline{AC}^2 : \overline{DF}^2$ :: $\overline{AB}^2 : \overline{DE}^2$. Q. E. D.

PROPOSITION IX.

363. Theorem.—*The areas of similar polygons are to each other as the squares of the homologous sides of the polygons.*

---

* The only object in taking the *largest* angle is to make the perpendicular fall *within* the triangle. Any two equal angles may be applied, and the demonstration is essentially the same.

DEM.—Let *abcdef* and ABCDEF be two similar polygons. Designate the former by p, and the latter by P. Then $p : P :: \overline{ab}^2 :: \overline{AB}^2$ or as the squares of any other two homologous sides.

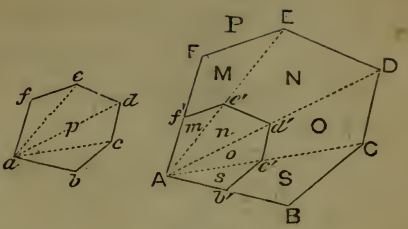


FIG. 251.

For, from the equal angles a and A drawing the diagonals, the corresponding partial angles into which a and A are divided are equal. [Let the student show why by ***342.***] Now take $Ab' = ab$, and draw $b'c'$, making angle $Ab'c' = b$. Then $b'c' = bc$, and $Ac' = ac$, since the triangles abc and $Ab'c'$ have two angles and the included side of one equal to two angles and the included side of the other. In like manner draw $c'd'$ making angle $b'c'd' = bcd$, $c'd' = cd$, and $Ad' = ad$. So, also, making angle $c'd'e' = cde$, and *angle* $d'e'f' = def$, $d'e' = de$, $e'f' = ef$, and $f'A = fa$. Hence the polygon $Ab'c'd'e'f' = p$, and its sides are respectively parallel to the corresponding sides of P. Now, let m, n, o, and s represent the triangles in which they stand, and M, N, O, and S the corresponding triangles of P, as AFE, etc. Triangles m and M being similar, and also n and N, we have

$$m : M :: \overline{Ae'}^2 : \overline{AE}^2, \text{ and } n : N :: \overline{Ac'}^2 : \overline{AE}^2.$$

Whence $m : M :: n : N$.

In like manner we can show that $n : N :: o : O$, and that $o : O :: s : S$.

Whence $m : M :: n : N :: o : O :: s : S$.

By composition, $(m+n+o+s)$ (or p) : $(M+N+O+S)$ (or P) :: $s : S$.

But $s : S :: \overline{Ab'}^2$ (or $\overline{ab}^2$) : $\overline{AB}^2$. Therefore $p : P :: \overline{ab}^2 : \overline{AB}^2$, or as the squares of any two homologous sides. Q. E. D.

364. COR. 1.—*Similar polygons* are to each other as the squares of their corresponding diagonals.*

In the demonstration we have $s : S :: \overline{Ac'}^2$ (or $\overline{ac}^2$) : $\overline{AC}^2$. Whence $p : P :: \overline{ac}^2 : \overline{AC}^2$. The same may be shown of any other corresponding diagonals.

365. COR. 2.—*Regular polygons* of the same number of sides are to each other as the squares of their homologous sides; since they are similar figures* (***349***).

366. COR. 3.—*Regular polygons* of the same number of sides are to each other as the squares of their apothems.*

For their apothems are to each other as their sides. Hence the squares of their apothems are to each other as the squares of their sides.

367. COR. 4.—*Circles are to each other as the squares of their radii* (***352***), *and as the squares of their diameters.*

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* This is a common elliptical form for "The *areas* of, etc."

OF PERIMETERS AND THE RECTIFICATION OF THE CIRCUMFERENCE.

368. The Rectification of a curve is the process of finding its length.

The term *rectification* signifies making straight, and is applied as above, under the conception that the process consists in finding a straight line equal in length to the curve.

PROPOSITION X.

369. Theorem.—*The perimeters of similar polygons are to each other as their homologous sides, and as their corresponding diagonals.*

Dem.—Let a, b, c, d, etc., and A, B, C, D, etc., be the homologous sides of two similar polygons whose perimeters are p and P; then $p : P :: a : A :: b : B :: c : C$, etc.; and r and R being corresponding diagonals, $p : P :: r : R$. Since the polygons are similar, $a : A :: b : B :: c : C :: d : D$, etc. By composition, $(a+b+c+d+\text{etc.})$ (or p) : (A + B + C + D + etc.) (or P) : : a : A, or as any other homologous sides. Also, as the homologous sides are to each other as the corresponding diagonals (***350***), $p : P :: r : R$. Q. E. D.

370. Cor. 1.—*The perimeters of regular polygons of the same number of sides are to each other as the apothems of the polygons.*

For the apothems are to each other as the sides of the polygons (***351***).

371. Cor. 2.—*The circumferences of circles are to each other as their radii, and as their diameters;* since they may be considered as regular polygons of the same number of sides (***352***).

PROPOSITION XI.

372. Theorem.—*The circumference of a circle whose radius is* 1, *is* 2π, *the numerical value of* π *being approximately* 3.1416.

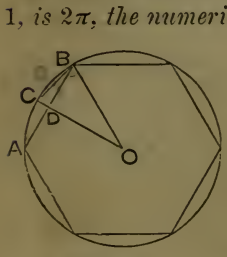


Fig. 252.

Dem.—We will approximate the circumference of a circle whose radius is 1, by obtaining, 1st, the perimeter of the regular inscribed hexagon; 2d, the perimeter of the regular inscribed dodecagon; 3d, the perimeter of the regular inscribed polygon of 24 sides; then of 48, etc.

In order to do this, let us find the relation between the chord of an arc and the chord of $\frac{1}{2}$ the arc in a circle whose radius is 1. Call the chord of an arc as

AB, C, and the chord of half the arc, as CB, c. Now, BDO is right angled at D, whence $\overline{BO}^2 = \overline{BD}^2 + \overline{DO}^2$ (*346*), or $DO = \sqrt{\overline{BO}^2 - \overline{BD}^2}$. But in the present case $BO = 1$; hence $DO = \sqrt{1 - \frac{1}{4}C^2}$. Taking DO from CO, we have $CD = 1 - \sqrt{1 - \frac{1}{4}C^2}$. From the right angled triangle BDC we have CB (or c) $= \sqrt{\overline{BD}^2 + \overline{CD}^2}$, or substituting $\frac{1}{2}C$ for BD, and $1 - \sqrt{1 - \frac{1}{4}C^2}$ for CD, this reduces to

$$c = \sqrt{2 - \sqrt{4 - C^2}}, \quad \text{or,} \quad \left[2 - (4 - C^2)^{\frac{1}{2}}\right]^{\frac{1}{2}}$$

By the use of this formula, we make the following computations:

No. sides.	Form of Computation.	Length of Side.	Perimeter.
6.	See (*271*)	1.00000000	6.00000000
12.	$c = \sqrt{2 - \sqrt{4 - 1^2}} = \sqrt{2 - \sqrt{3}}$, or $\left(2 - 3^{\frac{1}{2}}\right)^{\frac{1}{2}}$.	= .51763809	6.21165708
24.	$c = \left\{2 - \left[4 - \left(2 - 3^{\frac{1}{2}}\right)\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}} = \left[2 - \left(2 + 3^{\frac{1}{2}}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}$.	= .26105238	6.26525722
48.	$c = \left(2 - \left\{4 - \left[2 - \left(2 + 3^{\frac{1}{2}}\right)^{\frac{1}{2}}\right]\right\}^{\frac{1}{2}}\right)^{\frac{1}{2}}$ $= \left\{2 - \left[2 + \left(2 + 3^{\frac{1}{2}}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}$	= .13080626	6.27870041
96.	$c = \left(2 - \left\{2 + \left[2 + \left(2 + 3^{\frac{1}{2}}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}\right)^{\frac{1}{2}}$ . . .	= .06543817	6.28206396
192.	$c = \left[2 - \left(2 + \left\{2 + \left[2 + \left(2 + 3^{\frac{1}{2}}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}$ . .	= .03272346	6.28290510
384.	$c = \left\{2 - \left[2 + \left(2 + \left\{2 + \left[2 + \left(2 + 3^{\frac{1}{2}}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}$	= .01636228	6.28311544
768.	$c = \left(2 - \left\{2 + \left[2 + \left(2 + \left\{2 + \left[2 + \left(2 + 3^{\frac{1}{2}}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}\right)^{\frac{1}{2}}$	= .00818121	6.28316941

It now appears that the first four decimal figures do not change as the number of sides is increased, but will remain the same *how far soever we proceed.* We may therefore consider 6.28317, as *approximately* the circumference of a circle whose radius is 1, *i. e.*, $2\pi = 6.28317$, nearly; and $\pi = 3.1416$, nearly.

373. Scii.—The symbol π is much used in mathematics, and signifies, primarily, *the semi-circumference of a circle whose radius is* 1. $\frac{1}{2}\pi$ is therefore a symbol for a quadrant, 90°, or a right angle. $\frac{1}{4}\pi$ is equivalent to 45°, etc., the radius being always supposed 1, unless statement is made to the contrary. The numerical value of π has been sought in a great variety of ways, all of which agree in the conclusion that it cannot be exactly expressed in decimal numbers, but is approximately as given in the proposition. From the time of *Archimedes* (287 B.C.) to the present, much ingenious labor has been bestowed upon this problem. The most expeditious and elegant methods of approximation are furnished by the *Calculus.* The following is the value of π extended to fifteen places of decimals: 3.141592653589793.

PROPOSITION XII.

374. Theorem.—*The circumference of any circle is* $2\pi r$, r *being the radius.*

Dem.—The circumferences of circles being to each other as their radii, and

2π being the circumference of a circle whose radius is 1, we have $2\pi : circ. :: 1 : r$, whence $circ. = 2\pi r$.

375. COR.—*The circumference of a circle is πD, D being the diameter,* since $2\pi r = \pi 2r = \pi D$.

AREA OF THE CIRCLE.

PROPOSITION XIII.

376. Theorem.—*The area of a circle whose radius is* 1, *is* π.

DEM.—The area of a circle is $\frac{1}{2} r \times$ *circumference* (***328***). When $r = 1$, *circumference* $= 2\pi$ (***372***); hence

area of circle whose radius is $1 = \frac{1}{2} \times 2\pi = \pi$. Q. E. D.

PROPOSITION XIV.

377. Theorem.—*The area of any circle is πr^2, r being the radius.*

DEM.—The areas of circles being to each other as the squares of their radii, and π being the area of a circle whose radius is 1, we have

$$\pi : area\ of\ any\ circle :: 1^2 : r^2,$$

whence *area of any circle* $= \pi r^2$. Q. E. D.

378. COR.—*The area of any sector is such a part of the area of the circle as the angle of the sector is of four right angles.*

379. SCH.—As the value of π cannot be exactly expressed in numbers, it follows that the area cannot. Finding the area of a circle has long been known as the problem of *Squaring the Circle*, *i. e.*, finding a square equal in area to a circle of given radius. Doubtless many hare-brained visionaries or ignoramuses will still continue the chase after the phantom, although it has long ago been demonstrated that the diameter of a circle and its circumference are incommensurable. It is, however, an easy matter to conceive a square of the same area as any given circle. Thus, let there be a rectangle whose base is equal to the circumference of the circle, and whose altitude is half the radius; its area is exactly equal to the area of the circle. Now, let there be a square whose side is a mean proportional between the altitude and base of this rectangle; the area of the square is exactly equal to the area of the circle.

EXERCISES.

1. Show that if a chord of a circle is conceived to revolve, varying in length as it revolves, so as to keep its extremities in the circumference while it constantly passes through a fixed point, the rectangle of its segments remains constant.

2. The two segments of a chord intersected by another chord are 6 and 4, and one segment of the other chord is 3. What is the other segment of the latter chord?

3. Show how PROP'S I., II., and III. may be considered as different cases of one and the same proposition.

SUG'S.—By stating Propositions I. and II. thus, *The distances from the intersection of the lines to their intersections with the circumference*, what follows? In *Fig.* 245, if the secant AO becomes a tangent, what does OD become?

4. In a triangle whose sides are 48, 36, and 50, where do the bisectors of the angles intersect the sides?

5. In the last example find the lengths of the bisectors.

6. Review the examples under (***111, 112, 113, 114***), and give the reasons.

7. In a circle whose radius is 20, what is the length of the arc of a sector whose angle is 30°? What is the area of this sector?

8. If a circle whose radius is 24 is divided into 5 equal parts by concentric circumferences, what are the diameters of the several circles? If the radius is r, and number of parts n?

9. ***Prob.***—*To divide a line in extreme and mean ratio;* that is, *so that the whole line shall be to the greater segment, as the greater segment is to the less.*

SOLUTION.—Let it be proposed to divide the line AB in extreme and mean ratio. At one extremity of the line, as B, erect a perpendicular equal to half the line, that is, make BO = ½ AB. With O as a centre, describe a circumference passing through B. Draw AO, and take AC equal to AD. Then is AB divided in extreme and mean ratio at C, so that AB : AC :: AC : CB. To prove it, produce AO to E. Now, AE : AB :: AB : AD (***357***), or by inversion, AB : AE :: AD : AB. By division, AB : AE − AB (= AE − DE) (or AD) (= AC) :: AD (= AC) : AB − AD (= AB − AC) (or CB). That is, AB : AC :: AC : CB.

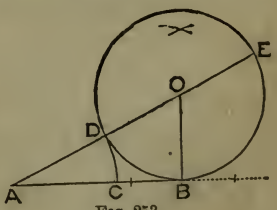


FIG. 253.

10. ***Prob.***—*To inscribe a regular decagon in a circle, and hence a regular pentagon, and regular polygons of* 20, 40, 80, *etc., sides.*

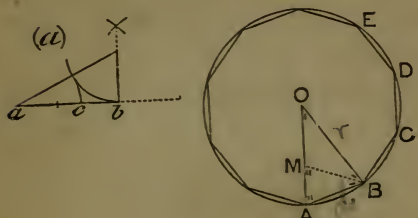


Fig. 254.

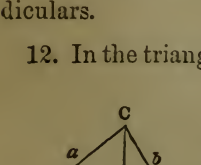
Solution.—Divide the radius in extreme and mean ratio, as at (*a*). Then is the greater segment *ac* the chord of a regular inscribed decagon, as ABCD, etc. To prove this, draw OA and OB, and taking OM = *ac* = AB, a side of the polygon, draw BM. Now, OA : OM :: OM : MA by construction. As OM = AB, we have OA : AB :: AB : MA. Hence, considering the antecedents as belonging to the triangle OAB, and the consequents to the triangle BAM we observe that the two sides about the angle A, which is common to both triangles, are proportional, hence the triangles are similar (***342***). Therefore, ABM is isosceles, since OAB is, and angle BMA = A = OBA, and MB = BA = OM This makes OMB also isosceles, and the angle O = OBM. Again the exterior angle BMA = O + OBM = 2O; hence A, which equals BMA = 2O. Hence also OBA, which equals A, = 2O. Wherefore O is $\frac{1}{5}$ the sum of the angles of the triangle OAB, or $\frac{1}{5}$ of 2 right angles, = $\frac{1}{10}$ of 4 right angles. The arc AB is, therefore the measure of $\frac{1}{10}$ of 4 right angles, and is consequently $\frac{1}{10}$ of the circumference.

To construct the pentagon, join the alternate angles of the decagon. To construct the regular polygon of 20 sides, bisect the arcs subtended by the sides of the decagon, etc.

11. The projection of one line upon another in the same plane is the distance between the feet of two perpendiculars let fall from the extremities of the former upon the latter. Show that this projection is equal to the square root of the difference between the square of the line and the square of the difference of the perpendiculars.

12. In the triangle ABC, p being a perpendicular upon BA, prove that

$$m + n\ (= c) : a + b :: a - b : m - n.$$



State the fact as a proposition. Give the necessary modification when the perpendicular falls without the triangle.

Sug. $a^2 - m^2 = b^2 - n^2$, whence $a^2 - b^2 = m^2 - n^2$, etc.

13. The three sides of a triangle being 4, 5, and 6, find the segments of the last side, made by a perpendicular from the opposite angle. *Ans.* 3.75, and 2.25.

14. Same as above when the sides are 10, 4, and 7, and the perpendicular is let fall from the angle included by the sides 10 and 4. Draw the figure. Why is one of the segments negative?

15. What is the area of a regular octagon inscribed in a circle whose radius is 1? What is its perimeter? What if the radius is 10?

16. What is the side of an equilateral triangle inscribed in a circle whose radius is 1?

SYNOPSIS.

Geometrical Properties developed by means of the Doctrine of Similarity.

- Importance of this doctrine.
- **Relations of the Segments.**
  - Prop. I. Of chords.
  - Prop. II. Of secants.
  - Prop. III. Of secants and tangents.
- **Bisectors of the Angles of Triangles.**
  - Prop. IV. How divide sides.
  - Prop. V. Of exterior angles.
  - Prop. VI. Length of in relation to other parts.
  - Prop. VII. All intersect at one point.
- **Areas of Similar Figures.**
  - Prop. VIII. Of triangles.
  - Prop. IX. Of polygons.
    - *Cor.* 1. As squares of diagonals.
    - *Cor.* 2. Regular polygons.
    - *Cor.* 3. As squares of apothems.
    - *Cor.* 4. Of circles.
- **Perimeters and Rectification.**
  - Definition of rectification.
  - Prop. X. Perimeters of similar polygons.
    - *Cor.* 1. Regular polygons.
    - *Cor.* 2. Circumferences as radii.
  - Prop. XI. Rectification of circumference whose radius is 1.
    - *Sch.* Signification and importance of π.
  - Prop. XII. Circumference of any circle $= 2\pi r$.
    - *Cor.* Also πD.
- **Area of Circle.**
  - Prop. XIII. Whose radius is 1.
  - Prop. XIV. Of any circle.
    - *Cor.* Of sector.
    - *Sch.* Squaring the circle.
- **Exercises.**
  - *Prob.* To divide a line in extreme and mean ratio.
  - *Prob.* To inscribe a regular decagon, etc.

CHAPTER II.

*SOLID GEOMETRY.**

SECTION I.

OF STRAIGHT LINES AND PLANES.

PERPENDICULAR AND OBLIQUE LINES.

380. Solid Geometry is that department of geometry in which the forms (or figures) treated are not limited to a single plane.

381. A Plane (or a *Plane Surface*) is a surface such that a straight line joining two points in it lies wholly in the surface.

ILL.—The surface of the blackboard is designed to be a plane. To ascertain whether it is truly so, take a ruler with a straight edge, and apply this edge in all directions upon it. If it always coincides, *i. e.*, touches throughout its whole length, the surface is a plane. Is the surface of the stove-pipe a plane? Will a straight line coincide with it in any direction? Will it in *every* direction?

PROPOSITION I.

382. Theorem.—*Three points not in the same straight line determine the position of a plane.*

DEM.—Let A, B, and C be three points not in the same straight line; then one plane can be passed through them, and only one; *i. e.*, they determine the position of a plane.

* In some respects, perhaps, "*Geometry of Space*" is preferable to this term; but, as neither is free from objections, and as this has the advantage of simplicity and long use, the author prefers to retain it.

For, pass a straight line through any two of these points, as A and B. Now, conceive any plane containing these two points; then will the line passing through them lie wholly in the plane (*381*). Conceive this plane to revolve on the line as an axis until the point C falls in the plane. Thus we have one plane passed through the three points. That there can be only one is evident, since when C falls in the plane, if it be revolved either way, C will not be in it.

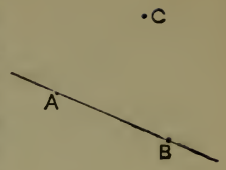


FIG. 255.

The same may be shown by first passing a plane through B and C, or A and C. There is, therefore, only one position of the plane in which it will contain the third point. Q. E. D.

383. COR. 1.—*Through one line, or two points, an infinite number of planes can be passed.*

384. COR. 2.—*Two intersecting lines determine the position of a plane.*

DEM.—For, the point of intersection may be taken as one of the three points requisite to determine the position of a plane, and any other two points, one in each of the lines, as the other two requisite points. Now, the plane passing through these points contains the lines, for it contains two points in each line.

385. COR. 3.—*Two parallel lines determine the position of a plane.*

DEM.—For, pass a plane through one of the parallels, and conceive it revolved until it contains some point of the second parallel; then as the plane cannot be revolved either way from this position without leaving this point without it, it is the *only* plane containing the first parallel and this point in the second. But parallels lie in the same plane (*66*), whence the plane of the parallels must contain the first line, and the specified point in the second. Therefore, the plane containing the first line and a point in the second is the plane of the parallels, and is fixed in position.

386. COR. 4.—*The intersection of two planes is a straight line.*

For *two* planes cannot have even three points, *not in the same straight line*, common, much less an indefinite number, which would be required if we conceived the intersection (that is, the common points) to be any other than a straight line.

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387. A Perpendicular to a Plane is a line which is perpendicular to all lines of the plane passing through its foot. Conversely, the plane is perpendicular to the line.

PROPOSITION II.

388. Theorem.—*A line which is perpendicular to two lines of a plane, at their intersection, is perpendicular to the plane.*

DEM.—Let PD be perpendicular to AB and CF at D; then is it perpendicular to MN, the plane of the lines AB and CF.

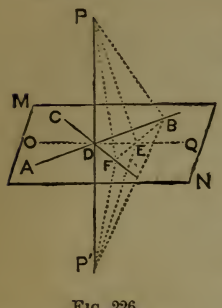

FIG. 226.

Let OQ be any other line of the plane MN, passing through D. Draw FB intersecting the three lines AB, CF, and OQ. Produce PD to P', making P'D = PD, and draw PF, PE, PB, P'F, P'E, and P'B. Then is PF = P'F, and PB = P'B, since FD and BD are perpendicular to PP', and PD = P'D (*284*). Hence, the triangles PFB and P'FB are equal (*292*); and, if PFB be revolved upon FB till P falls at P', PE will fall in P'E. Therefore OQ has E equally distant from P and P', and as D is also equidistant from the same points, OQ is perpendicular to PD at D (*130*). Now, as OQ is *any* line, PD is perpendicular to any line of the plane passing through its foot, and consequently perpendicular to the plane (*387*). Q. E. D.

389. COR.—*If one of two perpendiculars revolves about the other as an axis, its path is a plane perpendicular to the axis.*

Thus, if AB revolves about PP' as an axis, it describes the plane MN.

PROPOSITION III.

390. Theorem.—*At any point in a plane one perpendicular can be erected to the plane, and only one.*

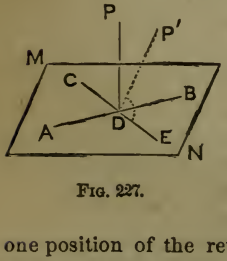

FIG. 227.

DEM.—Let it be required to show that one perpendicular, and only one, can be erected to the plane MN at D. Through D draw two lines of the plane, as AB and CE, at right angles to each other. CE being perpendicular to AB, let a line be conceived as starting from the position ED to revolve about AB as an axis. It will remain perpendicular to AB (*389*). Conceive it to have passed to P'D. Now, as it continues to revolve, P'DC diminishes continuously, and at the same rate as P'DE grows greater; hence, in one position of the revolving line, and in only one, as PD, PDE will equal PDC, and PD will be perpendicular to CE. Therefore, PD is perpendicular to two lines of the plane, at their intersection, and is the only line that can be thus perpendicular, whence it is perpendicular to the plane (*388*), and is the only perpendicular. Q. E. D.

PROPOSITION IV.

391. Theorem.—*If from any point in a perpendicular to a plane, oblique lines be drawn to the plane, those which pierce the plane at equal distances from the foot of the perpendicular are equal; and of those which pierce the plane at unequal distances from the foot of the perpendicular, those which pierce at the greater distances are the greater.*

DEM.—Let PD be a perpendicular to the plane MN, and PE, PE′, PE″, and PE‴, be oblique lines piercing the plane at equal distances ED, E′D, E″D, and E‴D, from the foot of the perpendicular; then PE = PE′ = PE″ = PE‴. For each of the triangles PDE, PDE′, etc., has two sides and the included angle equal to the corresponding parts in the other.

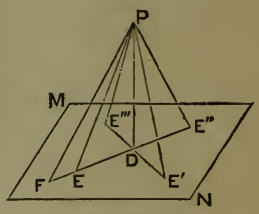

FIG. 258.

Again, let FD be longer than E′D. Then is PF > PE′. For, take ED = E′D; then PE = PE′, by the preceding part of the demonstration. But PF > PE by (*139*). Hence, PF > PE′. Q. E. D.

392. The Inclination of a line to a plane is measured by the angle which the line makes with a line of the plane passing through the point in which the line pierces the plane and the foot of a perpendicular to the plane from any point in the line.

Thus PFD is the inclination of PF to the plane MN.

393. COR. 1.—*The angles which oblique lines drawn from a common point in a perpendicular to a plane, and piercing the plane at equal distances from the foot of the perpendicular, make with the perpendicular, are equal; and the inclinations of such lines to the plane are equal.*

Thus the equality of the triangles, as shown in the demonstration, shows that EPD = E′PD = E″PD = E‴PD, and PED = PE′D = PE″D = PE‴D.

394. COR. 2.—Conversely, *If the angles which oblique lines drawn from a point in a perpendicular to a plane, make with the perpendicular, are equal, the lines are equal, and pierce the plane at equal distances from the foot of the perpendicular.*

DEM.—Thus, in the figure, let DPE′ = DPE″; then PE′ = PE″ and DE′ = DE″. For, revolve DE′P about PD; DE′ will continue in the plane MN, and when angle DPE′ coincides with its equal DPE″, PE′ coincides with PE″, and DE′ with DE″.

395. Cor. 3.—Also, conversely, *Equal oblique lines from the same point in the perpendicular, pierce the plane at equal distances from the foot of the perpendicular.*

Dem.—Let PE′ = PE″; then is DE′ = DE″. For, let PDE′ revolve upon PD until E′D falls in E″D; then, if DE′ were less than DE″, PE′ would be less than PE″; and, if DE′ were greater than DE″, PE′ would be greater than PE″. But both of these conclusions are contrary to the hypothesis. Hence, as DE′ can neither be less nor greater than DE″, it must equal it. This corollary follows also from (*297*).

396. Cor. 4.—*The perpendicular is the shortest line that can be drawn to a plane from a point without, and measures the distance of the point from the plane.*

PROPOSITION V.

397. Theorem.—*From a point without a plane one perpendicular can be drawn to the plane, and only one.*

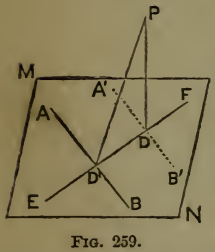

Fig. 259.

Dem.—Let it be required to show that one perpendicular can be drawn from P to the plane MN, and only one. Take AB, any line of the plane, and conceive PD′ perpendicular to it. Through D′ draw EF, a line of the plane, perpendicular to AB. Now, if PD′E = PD′F, they are both right angles, and PD′ is perpendicular to two lines of the plane passing through its foot, and hence perpendicular to the plane (*388*). If, however, PD′E does not equal PD′F, in the first instance, but PD′F < PD′E, conceive the line AB to move along the plane, continuing parallel to its primitive position, so as to cause D′ to move towards F, thus diminishing PD′E and increasing PD′F. At the same time observe that, if necessary in order to keep PD′A = PD′B*, EF can move along the plane parallel to its first position. Now, as PD′F increases, passing through all successive values, and PD′E diminishes in the same way, there will be some position of PD′, as PD, in which PDF = PDE, and as by hypothesis PDA′ remains = PDB′, PD becomes perpendicular to two lines passing through its foot, and hence perpendicular to the plane.

That there can be only one perpendicular is evident, since, if there were two, as PD′ and PD, there would be two right angles in the triangle PD′D.

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* According to the conception here used it would *not* be necessary.

398. COR.—*Through a given point in a line one plane can be passed perpendicular to the line, and only one.*

DEM.—Let **D** be the point in the line **PD**. Pass two lines through **D**, as **EF**, and **A′B′**, each perpendicular to **PD**; the plane of these lines is perpendicular to **PD**. Moreover, the required plane must contain both these lines, for if it passed through **D** and did not contain **DF**, there would be one line of the plane, at least, which would pass through **D** and not be perpendicular to **PD**, which is impossible. Hence, there can be no other plane than the plane of the two perpendiculars **EF** and **A′B′** which shall be perpendicular to **PD**, through **D**.

PROPOSITION VI.

399. Theorem.—*If from the foot of a perpendicular to a plane a line be drawn at right angles to any line of the plane, and this intersection be joined with any point in the perpendicular, the last line is perpendicular to the line of the plane.*

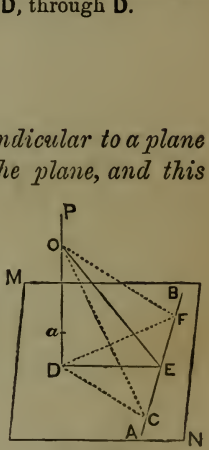

FIG. 260.

DEM.—From the foot of the perpendicular **PD** let **DE** be drawn perpendicular to **AB**, any line of the plane **MN**, and **E** joined with **O**, any point of the perpendicular; then is **OE** perpendicular to **AB**.

Take **EF** = **EC**, and draw **CD**, **FD**, **CO**, and **FO**. Now, **CD** = **DF** (?)*, whence **CO** = **FO** (?), and **OE** has **O** equally distant from **F** and **C**, and also **E**. Therefore, **OE** is perpendicular to **AB** (?). Q. E. D.

400. COR.—*The line* **DE** *measures the shortest distance between* **PD** *and* **AB**.

For, if from any other point in **AB**, as **C**, a line be drawn to **D**, it is longer than **DE** (?); and if drawn from **C** to *a*, any other point in **PD** than **D**, **C***a* is longer than **CD** (?), and consequently longer than **DE** (?).

PARALLEL LINES AND PLANES.

401. A Line is Parallel to a plane when the two will not meet, how far soever they be produced. The plane is also said to be parallel to the line.

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* Hereafter the reason will be often left out, and the mark (?) will be used to indicate that the student is to supply it.

PROPOSITION VII.

402. Theorem.—*One of two parallel lines is parallel to every plane containing the other.*

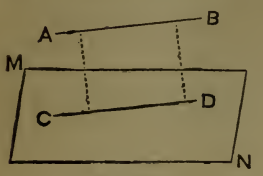

FIG. 261.

DEM.—AB being parallel to CD is parallel to any plane MN containing CD.

Since AB and CD are in the same plane (?), and as the intersection of their plane with MN is CD (?), if AB meets the plane MN, it must meet it in CD, or CD produced. But this is impossible (?). Whence AB is parallel to MN. Q. E. D.

403. COR. 1. *Through any given line a plane may be passed parallel to any other given line not in the plane of the first.*

For, through any point of the line through which the plane is to pass, conceive a line parallel to the second given line. The plane of the two intersecting lines is parallel to the second given line (?).

404. COR. 2.—*Through any point in space a plane may be passed parallel to any two lines in space.*

For, through the given point, conceive two lines parallel to the given lines; then is the plane of these intersecting lines parallel to the two given lines (?).

PROPOSITION VIII.

405. Theorem.—*If one of two parallels is perpendicular to a plane, the other is perpendicular also.*

DEM.—Let AB be parallel to CD, and perpendicular to the plane MN; then is CD perpendicular to MN.

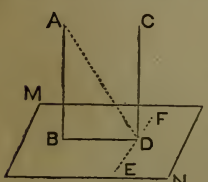

FIG 262.

For drawing BD in the plane MN, it is perpendicular to AB (?), and consequently to CD (?). Through D draw EF in the plane and perpendicular to BD, and join D with any point in AB, as A; then is EF perpendicular to AD (?). Now, EF being perpendicular to two lines, AD and BD of the plane ABDC, is perpendicular to the plane, and hence to any line of the plane passing through D, as CD. Therefore CD is perpendicular to BD and EF, and consequently to the plane MN (?). Q. E. D.

406. COR. 1.—*Two lines which are perpendicular to the same plane are parallel.*

Thus, AB and CD being perpendicular to the plane MN, if AB is not parallel to CD, draw a line through B which shall be. By the proposition this line is perpendicular to MN, and hence must coincide with AB (***390***).

407. COR. 2.—*Two lines parallel to a third not in their own plane are parallel to each other.*

DEM.—If AB and CD are parallel to EF, they are parallel to each other. Let MN be a plane passing through EF at F′, and to which EF is perpendicular; then are AB and CD respectively perpendicular to MN (?), and hence parallel (?). Q. E. D.

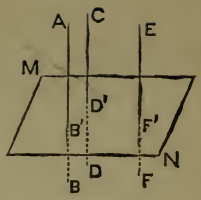

FIG. 263.

408. Parallel Planes are such as do not meet when indefinitely produced.

PROPOSITION IX.

409. Theorem.—*Two planes perpendicular to the same line are parallel to each other.*

DEM.—For, if they could meet in some point, as O, conceive two lines drawn from O, one in each plane, to the points where the perpendicular pierces the planes. We should then have two lines from the same point, perpendicular to the same line (?), which is impossible. Hence, as the planes cannot meet, they are parallel. Q. E. D.

PROPOSITION X.

410. Theorem.—*If a plane intersect two parallel planes, the lines of intersection are parallel.*

DEM.—Let AD intersect the parallel planes MN and PQ in AB and CD; then is AB parallel to CD. For, if AB and CD could meet, the planes MN and PQ would meet, as every point in AB is in MN, and every point in CD in PQ. Hence, AB and CD lie in the same plane, and do not meet how far soever they be produced; they are therefore parallel. Q. E. D.

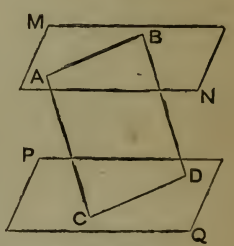

FIG. 264.

411. COR.—*Parallel lines intercepted between parallel planes are equal.*

Thus AC = BD if they are parallel. For, the intersections AB and CD, of the plane of these parallels, are parallel (?), and the figure ABDC is a parallelogram; whence AC = BD (?).

PROPOSITION XI.

412. Theorem.—*A line which is perpendicular to one of two parallel planes, is perpendicular to the other also.*

DEM.—Let AB be perpendicular to MN; then is it perpendicular to PQ, parallel to MN. For, pass two planes through AB, and let CA, DB, and EA, FB, be their intersections with the planes MN and PQ. Now CA and EA are perpendicular to AB (?); hence DB and FB being parallel to CA and EA (?) are perpendicular to AB (?). Therefore AB is perpendicular to two lines of the plane PQ, and consequently to the plane (?). Q. E. D.

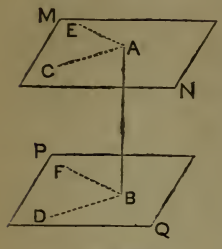

FIG. 265.

413. COR. 1.—*Through any point out of a plane, one plane can be passed parallel to the given plane, and only one.*

DEM.—To pass a plane through B parallel to MN, draw the perpendicular BA from B upon MN. Draw any two lines in the plane MN, through A, as CA and EA. Through B draw DB and FB parallel to CA and EA; then is PQ, the plane of these lines, perpendicular to AB, and hence parallel to MN. As the plane parallel to MN must contain FB and DB, and as but one plane can be passed through these lines, there can be only one plane through B parallel to MN.

414. The Distance *between two parallel planes at any point is measured by the perpendicular.*

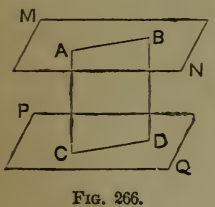

FIG. 266.

415. COR. 2.—*Parallel planes are everywhere equally distant from each other.*

DEM.—Let A and B be any two points in the plane MN, and AC and BD the perpendiculars from these points, let fall on the parallel plane PQ; then are they perpendicular to MN by the proposition, and since the figure ABCD is a parallelogram (?) [a rectangle, also (?)], AC = BD.

PROPOSITION XII.

416. Theorem.—*Two angles lying in different planes, but having their sides parallel and extending in the same direction, are equal, and their planes are parallel.*

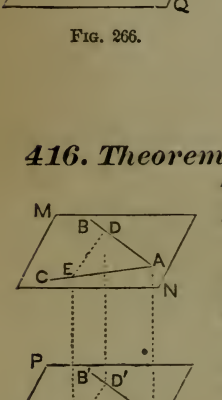

FIG. 267.

DEM.—Let A and A′ lie in the different planes MN and PQ, and have AB parallel to A′B′, and AC to A′C′; then A = A′, and MN and PQ are parallel.

For, take AD = A′D′, and AE = A′E′, and draw AA′, DD′, EE′, ED, and E′D′. Now, AD being equal and parallel to A′D′, AA′ = DD′ (?). For like reason AA′ = EE′, therefore EE′ = DD′. Again, since EE′ and DD′ are respectively parallel to AA′, they are parallel to each other (?), whence EDD′E′ is a parallelogram (?), and ED = E′D′. Hence the triangles ADE and A′D′E′ are

mutually equilateral, and A, opposite ED, is equal to A′, opposite E′D′ equal to ED.

Again, the plane of the angle BAC, MN, is parallel to PQ, the plane of B′A′C′. For, let a plane be passed through AC and revolved until it is parallel to PQ. It must cut DD′, which is parallel to AA′, and EE′, so that DD′ shall equal AA′ and EE′ (?); hence it must pass through D.

417. Cor. 1.—*If two intersecting planes be cut by parallel planes, the angles formed by the intersections are equal.*

Thus, AB′ and AE′ being cut by the parallel planes MN and PQ, AD is parallel to A′D′ (?), and lies in the same direction, and AE to A′E′. Hence BAC= B′A′C′ (?).

418. Cor. 2.—*If the corresponding extremities of three equal parallel lines not in the same plane, be joined, the triangles formed are equal, and their planes parallel.*

Thus, if AA′ = DD′ = EE′, the sides of the triangle AED are equal to the sides of A′E′D′, since the figures AD′, DE′, and EA′ are parallelograms (?), and the corollary comes under the proposition (?).

PROPOSITION XIII.

419. Theorem.—*The corresponding segments of lines cut by parallel planes are proportional.*

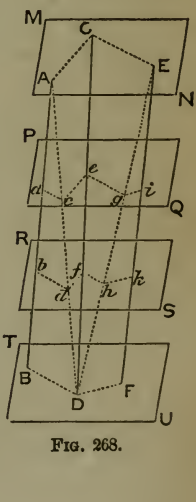

Fig. 268.

Dem.—Let AB, CD and EF be cut by the parallel planes MN, PQ, RS, and TU; then A*a* : C*e* :: *ab* : *ef* :: *b*B : *f*D, and A*a* : E*i* :: *ab* : *ik* :: *b*B : *k*F, and C*e* : E*i* :: *ef* : *ik* :: *f*D : *k*F.

For, join the extremities A and D, and E and D, and conceive the intersections of the plane of AB and AD with the parallel planes to be BD, *bd*, and *ac*. These lines are parallel (?), and A*a* : A*c* :: *ab* : *cd* :: *b*B : *d*D (?). For a similar reason, C*e* : A*c* :: *ef* : *cd* :: *f*D : *d*D (?). Whence, the consequents of the proportions being the same, the antecedents give A*a* : C*e* :: *ab* : *ef* :: *b*B : *f*D. In like manner we can show that C*e* : E*i* :: *ef* : *ik* :: *f*D : *k*F. [Let the student give the details.] From these proportions we have A*a* : E*i* :: *ab* : *ik* :: *b*B : *k*F (?). Q. E. D.

EXERCISES.

1. Designate any three points in the room, as one corner of the desk, a point on the stove, and some point in the ceiling, and show how you can conceive the plane of these points.

2. Show the position of two lines which will not meet, and yet are not parallel.

3. Conceive two lines, one line in the ceiling and one in the floor, which shall not be parallel to each other. What is the shortest distance between these lines?

4. The ceiling of my room is 10 feet above the floor. I have a 12 foot pole, by the aid of which I wish to determine a point in the floor directly under a certain point in the ceiling. How can I do it?

SUG.—Consult PROP. IV.

5. Upon what principle in this section is it that a stool with three legs always stands firm on a level floor, when one with four may not?

6. By the use of two carpenter's squares you can determine a perpendicular to a plane. How is it done?

7. If you wish to test the perpendicularity of a stud to a level floor, on how many sides of it is it necessary to measure the angle which it makes with the floor? By applying the right angle of the carpenter's square on *any* two sides of the stud, to test the angle which it makes with the floor, can you determine whether it is perpendicular or not?

8. We see in straight lines. If a line* be placed between our eye and a surface, it covers a certain space on the surface; this figure or space is said to be the *projection* of the line on that surface. Upon what principles in this section is it that the projections of straight lines are straight? Why is it that the projections of parallels which are parallel to the plane upon which we see them projected, are parallel, while parallel lines which are inclined to this plane are projected in oblique lines?

9. If a line is drawn at an inclination of 23° to a plane, what is the greatest angle which any line of the plane, drawn through the point where the inclined line pierces the plane, makes with the line? Can you conceive a line of the plane which makes an angle of 50° with the inclined line? Of 80°? Of 15°? Of 170°?

Hereafter, the student should make the synopses.

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* The term line is here used in its colloquial sense, and refers to a material representation, as a cord, the edge of a board, etc.

SECTION II.

OF SOLID ANGLES.

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420. A Solid Angle is the opening between two or more planes, each of which intersects all the others. The lines of intersection are called *Edges*, and the planes, or the portion of the planes between the edges, where there are more than two, are called *Faces*.

421. A Diedral Angle, or simply a *Diedral*, is the opening between *two* intersecting planes.

422. A Polyedral Angle, called also simply a *Polyedral*, is the opening between *three* or more planes which intersect so as to have one common point, and only one. In the case of three intersecting planes the angle is called a *Triedral*. The point common to all the planes is called the *Vertex*. The plane angles enclosing a polyedral are the *Facial* angles.

423. *A Diedral (Angle) is measured* by the plane angle included by lines drawn in its faces from any point in the edge, and perpendicular thereto. A diedral angle is called right, acute, or obtuse, according as its measure is right, acute, or obtuse. Of course the magnitude of a solid angle is independent of the distances to which the edges may chance to be produced.

ILL'S.—The opening between the two planes CABF and DABE is a *Diedral* (angle), AB is the *Edge*, and CABF and DABE are the *Faces*. Let MO lie in the plane AF, perpendicular to the edge; and NO in AE, and also perpendicular to the edge; then the plane angle MON is the measure of the diedral. A diedral may be read by the letters on the edge, when there

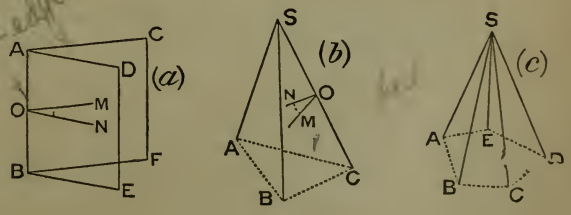


FIG. 269.

would be no ambiguity, or otherwise by these letters and one in each face; thus, the diedral in (*a*) may be designated as AB, or as C-AB-D.

In (*b*) we have a *Triedral* (angle). The *edges* are SA, SB, and SC; the *faces* ASB, BSC, and ASC: the *facial angles* are ASB, ASC, and BSC; and S is the *vertex*. Such an angle, and any polyedral (angle), may be read by naming the angle at the vertex, when there would be no ambiguity, or otherwise by naming the letter at the vertex, and then one in each edge; thus S-ABCDE designates the polyedral (*c*). The opening between the planes is the angle, in each case.

OF DIEDRALS.

424. A Diedral may be considered as generated by the revolution of a plane about a line of the plane, and hence we may see the propriety of measuring it by the angle included by two lines in its faces perpendicular to its edge, as stated in the preceding article.

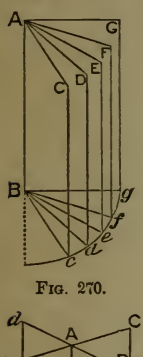


FIG. 270.

ILL.—Let AB be a line of the plane CB. Conceive *g*B perpendicular to AB. Now, let the plane revolve upon AB as an axis, whence *g*B describes a circle (?); and at any position of the revolving plane, as *f*BAF, since *f*B*g* measures the amount of revolution, it may be taken as the measure of the diedral *f*-BA-*g*. When *g*B has made $\frac{1}{4}$ of a revolution, the plane will have made $\frac{1}{4}$ of a revolution, and the diedral will be right.

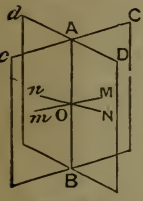


FIG. 271.

425. COR.—*Opposite diedrals are equal.*

Thus, if C-AB-D is measured by MON, *c*-AB-*d* is measured by the equal angle *n*O*m*.

PROPOSITION I.

426. Theorem.—*Any line in one face of a right diedral, perpendicular to its edge, is perpendicular to the other face.*

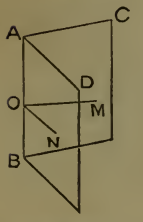


FIG. 272.

DEM.—In the face CB of the right diedral C-AB-D, let MO be perpendicular to the edge AB; then is it perpendicular to the face DB. For, draw ON in the face DB, and perpendicular to AB. Now, since the diedral is right, and MON measures its angle, MON is a right angle; whence MO is perpendicular to two lines of the plane DB, and consequently perpendicular to the plane. Q. E. D.

427. COR.—Conversely, *If one plane contain a line which is perpendicular to another plane, the diedral is right.*

Thus, if MO is perpendicular to the plane DB, C-AB-D is a right diedral. For MO is perpendicular to every line of DB passing through its foot (?); and hence is perpendicular to ON, drawn at right angles to AB. Whence C-AB-D is a right diedral, for it is measured by a right plane angle.

PROPOSITION II.

428. Theorem.—*If two planes are perpendicular to a third, their intersection is perpendicular to the third plane.*

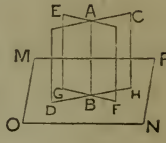


FIG. 273.

DEM.—If CD and EF are perpendicular to the plane MN, then is AB perpendicular to MN. For, EF being perpendicular to MN, D-FG-E is a right diedral, and a line in EF and perpendicular to FG at B is perpendicular to MN; also a line in the plane CD, and perpendicular to DH at B, is perpendicular to MN (?). Hence, as there can be one and only one perpendicular to MN at B, and as this perpendicular is in both planes, CD and EF, it is their intersection. Q. E. D.

PROPOSITION III.

429. Theorem.—*If from any point perpendiculars be drawn to the faces of a diedral angle, their included angle will be the supplement of the angle which measures the diedral, or equal to it.*

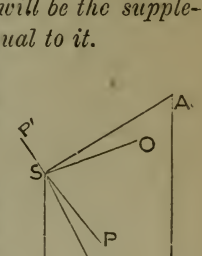


FIG. 274.

DEM.—Let BD and AD be any two planes including the diedral A-SD-B, then will two lines drawn from any point, perpendicular to these planes, include an angle which is the supplement of the measure of the diedral, or equal to it.

If the point from which the lines are drawn is not in the edge SD, we may conceive two lines drawn through any point, as S, in this edge, which shall be parallel to the two proposed, and hence include an equal angle, and have their plane parallel to the plane of the proposed angle (***416***). Let the latter lines be SO and SP. We are to show that OSP is supplemental to the measure of A-SD-B. A plane passed through S, perpendicular to the edge SD, will contain the lines SO and SP (***388***); and its intersections with the faces, as SB and SA, will form an angle (ASB) which is the measure of the diedral (***423***). Now, PSA = *a right angle* (?), and OSB = *a right angle* (?). Hence, PSO and ASB are either equal or supplemental (***283***). Q. E. D.

OF TRIEDRALS.

430. Triedrals are *Rectangular*, *Birectangular*, or *Trirectangular*, according as they have one, two, or three, right diedral angles.

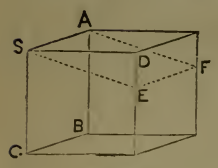


FIG. 275.

ILL'S.—The corner of a cube is a *Trirectangular* triedral, as S-ADC. Conceive the upper portion of the cube removed by the plane ASEF; then the angle at S, *i. e.*, S-AEC is a *Birectangular* triedral, A-SC-E and A-SE-C being right diedrals.

431. ***An Isosceles Triedral*** is one that has two of its facial angles equal. ***An Equilateral Triedral*** is one that has all three of its facial angles equal.

432. ***Two Symmetrical Triedrals*** are such as have the facial angles of the one equal to the facial angles of the other, each to each; but in which the equal facial angles are not similarly situated, and hence the triedrals are not necessarily capable of superposition.

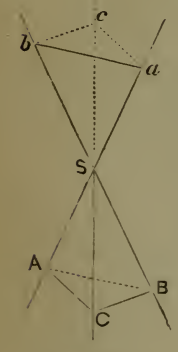


FIG. 276.

ILL'S.—Let the edges of the triedral S-ABC, be produced beyond the vertex, forming a second triedral S-*abc*; then are the two triedrals symmetrical, *i. e.*, the faces are equal plane angles, but disposed in a different order. Thus, ASB = *a*S*b*, ASC = *a*S*c*, and BSC = *b*S*c*; but the triedrals cannot be made to coincide. To show this fact, conceive the upper triedral detached, and the face *a*S*c* placed in its equal face ASC, S*a* in SA, and S*c* in SC. Now, the edge S*b*, instead of falling in SB will fall on the left of the plane ASC.

Symmetrical solids are of frequent occurrence: the two hands form an illustration; for, though the parts may be exactly alike, the hands cannot be placed so that their like parts will be similarly situated; in short, the left glove will not fit the right hand.

433. Two triedrals are ***Supplementary*** when the edges of one are perpendicular to the faces of the other. (See ***438a.***)

PROPOSITION IV.

434. ***Theorem.***—*The sum of any two facial angles of a triedral is greater than the third.*

DEM.—This proposition needs demonstration only in case of the sum of the two smaller facial angles as compared with the greatest (?). Let ASB and BSC each be less than ASC; then is ASB + BSC > ASC. For, make the angle ASb' = ASB, and Sb' = Sb, and pass a plane through b and b', cutting SA and SC in a and c. The two triangles aSb and aSb' are equal (?), whence $ab' = ab$. Now, $ab + bc > ac$ (?), and subtracting ab from the first member, and its equal ab' from the second, we have $bc > b'c$. Whence the two triangles bSc and b'Sc have two sides equal, but the third side $bc >$ than the third side $b'c$, and consequently angle bSc > b'Sc. Adding ASB to the former, and its equal ASb' to the latter, we have ASB + BSC > ASC. Q. E. D.

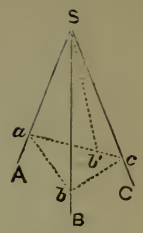


FIG. 277.

435. COR.—*The difference between any two facial angles of a triedral is less than the third facial angle* (?).

PROPOSITION V.

436. Theorem.—*The sum of the facial angles of a triedral may be anything between* 0 *and four right angles.*

DEM.—Let ASB, BSC, and ASC be the facial angles enclosing a triedral; then, as each must have some value, the sum is greater than 0, and we have only to show that ASB + ASC + BSC < 4 *right angles.* Produce either edge, as AS, to D. Now, in the triedral S-BCD, BSC < BSD + CSD. To each member of this inequality add ASB + ASC, and we have

ASB + ASC + BSC < ASB + ASC + BSD + CSD.

But, ASB + BSD = 2 right angles (?), and ASC + CSD = 2 *right angles;* whence ASB + ASC + BSD + CSD = 4 *right angles;* and consequently, ASB + ASC + BSC < 4 *right angles.* Q. E. D.

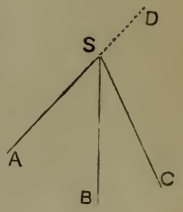


FIG. 278.

PROPOSITION VI.

437. Theorem.—*Two triedrals having the facial angles of the one equal to the facial angles of the other, each to each, and similarly arranged, are equal.*

DEM.—In the triedrals S and *s*, let ASB = *asb*, BSC = *bsc*, and CSA = *csa*; and let these facial angles occur in the same order, then is S-ABC = *s-abc*.

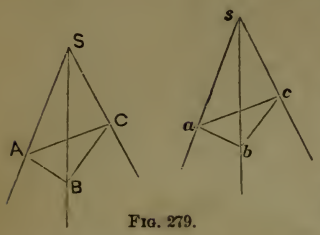


FIG. 279.

Take C, any point in SC, and make *sc*=SC, and draw AC and BC perpendicular to SC, and *ac* and *bc* perpendicular to *sc*; then ACB measures the diedral A-SC-B, and *acb* the diedral *a-sc-b*. Now the triangles SCB and *scb* are mutually equiangular (?) and have SC = *sc*; hence SB = *sb*, and CB = *cb*. For a like reason AC = *ac*, and SA = *sa*. Hence AB = *ab* (?), and the triangles ACB and *acb* are equal (?). Now, since angle ACB, measuring the diedral A-SC-B, equals angle *acb*, measuring the diedral *a-sc-b*, these diedrals are equal, and can be applied to each other. Applying these diedrals, since angle ASC = *asc*, and BSC = *bsc*, the edges AS and *as* coincide, as also do BS and *bs*, whence the triedrals coincide throughout, and are consequently equal. Q. E. D.

PROPOSITION VII.

438. Theorem.—*Of two supplementary triedrals, the facial angles of the one are the supplements of the diedrals of the other.*

DEM.—Let S-ABC be any triedral; if a second triedral be formed with its edges perpendicular to the faces of S-ABC, one to each face respectively, then are the facial angles of the one, supplements of the diedrals of the other.

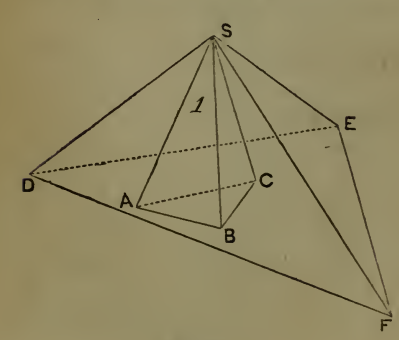


FIG. 280.

If the vertex of the second triedral is not at the vertex of the first, we may conceive a triedral formed by drawing three lines through the vertex S, as SE, SD, and SF, parallel to the edges of the given triedral; then will these edges be perpendicular to the same planes as the edges to which they are parallel (***405***), and hence the angle thus formed (S-EDF) will be a triedral supplemental to S-ABC (***433***), and the facial angles of the two having their edges parallel will be equal (***416***), and consequently the triedrals equal (***437***). Now, SE being perpendicular to ASB, and SF to ASC, angle ESF is supplemental to the diedral B-SA-C (***429***.) In like manner, SD being perpendicular to BSC, DSE is supplemental to A-SB-C, and DSF to A-SC-B.

Thus we have shown that the facial angles of S-EDF are the supplements

of the diedrals of S-ABC. We are now to show that the facial angles of S-ABC are supplements of the diedrals of S-EDF; *i. e.*, that ASB is the supplement of D-SE-F, BSC of E-SD-F, and ASC of D-SF-E. Since SE is by hypothesis perpendicular to ASB, it is perpendicular to AS (*387*); and since SF is perpendicular to ASC, it is perpendicular to AS (*387*). Hence AS is perpendicular to the face FSE (?). In like manner we may show that SB is perpendicular to DSE, and SC to DSF; whence it follows from the preceding part of the demonstration, or directly from (*429*), that angle ASB is the supplement of D-SE-F, BSC of E-SD-F, and ASC of D-SF-E.

438a. Sch.—If any edge of S-EDF, as DS, is produced beyond S, another triedral is formed which has its edges perpendicular to the faces of S-ABC. Thus in all 4 triedrals can be formed with their edges perpendicular to the faces of S-ABC; but the proposition holds only for S-EDF.

PROPOSITION VIII.

439. Theorem.—*The sum of the diedrals of a triedral may be anything between two and six right angles.*

Dem.—Each diedral being the supplement of a plane angle of the supplementary triedral, the sum of the three diedrals is 3 times 2 right angles, or 6 *right angles — the sum of the angles of the supplementary triedral.* But this latter sum may be anything between 0 and 4 right angles (?). Hence the sum of the diedrals may be anything between 2 and 6 right angles. Q. E. D.

PROPOSITION IX.

440. Theorem.—*An isosceles triedral and its symmetrical triedral are equal.*

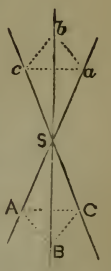


Fig. 281.

Dem.—Let S-ABC be an isosceles triedral with the facial angle ASB = BSC; then is it equal to its symmetrical triedral S-*abc*.

For, revolve S-*abc* about S until S*b* falls in SB, and bring the plane S*ba* into the plane SBC; then, since the diedrals C-SB-A and *a*-S*b*-*c* are opposite, they are equal (*425*),* and the plane S*bc* will fall in SBA. Moreover, S*a* will fall in SC, since angle BSC = ASB (by hypothesis) = *b*S*a* (vertical to ASB). In like manner S*c* will fall in SA, and the triedrals will coincide, and are therefore equal. Q. E. D.

441. Sch.—If angle ASB is not equal to BSC, it is easy to see that the ap

* Should the pupil have difficulty in perceiving this, let him notice that CSB and *c*S*b* are parts of one and the same plane; and ASB and *a*S*b* are parts of another. Now *b*B is the intersection of these planes, and the diedrals mentioned are on opposite sides of this line of intersection.

plication will fail, notwithstanding the diedrals are equal, and the triedrals symmetrical.

442. Cor. 1.—*The diedrals opposite the equal facial angles of an isosceles triedral are equal.*

The diedral b-Sa-c = B-SA-C being opposite, and b-Sa-c = B-SC-A as shown in the demonstration; hence B-SA-C = B-SC-A.

443. Cor. 2.—Conversely, *If two diedrals of a triedral are equal, the triedral is isosceles.*

Dem.—If B-SA-C = B-SC-A, S-ABC is isosceles. For, revolving S-abc as before till the facial angle aSc falls in its equal (?) ASC, since the diedral B-SC-A = B-SA-C (by hypothesis) and the latter equals its opposite b-Sa-c, the plane bSa will fall in the plane BSC; and, for like reasons, the plane bSc will fall in BSA. Now, as these planes coincide, their intersections Sb and SB coincide, and the triedrals are equal; and the facial angle BSC = bSa. But bSa = ASB (?); hence ASB = BSC; *i. e.*, the triedral S-ABC is isosceles.

PROPOSITION X.

444. ***Theorem.***—*Two symmetrical triedrals are equivalent.*

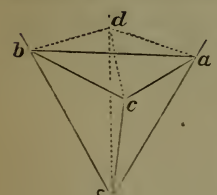


Fig. 282.

Dem.—S-ABC is equivalent to its symmetrical triedral S-abc.

For, let dD be so drawn that the angles DSA, DSC, and DSB shall be equal, and consequently dSa = dSc = dSb, and the latter respectively equal to the former. Then the isosceles symmetrical triedral S-DCB = S-dcb, S-DCA = S-dca, and S-ADB = S-adb. Whence the polyedral S-ABCD = S-$abcd$. Now, from the former subtracting S-ADB, and from the latter S-adb, there remains S-ABC = S-abc. Q. E. D.

445. Sch.—If dD fell within the given triedrals, they would be made up of the three equal isosceles triedrals, and hence equivalent.

PROPOSITION XI.

446. ***Theorem.***—*Two triedrals which have two facial angles and the included diedral equal, each to each, are equal, or symmetrical and equivalent.*

Dem.—If the equal faces are on the same sides of the diedral in the two triedrals, the one figure can be applied to the other; and if they are on different

sides, the edges of one triedral may be produced, forming the symmetrical triedral, to which the other given triedral may be applied. [Let the student construct figures, and go through with the application.]

PROPOSITION XII.

447. Theorem.—*Two triedrals which have two diedrals and the included facial angle equal, are equal, or symmetrical and equivalent.*

DEM.—[Same as in the preceding. Let the student draw figures like those for the preceding, and go through with the details of the application.]

448. COR.—*It will be observed that in equal or in symmetrical triedrals, the equal facial angles are opposite the equal diedrals.*

PROPOSITION XIII.

449. Theorem.—*Two triedrals which have two facial angles of the one equal to two facial angles of the other, each to each, and the included diedrals unequal, have the third facial angles unequal, and the greater facial angle belongs to the triedral having the greater included diedral.*

DEM.—Let ASC = *asc*, and ASB = *asb*, while the diedral C-SA-B > *c-sa-b*; then CSB > *csb*.

For, make the diedral C-SA-*o* = *c-sa-b*: and taking AS*o* = *asb*, bisect the diedral *o*-SA-B with the plane ISA. Draw *o*I and *o*C, and conceive the planes *o*SI and *o*SC. Now, the triedral S-A*o*C = *s-abc*, since they have two facial angles and the included diedral equal (***446***). For a like reason S-AI*o* = S-AIB, and the facial angle *o*SI = ISB. Again, in the triedral S-I*o*C, *o*SI + ISC > *o*SC (***434***), and substituting ISB for *o*SI, we have ISB + ISC (or BSC) > *o*SC, or its equal *bsc*. Q. E. D.

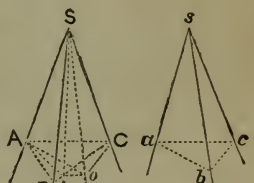


FIG. 283.

450. COR.—Conversely, *If the two facial angles are equal, each to each, in the two triedrals, and the third facial angles unequal, the diedral opposite the greater facial angle is the greater.*

That is, if ASB = *asb*, and ASC = *asc*, while BSC > *bsc*, the diedral B-AS-C > *b-as-c*. For, if B-AS-C = *b-as-c*, BSC = *bsc* (***446***), and if B-AS-C < *b-as-c*, BSC < *bsc*, by the proposition. Therefore, as B-AS-C cannot be equal to nor less than *b-as-c*, it must be greater.

PROPOSITION XIV.

451. Theorem.—*Two triedrals which have the three facial angles of the one equal to the three facial angles of the other, each to each, are equal, or symmetrical and equivalent.*

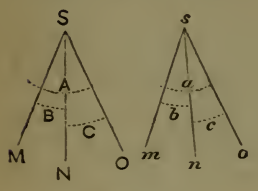


FIG. 284.

DEM.—Let A, B, and C represent the facial angles of one, and a, b, and c the corresponding facial angles of the other. If A = a, B = b, and C = c, the triedrals are equal. For A being equal to a, and B to b, if, of their included diedrals, SM were greater than sm, C would be greater than c; and if diedral SM were less than diedral sm, C would be less than c, by the last corollary. Hence, as diedral SM can neither be greater nor less than diedral sm, it must be equal to it. For like reasons, diedral SN = diedral sn, and diedral SO = diedral so. Therefore, the triedrals are equal, or symmetrical, according to the arrangement of the faces. Thus, if SN and sn are both considered as lying on the same side of the planes MSO and mso, the triedrals are equal; but, if one lies on one side and the other on the opposite side of those planes (SN in front, and sn behind, for example), the diedrals are symmetrical, and hence equivalent.

PROPOSITION XV.

452. Theorem.—*Two triedrals which have the three diedrals of the one equal to the three diedrals of the other, each to each, are equal, or symmetrical and equivalent.*

DEM.—In the two triedrals supplementary to the given triedrals, the facial angles of the one will be equal to the facial angles of the other, each to each, since they are supplements of equal diedrals (***438***). Hence, the supplementary triedrals are equal or equivalent, by the last proposition. Now, the facial angles of the first triedrals are supplements of the diedrals of the supplementary; whence the corresponding facial angles, being the supplements of equal diedrals, are equal. Therefore, the proposed triedrals have their facial angles equal, each to each, and are consequently equal, or symmetrical and equivalent. Q. E. D.

453. COR.—*All trirectangular triedrals are equal.*

454. SCH.—The proof that two forms are equal, includes the fact that corresponding parts are equal.

OF POLYEDRALS.

455. A Convex Polyedral is a polyedral in which none of the faces, when produced, can enter the solid angle. A section of such a polyedral made by a plane cutting all its edges is a convex polygon. [See *Fig.* 285.]

PROPOSITION XVI.

456. Theorem.—*The sum of the facial angles of any convex polyedral is less than four right angles.*

DEM.—Let S be the vertex of any convex polyedral. Let the edges of this polyedral be cut by any plane, as ABCDE, which section will be a convex polygon, since the polyedral is convex. From any point within this polygon, as O, draw lines to its vertices, as OA, OB, OC, etc. There will thus be formed two sets of triangles, one with their vertices at S, and the other with their vertices at O; and there will be an equal number in each set, for the sides of the polygon form the bases of both sets. Now, the sum of the angles of one set of these triangles is equal to the sum of the angles of the other set. But the sum of the angles at the bases of the triangles having their vertices at S is greater than the sum of the angles at the bases of the triangles having their vertices at O, since SBA + SBC > ABC, SCB + SCD > BCD, etc. (***434***). Therefore the sum of the angles at S is less than the sum of the angles at O, *i. e.*, less than 4 right angles. Q. E. D.

FIG. 285.

EXERCISES.

1. I have an iron block whose corners are all square (edges right diedrals, and the vertices trirectangular, or right, triedrals). If I bend a wire square around one of its edges, as *c*S′*d*, at what angle do I bend the wire? If I bend a wire obliquely around the edge, as *a*S*b*, at what angle can I bend it? If I bend it obliquely, as *e*S″*f*, at what angle can I bend it?

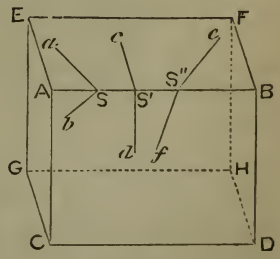


FIG. 286.

2. *Fig.* 286 represents the appearance of a rectangular parallelopiped, as seen from a certain position. Now, all the angles of such a solid are right angles: why is it that they nearly all *appear* oblique? Can you see a right parallelopiped from such a position that all the angles seen shall *appear* as right angles?

3. The diedral angles of crystals are measured with great care, in order to determine the substance of which the crystals consist. How must the measure be taken? If we measure obliquely around the edge, shall we get the true value of the angle?

4. Cut out any triedral from a block of wood (or a potato), and stick three pins into it, as near the vertex as you can, one in each face, and perpendicular to that face. What figure do the three pins form? What relation does the angle included between any two adjacent pins sustain to one of the diedrals of the block? Which ones are they that sustain this relation?

5. Can three planes intersect each other and yet not form a triedral angle? In how many ways? Can they all three have a common point, and yet not form a triedral?

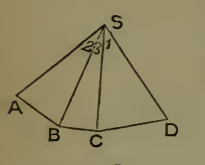


FIG. 287.

6. From a piece of pasteboard cut two figures of the same size, like ABCDS and *abcds*. Then drawing SB and SC so as to make 1 the largest angle and 3 the smallest, cut the pasteboard almost through in these lines, so that it will readily bend in them. Now fold the edges AS and DS together, and a triedral will be formed. From the piece *bcads* form a triedral in like manner, only let the lines *sc* and *sa* be drawn so as to make the angles 1, 2, and 3 of the same size as before, while they occur in the order given in *bcads*. Now, see if you can slip one triedral into the other, so that they will fit. What is the difficulty?

7. In the last case, if 1 equals $\frac{4}{6}$ of a right angle, $2 = \frac{1}{3}$ of a right angle, and $3 = \frac{2}{5}$ of a right angle, can you form the triedral? Why? If you keep increasing the size of 1, 2, and 3, until the sum becomes equal to 4 right angles, will it always be possible to form a triedral? How is it when the sum equals 4 right angles?

SECTION III.

OF PRISMS AND CYLINDERS.

457. A Prism is a solid, two of whose faces are equal, parallel polygons, while the other faces are parallelograms. The equal parallel polygons are the *Bases*, and the parallelograms make up the *Lateral* or *Convex Surface*. Prisms are triangular, quadrangular, pentagonal, etc., according to the number of sides of the polygon forming a base.

458. A Right Prism is a prism whose lateral edges are perpendicular to its bases. ***An Oblique*** prism is a prism whose lateral edges are oblique to its bases.

459. A Regular Prism is a right prism whose bases are regular polygons; whence its faces are equal rectangles.

460. The Altitude of a prism is the perpendicular distance between its bases: the altitude of a right prism is equal to any one of its lateral edges.

461. A Truncated Prism is a portion of a prism cut off by a plane not parallel to its base. A section of a prism made by a plane perpendicular to its lateral edges is called a *Right Section*.

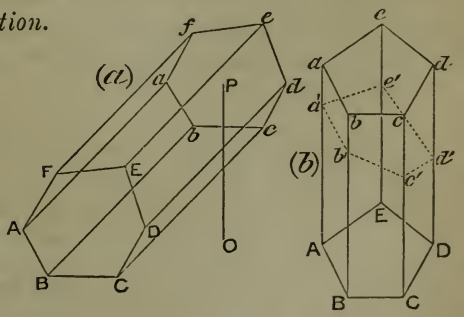


Fig. 288.

Ill's.—In the figure, (*a*) and (*b*) are both prisms: (*a*) is oblique and (*b*) right. PO represents the altitude of (*a*); and any edge of (*b*), as *b*B, is its altitude. ABCDEF, and *abcdef*, are lower and upper bases, respectively. Either portion of (*b*) cut off by an oblique plane, as *a'b'c'd'e'*, is a truncated prism.

462. A Parallelopiped is a prism whose bases are parallelograms: its faces, inclusive of the bases, are consequently all parallel-

ograms. If its faces are all rectangular, it is a *rectangular* parallelopiped.

463. A Cube is a rectangular parallelopiped whose faces are all equal squares.

PROPOSITION I.

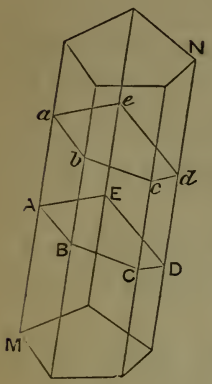


Fig. 289.

464. Theorem.—*Parallel plane sections of any prism are equal polygons.*

Dem.—Let ABCDE and *abcde* be parallel sections of the prism MN; then are they equal polygons.

For, the intersections with the lateral faces, as *ab* and AB, etc., are parallel, since they are intersections of parallel planes by a third plane (***410***). Moreover, these intersections are equal, that is, $ab = AB$, $bc = BC$, $cd = CD$, etc., since they are parallels included between parallels (***242***). Again, the corresponding angles of these polygons are equal, that is, $a = A$, $b = B$, $c = C$, etc., since their sides are parallel and lie in the same direction (***416***). Therefore the polygons ABCDE, and *abcde*, are mutually equilateral and equiangular; that is, they are equal. Q. E. D.

465. Cor.—*Any plane section of a prism, parallel to its base, is equal to the base; and all right sections are equal.*

PROPOSITION II.

466. Theorem.—*If three faces including any triedral of one prism are equal respectively to three faces including a triedral of the other, and similarly placed, the prisms are equal.*

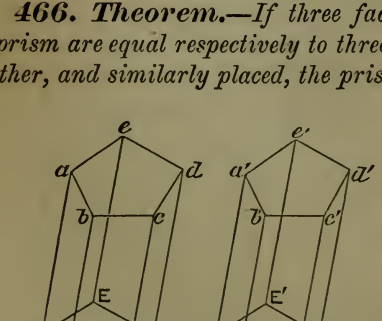


Fig. 290.

Dem.—In the prisms A*d*, and A′*d*′, let ABCDE equal A′B′C′D′E′, AB*ba* = A′B′*b*′*a*′, and BC*cb* = B′C′*c*′*b*′; then are the prisms equal.

For, since the facial angles of the triedrals B and B′ are equal, the triedrals are equal (***451***), and being applied they will coincide. Now, conceiving A′*d*′ as applied to A*d*, with B′ in B, since the bases are equal polygons, they will coincide throughout; and the faces *a*B and *a*′B′ will also coincide. Whence, as *a*′*b*′ falls in *ab*,

and $b'c'$ in bc, the upper bases, which are equal because equal to the equal lower bases, will coincide. Therefore the remaining edges will have two points common in each, and will consequently coincide.

467. COR. 1.—*Two right prisms having equal bases and equal altitudes are equal.*

If the faces are not similarly arranged, one prism can be inverted.

468. COR. 2.—*The above proposition applies also to truncated prisms.*

PROPOSITION III.

469. Theorem.—*Any oblique prism is equivalent to a right prism, whose bases are right sections of the oblique prism, and whose edge is equal to the edge of the oblique prism.*

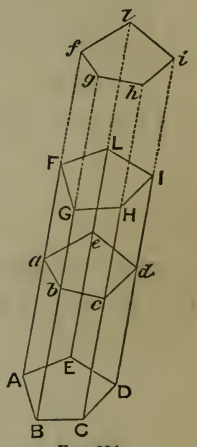


FIG. 291.

DEM.—Let LB be an oblique prism, of which *abcde* and *fghil* are right sections, and gb = GB; then is lb equivalent to LB. For the truncated prisms lC and eB have the faces including any triedral, as C and B, equal and similarly placed (?), whence these prisms are equal (***466***). Now, from the whole figure, take away prism lC, and there remains the oblique prism LB; also, from the whole take away the prism eB, and there remains the right prism lb. Therefore, the right prism lb is equivalent to the oblique prism LB. Q. E. D.

PROPOSITION IV.

470. Theorem.—*The opposite faces of a parallelopiped are equal and parallel.*

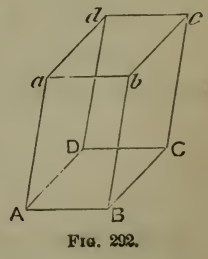


FIG. 292.

DEM.—Let Ac be a parallelopiped, AC and ac being its equal bases (***462***); then are its opposite faces equal and parallel.

Since the bases are parallelograms, AB is equal and parallel to DC; and, since the faces are parallelograms, aA is equal and parallel to dD. Hence angle aAB = dDC, and their planes are parallel, since their sides are parallel and extend in the same directions. Therefore aB and dC are equal (***301***) and parallel parallelograms. In like manner it may be shown that aD is equal and parallel to bC.

PROPOSITION V.

471. Theorem.—*The diagonals of a parallelopiped bisect each other.*

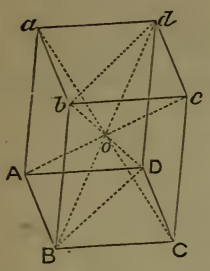


Fig. 293.

Dem.—Pass a plane through two opposite edges, as bB and dD. Since the bases are parallel, bd and BD will be parallel (***410***), and bBDd will be a parallelogram. Hence, bD and dB are bisected at o (?). For a like reason, passing a plane through dc and AB, we may show that dB and cA bisect each other, and hence that cA passes through the common centre of dB and bD. So also aC is bisected by bD, as appears from passing a plane through ab and DC. Hence, all the diagonals are bisected at o. Q. E. D.

472. Cor.—*The diagonals of a rectangular parallelopiped are equal.*

PROPOSITION VI.

473. Theorem.—*A parallelopiped is divided into two equivalent triangular prisms by a plane passing through its diagonally opposite edges.*

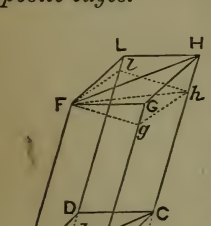


Fig. 294.

Dem.—Let H-ABCD be a parallelopiped, divided through its diagonally opposite edges FA and HC; then are the triangular prisms H-ABC, and L-ADC equivalent.

For this parallelopiped is equivalent to a right parallelopiped having a right section Abcd for its base, and AF for its edge (***469***), *i. e.*, H-ABCD is equivalent to h-Abcd. For the same reason the oblique triangular prism H-ABC is equivalent to the right triangular prism h-Abc; and L-ADC is equivalent to l-Adc. But h-Abc is equal to h-Adc, as they are right prisms with equal bases (***467***) and a common altitude. Hence, H-ABC is equivalent to L-ADC, as they are equivalent to two equal prisms. Q. E. D.

PROPOSITION VII.

474. Theorem.—*Any parallelopiped is equivalent to a rectangular parallelopiped having an equivalent base and the same altitude.*

DEM.—Let H-ABCD be any parallelopiped with all its faces oblique. 1st. By making the right section *ad*He, and completing the parallelopiped *ad*H*ebc*G*f*, we have an equivalent right parallelopiped (***469***). 2d. Through the edge *ef* of this right parallelopiped make the right section *ea'b'f* and complete the parallelopiped *ea'b'f*H*d'c'*G, and we have a rectangular parallelopiped equivalent to the one previously formed (***469***), and hence equivalent to the given one. Now, the base of this rectangular parallelopiped, *i. e.*, *a'b'c'd'*, is equal to *abcd* (?), which in turn is equivalent to ABCD (?). Moreover, the altitude of the rectangular parallelopiped is the same as that of the given one, since their bases lie in the same parallel planes A*c'* and EG. Therefore, the parallelopiped H-ABCD is equivalent to the rectangular parallelopiped H-*a'b'c'd'*, which has an equivalent base and the same altitude. Q. E. D.

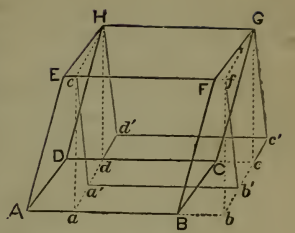


FIG. 295.

PROPOSITION VIII.

475. Theorem.—*The area of the lateral surface of a right prism is equal to the product of its altitude into the perimeter of its base.*

DEM.—The lateral faces are all rectangles, having for their common altitude the altitude of the prism (***460***). Whence the area of any face is the product of the altitude into the side of the base which forms its base; and the sum of the areas of the faces is the common altitude into the sum of the bases of the faces, that is, into the perimeter of the base of the prism. Q. E. D.

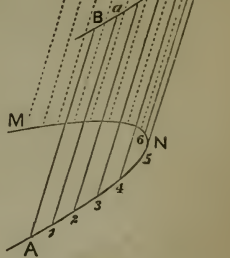


FIG. 296.

476. A Cylindrical Surface is a curved surface traced by a straight line moving so as to remain constantly parallel to its first position, while any point in it traces some curve. The moving line is called the *Generatrix*, and the curve traced by a point of the line is the *Directrix*.

ILL.—Suppose a line to start from the position AB, and move towards N in

such a manner as to remain all the time parallel to its first position AB, while A traces the curve A123456----M. The surface thus traced is a *Cylindrical Surface;* AB is the *Generatrix*, and the curve ANM the *Directrix.*

477. A Circular Cylinder, called also a *Cylinder of Revolution,* is a solid generated by the revolution of a rectangle around one of its sides as an axis.

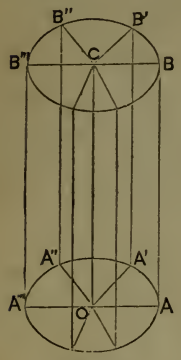


FIG. 297.

ILL.—Let COAB be a rectangle, and conceive it revolved about CO as an axis, taking successively the positions COA'B', COA''B'', etc.; the solid generated is a *Circular Cylinder*, or a cylinder of revolution. The revolving side AB is the generatrix of the surface, and the circumference OA (or CB) is the directrix. This is the only cylinder treated in Elementary Geometry, and is usually meant when the word *Cylinder* is used without specifying the kind of cylinder.

478. The Axis of the cylinder is the fixed side of the rectangle. The side of the rectangle opposite the axis generates the *Convex Surface;* while the other sides of the rectangle, as OA and CB, generate the *Bases,* which in the cylinder of revolution are circles. Any line of the surface corresponding to some position of the generatrix is called an *Element* of the surface.

479. A Right Cylinder is one whose elements are perpendicular to its base. In such a cylinder any element is equal to the axis. *A Cylinder of Revolution* (***477***) is *right.*

480. A prism is said to be inscribed in a cylinder, when the bases of the prism are inscribed in the bases of the cylinder, and the edges of the prism coincide with elements of the cylinder.

PROPOSITION IX.

481. Theorem.—*The area of the convex surface of a cylinder of revolution is equal to the product of its axis into the circumference of its base,* i. e., $2\pi RH$, H *being the axis and* R *the radius of the base.*

DEM.—Let a right prism, with any regular polygon for its base, be inscribed in the cylinder, as *k-abcdef*, in the cylinder whose axis is HO. The area of the lateral surface of the prism is HO (= *hb*) into the perimeter of its base, *i. e.*, HO × (*ab* + *bc* + *cd* + *de* + *ef* + *fa*). Now, bisect the arcs *ab*, *bc*, etc., and inscribe a regular polygon of twice the number of sides of the preceding, and on this polygon as a base construct the right inscribed prism with double the number of faces that the first had. The area of the lateral surface of this prism is HO × *the perimeter of its base*. In like manner conceive the operation of inscribing right prisms with regular polygonal bases continually repeated; it will *always* be true that the area of the lateral surface is equal to HO × *the perimeter of the base*. But the circumference of the base of the cylinder is the limit toward which the perimeters of the inscribed polygons forming the bases of the prisms constantly approach, and the convex surface of the cylinder is the limit of the lateral surface of the inscribed prism. Therefore, the area of the convex surface of the cylinder is HO into the circumference of the base. Finally, if R is the radius of the base, $2\pi R$ is its circumference. This multiplied by H the altitude, *i. e.*, H × 2πR, or 2πRH, is the area of the convex surface of the cylinder.

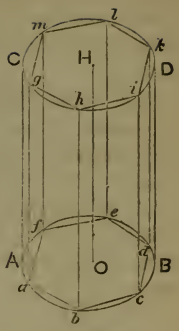


FIG. 298.

PROPOSITION X.

482. Theorem.—*The volume of a rectangular parallelopiped is equal to the product of the three edges of one of its triedrals.*

DEM.—Let H-CBFE be a rectangular parallelopiped. 1st. Suppose the edges commensurable, and let BC be 5 units in length, BA 4, and BF 7. Now conceive a cube, as *d-f b*B*g*, whose edge is one of these linear units. This cube may be used as the unit of volume. Conceive the parallelopiped O-*ca*B*b*, whose length is 7, and whose edges *ca* and *cb* are 1 (the linear unit of measure assumed). This parallelopiped will contain as many of the units of volume as there are linear units in BF: we suppose 7. Again, conceive the parallelopiped whose base is ECBF and altitude PE, one of the linear units. This parallelopiped will contain as many of the former as there are linear units in BC: we suppose 5. Hence this last volume is 5 × 7 = 35. Finally, there will

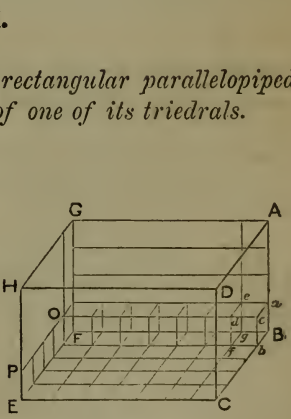


FIG. 299.

be as many times this number of units of volume in the whole parallelopiped as AB contains linear units, or $4 \times 35 = 140$. Hence, when the edges are commensurable, the volume is the product of the three edges including a triedral.

2nd. When the edges are not commensurable, we reach the same conclusion by taking successively a smaller and smaller linear unit. Thus, for a first approximation take some aliquot part of one edge, as $\frac{1}{10}$ of FB. Now, by hypothesis this is not contained an exact number of times in BC, nor in BA. But conceive it as applied to BC as many times as it can be; the remainder will be less than $\frac{1}{10}$ FB. In like manner conceive it applied to AB. The volume of the parallelopiped included by these edges will be measured by the product of the edges. Now conceive the linear unit smaller. The unmeasured portion will be less. Thus, by supposing the linear unit to diminish indefinitely, we see that *it will always remain true* that the measure is the product of the three edges forming a triedral.

483. COR. 1.—*The volume of a cube is the third power of its edge.*

484. SCH.—This fact gives rise to the term *cube*, as used in arithmetic and algebra, for "third power."

485. COR. 2.—*The volume of a rectangular parallelopiped is equal to the product of its altitude into the area of its base, the linear unit being the same for the measure of all the edges.*

486. COR. 3.—*The volume of any parallelopiped is equal to the product of its altitude and the area of its base.*

For any parallelopiped is equivalent to a rectangular parallelopiped having an equivalent base and the same altitude (***474***).

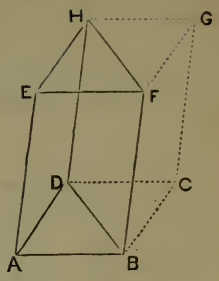
487. COR. 4.—*Parallelopipeds of the same or equivalent bases are to each other as their altitudes, and those of the same altitudes are to each other as their bases. And, in general, parallelopipeds are to each other as the products of their bases and altitudes.*

PROPOSITION XI.

488. ***Theorem.***—*The volume of any prism is equal to the product of its altitude into its base.*

DEM.—1st. Let E-ABD be a triangular prism. Complete the parallelopiped E-ABCD. Then is E-ABD = ½ E-ABCD (*473*). But the volume of E-ABCD is equal to its altitude into its base; hence the volume of E-ABD is equal to its altitude into ½ ABCD, or ABD.

2d. Any prism may be divided into partial, triangular prisms, by passing planes through one edge and all the other non-adjacent edges, as in the figure. Let H be the altitude of the whole prism, then is it also the common altitude of the partial prisms. Now, the volume of each triangular prism is H into its base; hence, the sum of the volumes is H into the sum of the bases, *i.e.*, H into the base of the whole prism.



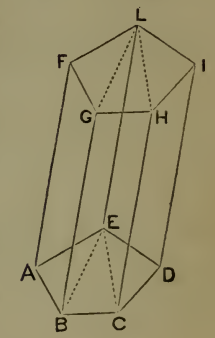


FIG. 300.

489. COR. 1.—*The volume of a right prism is equal to the product of its edge into its base.*

490. COR. 2.—*Prisms of the same altitude are to each other as their bases; and prisms of the same or equivalent bases are to each other as their altitudes; and, in general, prisms are to each other as the products of their bases and altitudes.*

PROPOSITION XII.

491. Theorem.—*The volume of a cylinder of revolution is equal to the product of its base and altitude,* i. e., $\pi R^2 H$, H *being the altitude and* R *the radius of the base.*

DEM.—Inscribe any regular right prism in the cylinder, as in (*481*). The volume of this prism is equal to the product of its base and altitude; and this continues to be the fact as the number of sides of the polygon forming the base is successively doubled, and the prism approaches equality with the cylinder. Hence, as the volume of the prism is *always* equal to the product of its base and altitude, and as the altitude of the prism remains equal to the altitude of the cylinder, this fact is true when the number of the sides of the base of the prism is *infinitely* multiplied; whence the volume of the cylinder is equal to the product of its base and altitude. Now, R being the radius of the base, the area of the base is πR^2 (?): hence, the volume of the cylinder is equal to $\pi R^2 H$.

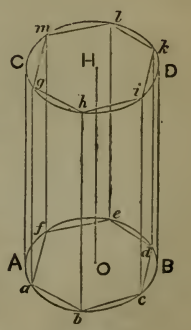


FIG. 301.

492. Cor.—*The volume of any cylinder is equal to the product of its base into its altitude.*

This can be demonstrated in a manner altogether analogous to the case given in the proposition.

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493. Similar Solids are such as have their corresponding solid angles equal and their homologous edges proportional.

494. Similar Cylinders of revolution are such as have their altitudes in the same ratio as the radii of their bases.

495. Homologous Edges of similar solids are such as are included between equal plane angles in corresponding faces.

Ill's.—The idea of similarity in the case of solids is the same as in the case of plane figures, viz., that of *likeness of form*. Thus, one would not think such a cylinder as *one* joint of stovepipe, similar to another composed of a hundred joints of the same pipe. One would be *long* and *very slim* in proportion to its length, while the other would not be thought of as *slim*. But, if we have two cylinders the radii of whose bases are 2 and 4, and whose lengths are respectively 6 and 12, we readily recognize them as of the same shape: they are similar.

PROPOSITION XIII.

496. Theorem.—*The lateral surfaces of similar right prisms are to each other as the squares of their edges (or altitudes) and as the squares of any two homologous sides of their bases,* i. e., *as the squares of any two homologous lines.*

Dem.—Let A, B, C, D, and E, be the sides of the base of one right prism whose edge (equal to its altitude) is H, and a, b, c, d, and e, the homologous sides of a similar prism whose edge is h. Letting $A + B + C + D + E = P$, and $a + b + c + d + e = p$, we have

$P : p :: A : a :: B : b :: C : c$, etc. (?).

But by hypothesis, $H : h :: A : a :: B : b$, etc.

Hence, $P : p :: H : h$ (?).

Now, $H : h :: H : h$ (?).

Whence, $P \times H : p \times h :: H^2 : h^2$ (?).

And as $H^2 : h^2 :: A^2 : a^2 :: B^2 : b^2$, etc.,

we have $P \times H : p \times h :: A^2 : a^2 :: B^2 : b^2$, etc.

But P × H is the area of the lateral surface of one prism and $p \times h$ of the other, whence the truth of the theorem appears.

PROPOSITION XIV.

497. Theorem.—*The volumes of similar prisms are to each other as the cubes of their homologous edges, and as the cubes of their altitudes.*

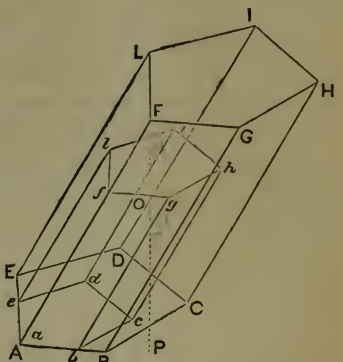

FIG. 302.

DEM.—Let H-ABCDE and *h-abcde* be two similar prisms, of which A and *a* are corresponding triedrals. Placing *a* so that it will coincide with A, all the faces and edges of one will be parallel to or coincident with the corresponding parts of the other, by definition (***493***). Let fall the perpendicular FP upon the common base, or its plane produced, so that FP shall equal the altitude of H-ABCDE, and OP, intercepted between the planes of the upper and lower bases of *h-abcde*, shall be its altitude. Call the former altitude H, and the latter *h*. Since FP and AF are cut by parallel planes, we have

$AF : af :: H : h$; and $AB : ab :: H : h$, since by definition $AF : af :: AB : ab$, etc. Call the base of H-ABCDE B, and of *h-abcde b*. Now, as the bases are similar polygons,

$$B : b :: \overline{AB}^2 : \overline{ab}^2 :: H^2 : h^2.$$

But $$H : h :: AB : ab :: H : h.$$

Hence, $$B \times H : b \times h :: \overline{AB}^3 : \overline{ab}^3 :: H^3 : h^3.$$

Now, as $B \times H$ and $b \times h$ are the volumes of the respective prisms, and as $\overline{AB}^3 : \overline{ab}^3$ as the cubes of any other homologous edges are to each other, the truth of the theorem is demonstrated.

PROPOSITION XV.

498. Theorem.—*The convex surfaces of similar cylinders of revolution are to each other as the squares of their altitudes, and as the squares of the radii of their bases.*

DEM.—Let H and *h* be the altitudes, and R and *r* the radii of the bases of two similar cylinders; the convex surfaces are $2\pi RH$ and $2\pi rh$ (***481***). Now,

$$2\pi RH : 2\pi rh :: RH : rh \text{ (?) (1).}$$

By hypothesis, $H : h :: R : r$, or $\frac{H}{R} = \frac{h}{r}$ and $\frac{R}{H} = \frac{r}{h}$.

Multiplying the terms of the second couplet of (1) by these equals, we have,

$$2\pi RH : 2\pi rh :: H^2 : h^2,$$

and $$2\pi RH : 2\pi rh :: R^2 : r^2.$$ Q. E. D.

PROPOSITION XVI.

499. Theorem.—*The volumes of similar cylinders of revolution are to each other as the cubes of their altitudes, or as the cubes of the radii of their bases.*

DEM.—Using the same notation as in the last demonstration, the student should be able to give the reasons for the following steps.

$R : r :: H : h$ (?), whence $\pi R^2 : \pi r^2 :: H^2 : h^2$ (?). Multiplying the last proportion by $H : h :: H : h$, we have $\pi R^2 H : \pi r^2 h :: H^3 : h^3$, or as $R^3 : r^3$, since $H^3 : h^3 :: R^3 : r^3$ (?). Now, $\pi R^2 H$ and $\pi r^2 h$ are the volumes of the cylinders (?); hence the volumes are to each other as the cubes of the altitudes, or as the cubes of the radii of the bases. Q. E. D.

SCH.—It is a general truth, that the surfaces of similar solids, *of any form*, are to each other as the squares of homologous lines; and their volumes are as the cubes of such lines.

EXERCISES.

1. A farmer has two grain bins which are parallelopipeds. The front of one bin is a rectangle 6 feet long by 4 high, and the front of the other a rectangle 8 feet long by 4 high. They are built between parallel walls 5 feet apart. The bottom and ends of the first, he says, are "square" (he means, it is a rectangular parallelopiped), while the bottom and ends of the other slope, *i. e.*, are oblique to the front. What are the relative capacities of the bins?

2. How many square feet of boards in the walls and bottom of the first bin mentioned in *Ex.* 1?

3. An average sized honey bee's cell is a right hexagonal prism, .8 of an inch long, with faces $\frac{3}{20}$ of an inch wide. The width of the face is always the same, but the length of the cell varies according to the space the bee has to fill. Are honey bee's cells similar? Is a honey bee's cell of the dimensions given above, similar to a wasp's cell which is 1.6 inches long, and whose face is .3 of an inch wide? How much more honey will the wasp's cell hold than the honey bee's?

4. How many square inches of sheet-iron does it take to make a joint of 7-inch stovepipe 2 feet 4 inches long, allowing an inch and a half for making the seam?

5. A certain water-pipe is 3 inches in diameter. How much water is discharged through it in 24 hours, if the current flows 3 feet per

minute? How much through a pipe of twice as great diameter, at the same rate of flow?

6. What is the ratio of the length of a hogshead holding 125 gallons, to the length of a keg of the same shape, holding 8 gallons?

7. What are the relative amounts of cloth required to clothe 3 men of the same form (similar solids), one being 5 feet high, another 5 feet 9 inches, and the other 6 feet, provided they dress in the same style? If the second of these men weighs 156 lbs., what do the others weigh?

8. If a man $5\frac{1}{2}$ feet high weighs 160 lbs., and a man 3 inches taller weighs 180 lbs., which is the stouter in proportion to his height?

9. I have a prismatic piece of timber from which I cut two blocks both 5 feet long measured along one edge of the stick; but one block is made by cutting the stick square across (a right section), and the other by cutting both ends of it obliquely, making an angle of 45° with the same face of the timber. Which block is the greater? Which has the greater lateral surface?

10. How many cubic feet in a log 12 feet long and 2 feet 5 inches in diameter? How many square feet of inch boards can be cut from such a log, allowing $\frac{1}{4}$ for waste in slabs and sawing?

SECTION IV.

OF PYRAMIDS AND CONES.

500. A Pyramid is a solid having a polygon for its base, and triangles for its lateral faces. If the base is also a triangle, it is called a triangular pyramid, or a tetraedron (*i. e.*, a solid with four faces). The vertex of the polyedral angle formed by the faces is the *vertex* of the pyramid.

501. The Altitude of a pyramid is the perpendicular distance from its vertex to the plane of its base.

502. A Right Pyramid is one whose base is a regular

polygon, and the perpendicular from whose vertex falls at the middle of the base. This perpendicular is called the *axis*.

503. A Frustum of a pyramid is a portion of the pyramid intercepted between the base and a plane parallel to the base. If the cutting plane is not parallel to the base, the portion intercepted is called a *Truncated* pyramid.

504. The Slant Height of a right pyramid is the altitude of one of the triangles which form its faces. The *Slant Height of a Frustum* of a right pyramid is the portion of the slant height of the pyramid intercepted between the bases of the frustum.

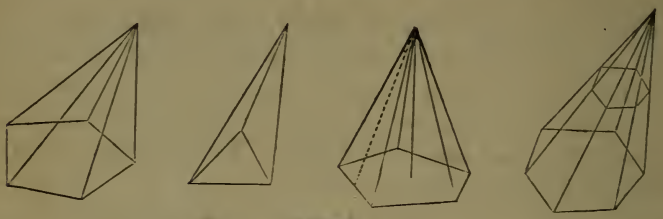

FIG. 303.

ILL'S.—The student will be able to find illustrations of the definitions in the accompanying figures.

505. A Conical Surface is a surface traced by a line which passes through a fixed point, while any other point traces a curve. The line is the *Generatrix*, and the curve the *Directrix*. The fixed point is the *Vertex*. Any line of the surface corresponding to some position of the generatrix is called an *Element* of the surface.

506. A Cone of Revolution is a solid generated by the revolution of a right angled triangle around one of its sides, called the *Axis*. The hypotenuse describes the *Convex Surface* of the cone, and corresponds to the generatrix in the preceding definition. The other side of the triangle describes the *Base*. This cone is *right*, since the perpendicular (the axis) falls at the middle of the base. The *Slant Height* is the distance from the vertex to the circumference of the base, and is the same as the hypotenuse of the generating triangle.

507. The terms *Frustum* and *Truncated* are applied to the cone in the same manner as to the pyramid.

508. A pyramid is said to be *Inscribed* in a cone when the base of the pyramid is inscribed in the base of the cone, and the edges of the pyramid are elements of the surface of the cone. The two solids have a common vertex and a common altitude.

509. If the generatrix be considered as an indefinite straight line passing through a fixed point, the portions of the line on opposite sides of the point will each describe a conical surface. These two surfaces, which in *general* discussions are considered but one, are called ***Nappes.*** The two nappes of the same cone are evidently alike.

ILL'S.—In the figure, (*a*) represents a conical surface which has the curve ACB for its directrix, and SA for its generatrix. The figures indicate the suc-

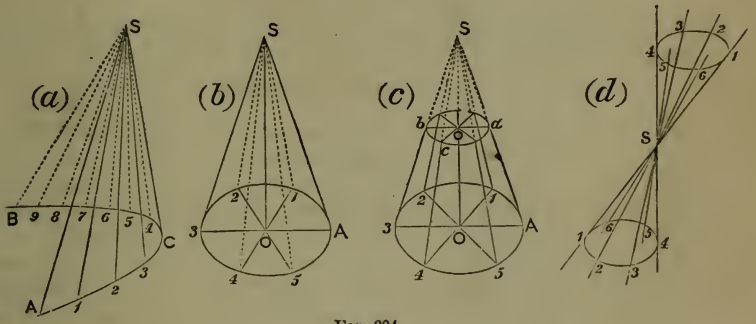

FIG. 304.

cessive positions of the point A, as it passes around the curve, while the point S remains fixed. (*b*) represents a *Cone of Revolution*, or a right cone with a circular base. It may be considered as generated in the general way, or by the right angled triangle SOA revolving about SO as an axis. SA describes the convex surface, and OA the base. The figure (*c*) represents the *Frustum* of a cone, the portion above the plane *abc* being supposed removed. Figure (*d*) represents the two nappes of an oblique cone.

PROPOSITION I.

510. Theorem.—*Any section of a pyramid made by a plane parallel to its base is a polygon similar to the base.*

DEM.—The section *abcde* of the pyramid S-ABCDE, made by a plane parallel to ABCDE, is similar to ABCDE.

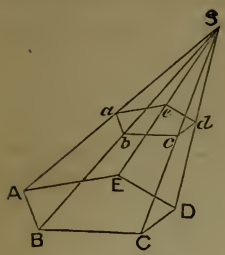

FIG. 305.

Since AB and *ab* are intersections of two parallel planes by a third plane, they are parallel (?). So also *bc* is parallel to BC, *cd* to CD, etc. Hence, angle $b = B$, $c = C$, etc. (?), and the polygons are mutually equiangular. Again, $ab : AB :: Sb : SB$, and $bc : BC :: Sb : SB$ (?). Hence, $ab : bc :: AB : BC$ (?). In like manner, we can show that $bc : cd :: BC : CD$, etc. Therefore, *abcde* and ABCDE are mutually equiangular, and have their corresponding sides proportional, and are consequently similar. Q. E. D.

PROPOSITION II.

511. Theorem.—*If two pyramids of the same altitude are cut by planes equally distant from and parallel to their bases, the sections are to each other as the bases.*

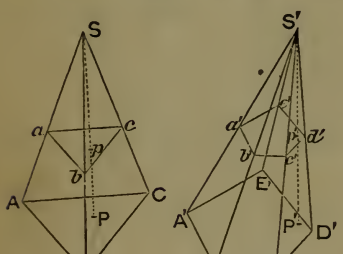

FIG. 306.

DEM.—Let S-ABC and S'-A'B'C'D'E' be two pyramids of the same altitude, cut by the planes *abc* and *a'b'c'd'e'*, parallel to and at equal distances from their bases; then is $abc : a'b'c'd'e' :: ABC : A'B'C'D'E'$.

For, conceive the bases in the same plane. Let $SP = S'P'$ be the common altitude, and $Sp = S'p'$ the distances of the cutting planes from the vertex. We have

$$ABC : abc :: \overline{AB}^2 : \overline{ab}^2 :: \overline{SP}^2 : \overline{Sp}^2 \text{ (?)}.$$

Also, $A'B'C'D'E' : a'b'c'd'e' :: \overline{A'B'}^2 : \overline{a'b'}^2 :: \overline{S'P'}^2 : \overline{S'p'}^2$ (?).

Whence, as $SP = S'P'$, and $Sp = S'p'$ (?), we have

$$abc : a'b'c'd'e' :: ABC : A'B'C'D'E' \text{ (?)}.$$ Q. E. D.

512. COR.—*If the bases are equivalent, the sections are also equivalent.*

PROPOSITION III.

513. Theorem.—*The area of the lateral surface of a right pyramid is equal to the perimeter of the base multiplied by one-half the slant height.*

DEM.—The faces of such a pyramid are equal isosceles triangles (?), whose common altitude is the slant height of the pyramid (?). Hence, the area of

these triangles is the product of one-half the slant height into the sum of their bases. But this is the lateral surface of the pyramid. (See the third cut in *Fig.* 303.)

514. Cor.—*The area of the lateral surface of the frustum of a right pyramid is equal to the product of its slant height into half the sum of the perimeters of its bases.*

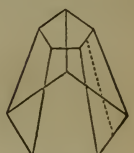

Fig. 307.

The student will be able to give the proof. It is based upon (***325***) and definitions.

PROPOSITION IV.

515. Theorem.—*The area of the convex surface of a cone of revolution (a right cone with a circular base) is equal to the product of the circumference of its base and one-half its slant height,* i. e., $\pi RH'$, R *being the radius of the base, and* H′ *the slant height.*

Dem.—In the circle which forms the base of the cone, conceive a regular polygon inscribed, as *abcde*. Joining the vertices of the angles of this polygon with the vertex of the cone, there will be constructed a right pyramid inscribed in the cone. Now, if the arcs subtended by the sides of this polygon are bisected, and these again bisected, etc., and at every step a right pyramid conceived as inscribed, it will *always* remain true that the lateral surface of the pyramid is the perimeter of its base into half its slant height. But, as the number of faces of the pyramid is increased, the perimeter of the base approaches the circumference of the base of the cone, the slant height of the pyramid approaches the slant height of the cone, and the lateral surface of the pyramid approaches the convex surface of the cone. Hence, *at* the limit we still have the same expression for the area of the convex surface, that is, the circumference of the base multiplied by half the slant height. Finally, if R is the radius of the base, its circumference is $2\pi R$, and H′ being the slant height, we have for the area of the convex surface $2\pi R \times \frac{1}{2}H'$, or $\pi RH'$.

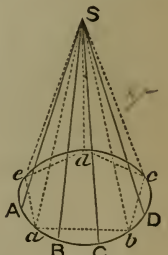

Fig. 308.

516. Cor. 1.—*The area of the convex surface of a cone is also equal to the product of the slant height into the circumference of the circle parallel to the base, and midway between the base and vertex.*

This follows directly from the fact that the radius of the circle midway between the base and vertex is one-half the radius of the base, *i. e.*, $\frac{1}{2}R$, whence its circumference is πR. Now, $\pi R \times H'$ is the area of the convex surface, by the proposition.

517. Cor. 2.—*The area of the convex surface of the frustum of a cone is equal to the product of its slant height into half the sum of the circumferences of its bases;* i. e., π (R + r) H′, R *and* r *being the radii of its bases, and* H′ *its slant height.*

From the corresponding property of the frustum of a pyramid, the student will be able to deduce the fact that $\frac{1}{2}$ (2πR + $2\pi r$) H′, or π (R + r) H′, is the area of this surface.

518. Cor. 3.—*The area of the convex surface of the frustum of a cone is equal to the product of its slant height into the circumference of the circle midway between the bases.*

The radius of the circle midway between the bases is $\frac{1}{2}$ (r + R), whence its circumference is π (r + R). Now, π (r + R) × H′ is the area of the convex surface of the frustum, by the preceding corollary.

PROPOSITION V.

519. Theorem.—*Two pyramids having equivalent bases and the same altitudes are equivalent,* i. e., *equal in volume.*

Dem.—Let S-ABCD and S′-A′B′C′D′E′ be two pyramids having the same altitudes, and base ABCD equivalent to base A′B′C′D′E′, *i. e.*, equal in area; then is pyramid S-ABCD equivalent to S′-A′B′C′D′E′, *i. e.*, equal in volume.

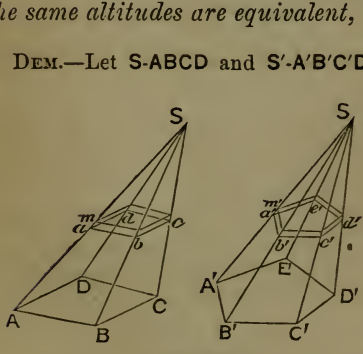

Fig. 309.

For, conceive the bases to be in the same plane, and a plane to start from coincidence with the plane of the bases, and move toward the vertices, remaining all the time parallel to the bases. At every stage of its progress the sections are equivalent, and as the plane reaches both vertices at the same time, by reason of the common altitude, it is evident that the volumes are equal.

Or, if desired, we may consider the two pyramids as divided into an equal number of infinitely thin *laminæ* parallel to the bases. Each lamina in one has its corresponding equivalent lamina in the other; hence the sum of all the *laminæ* in one equals the sum of all the *laminæ* in the other; *i. e.*, the pyramids are equivalent.

PROPOSITION VI.

520. Theorem.—*The volume of a triangular pyramid is equal to one-third the product of its base and altitude.*

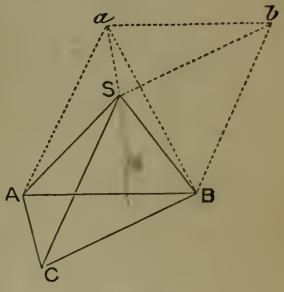

Fig. 310.

Dem.—Let S-ABC be a triangular pyramid, whose altitude is H*; then is the volume equal to $\frac{1}{3}$ H × *area* ABC.

For, through A and B draw A*a* and B*b* parallel to SC; and through S draw S*a* and S*b* parallel to CA and CB, and join *a* and *b*; then S*ab*-ABC is a prism with its bases equal to the base of the pyramid. Now, the solid added to the given pyramid is a quadrangular pyramid with *ab*BA as its base, and its vertex at S. Divide this into two triangular pyramids by drawing *a*B and passing a plane through SB and *a*B. These triangular pyramids are equivalent, since they have equal bases *a*AB and *ab*B and a common altitude, the vertices of both being at S. Again, S-*ab*B may be considered as having *ab*S (equal to ABC) as its base, and the altitude of the first pyramid (equal to the altitude of the prism) for its altitude, and hence as equivalent to the given pyramid. Therefore S-ABC is one third of the prism S*ab*-ABC. But the volume of the prism is H × *area* ABC. Therefore the volume of the pyramid S-ABC is $\frac{1}{3}$ H × *area* ABC. Q. E. D.

521. Cor. 1.—*The volume of any pyramid is equal to one-third the product of its base and altitude.*

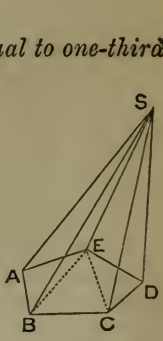

Fig. 311.

Dem.—Since any pyramid can be divided into triangular pyramids by passing planes through any one edge, as SE, and each of the other edges not adjacent, as SB and SC, the volume of the pyramid is equal to the sum of the volumes of several triangular pyramids having the same altitude as the given pyramid, and the sum of whose bases is the base of the given pyramid. Hence the truth of the corollary.

522. Cor. 2.—*Pyramids having equivalent bases are to each other as their altitudes; such as have equal altitudes are to each other as their bases; and, in general, pyramids are to each other as the products of their bases and altitudes.*

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* Not drawn in the figure, lest it might confuse.

PROPOSITION VII.

523. Theorem.—*The volume of the frustum of a triangular pyramid is equal to the volume of three pyramids of the same altitude as the frustum, and whose bases are the upper base, the lower base, and a mean proportional between the two bases of the frustum.*

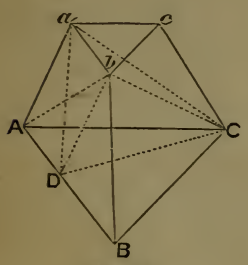

FIG. 312

DEM.—Let *abc*-ABC be the frustum of a triangular pyramid. Through *ab* and C pass a plane cutting off the pyramid C-*abc*. This has for its base the upper base of the frustum, and for its altitude the altitude of the frustum. Again, draw A*b*, and pass a plane through A*b* and *b*C, cutting off the pyramid *b*-ABC, which has the same altitude as the frustum, and for its base the lower base of the frustum. There now remains a third pyramid, *b*-AC*a*, to be examined. Through *b* draw *b*D parallel to *a*A, and draw DC and *a*D. The pyramid D-AC*a* is equivalent to *b*-AC*a*, since it has the same base and the same altitude. But the former may be considered as having ADC for its base, and the altitude of the frustum for its altitude, *i. e.*, as pyramid *a*-ADC. We are now to show that ADC is a mean proportional between *abc* and ABC.

$$\text{ABC} : abc :: \overline{\text{AB}}^2 : \overline{ab}^2 :: \overline{\text{AB}}^2 : \overline{\text{AD}}^2 \text{ (?)}.$$

Also, $\text{ABC} : \text{ADC} :: \text{AB} : \text{AD}$ (?);

whence $\overline{\text{ABC}}^2 : \overline{\text{ADC}}^2 :: \overline{\text{AB}}^2 : \overline{\text{AD}}^2$ (?).

By equality of ratios, $\text{ABC} : abc :: \overline{\text{ABC}}^2 : \overline{\text{ADC}}^2$;

whence $\overline{\text{ADC}}^2 = abc \times \text{ABC}$, *i. e.*, ADC is a mean proportional between the upper and lower bases of the frustum.

524. COR.—*The volume of the frustum of any pyramid is equal to the volume of three pyramids having the same altitude as the frustum, and for bases, the upper base, the lower base, and a mean proportional between the bases of the frustum.*

For, the frustum of any pyramid is equivalent to the corresponding frustum of a triangular pyramid of the same altitude and an equivalent base (?); and the bases of the frustum of the triangular pyramid being both equivalent to the corresponding bases of the given frustum, a mean proportional between the triangular bases is a mean proportional between their equivalents.

PROPOSITION VIII.

525. Theorem.—*The volume of a cone of revolution is equal to one-third the product of its base and altitude;* i. e., $\frac{1}{3}\pi \text{R}^2\text{H}$, R *being the radius of the base and* H *the altitude.*

DEM.—This follows from the volume of a pyramid, by a course of reasoning precisely the same as in (*515*). The volume of a pyramid being equal to one-third the product of the base and altitude, and the cone being the limit of the pyramid, the volume of the cone is one-third the product of its base and altitude. Now, R being the radius of the base of a cone of revolution, the base (area of) is πR^2, whence $\frac{1}{3}\pi R^2 H$ is the volume, H being the altitude.

526. COR. 1.—*The volume of any cone is equal to one-third the product of its base and altitude.*

527. COR. 2.—*The volume of the frustum of a cone is equal to the volume of three cones having the same altitude as the frustum, and for bases, the upper base, the lower base, and a mean proportional between the two bases of the frustum.*

The truth of this appears from the fact that the frustum of a cone is the limit of the frustum of a pyramid.

PROPOSITION IX.

528. Theorem.—*The lateral surfaces of similar right pyramids are to each other as the squares of their homologous edges, their slant heights, and their altitudes;* i. e., *as the squares of any two homologous dimensions.*

DEM.—Let A and a be homologous sides of the bases of two similar right pyramids, H′ and h' their slant heights, H and h their altitudes, and P and p the perimeters of their bases; then—

(1) $P : p :: A : a$, because the bases are similar polygons;

(2) $A : a :: H' : h'$, because the faces are similar triangles;

(3) $H' : h' :: H : h$ (?).

Whence, $P : p :: H' : h'$;

and, as $\frac{1}{2}H' : \frac{1}{2}h' :: H' : h'$,

multiplying, we have $\frac{1}{2}P \times H' : \frac{1}{2}p \times h' :: H'^2 : h'^2 :: A^2 : a^2 :: H^2 : h^2$. But $\frac{1}{2}P \times H'$ and $\frac{1}{2}p \times h'$ are the areas of the lateral surfaces.

PROPOSITION X.

529. Theorem.—*The convex surfaces of similar cones of revolution are to each other as the squares of their slant heights, the radii of their bases, and their altitudes;* i. e., *as the squares of any two homologous dimensions.*

DEM.—Let H′ and h' be the slant heights of two similar cones of revolution, R and r the radii of their bases, and H and h their altitudes; their convex surfaces are $\pi RH'$ and $\pi rh'$. Now, since the cones are similar $R : r :: H' : h'$.

Multiplying the terms of this proportion by the corresponding terms of $\pi H' \cdot \pi h' :: H' : h'$, we have—

$$\pi RH' : \pi rh' :: H'^2 : h'^2.$$

Hence the convex surfaces are as the squares of the slant heights, and since $R : r :: H' : h' :: H : h$ (?), $R^2 : r^2 :: H'^2 : h'^2 :: H^2 : h^2$; and consequently $\pi RH' : \pi rh' :: R^2 : r^2 :: H^2 : h^2$.

PROPOSITION XI.

530. Theorem.—*The volumes of similar pyramids are to each other as the cubes of their homologous dimensions.*

DEM.—Letting A and a be homologous sides of the bases of two similar pyramids, B and b their bases, and H and h their altitudes, the student should be able to give the reasons for the following proportions :

$$B : b :: A^2 : a^2 :: H^2 : h^2.$$

$$\tfrac{1}{3}H : \tfrac{1}{3}h :: A : a :: H : h.$$

Whence $\tfrac{1}{3}BH : \tfrac{1}{3}bh :: A^3 : a^3 :: H^3 : h^3$. Q. E. D.

PROPOSITION XII.

531. Theorem.—*The volumes of similar cones are to each other as the cubes of their altitudes, or as the cubes of the radii of their bases.*

DEM. R and r being the radii of their bases, and H and h their altitudes,

$$R^2 : r^2 :: H^2 : h^2 \text{ (?), and } R^3 : r^3 :: H^3 : h^3.$$

Also, $\tfrac{1}{3}\pi H : \tfrac{1}{3}\pi h :: H : h.$

Multiplying, $\tfrac{1}{3}\pi R^2 H : \tfrac{1}{3}\pi r^2 h :: H^3 : h^3$, or as $R^3 : r^3$. Q. E. D.

EXERCISES.

1. What is the area of the lateral surface of a right hexagonal pyramid whose base is inscribed in a circle whose diameter is 20 feet, the altitude of the pyramid being 8 feet? What is the volume of this pyramid?

2. What is the area of the lateral surface of a right pentagonal pyramid whose base is inscribed in a circle whose radius is 6 yards, the slant height of the pyramid being 10 yards? What is the volume of this pyramid?

3. How many quarts will a can contain, whose entire height is 10 inches, the body being a cylinder 6 inches in diameter and $6\frac{1}{2}$ inches

high, and the top a cone? How much tin does it take to make such a can, allowing nothing for waste and the seams?

4. If very fine dry sand is piled upon a smooth horizontal surface, without any lateral support, the angle of slope (*i. e.*, the angle of inclination of the sloping side of the pile with the plane) is about 31°. Suppose two circles be drawn on the floor, one 4 feet in diameter and the other 3, and sand piles be made as large as possible on these circles as bases, no other support being given. What is the relative magnitude of the piles?

5. In the case of sand piles, as given in the last example, the ratio of the radius of the base to the altitude of the pile is $\frac{5}{3}$. How many cubic feet in each of the above piles?

6. The frustum of a right pyramid was 72 feet square at the lower base and 48 at the upper; and its altitude was 60 feet. What was the lateral surface? What the volume?

THE STUDENT SHOULD FURNISH A SYNOPSIS OF EACH SECTION AT ITS CLOSE.

SECTION V.

OF THE SPHERE.*

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532. A Sphere is a solid bounded by a surface every point in which is equally distant from a point within called the *Centre.* The distance from the centre to the surface is the *Radius,* and a line passing through the centre and limited by the surface is a *Diameter.* The diameter is equal to twice the radius.

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* A spherical blackboard is almost indispensable in teaching this section, as well as in teaching Spherical Trigonometry. A sphere about 2 feet in diameter, mounted on a pedestal, and having its surface slated or painted as a blackboard, is what is needed. It can be obtained of the manufacturers of school apparatus, or made in any good turning-shop.

CIRCLES OF THE SPHERE.

PROPOSITION I.

533. Theorem.—*Every section of a sphere, made by a plane, is a circle.*

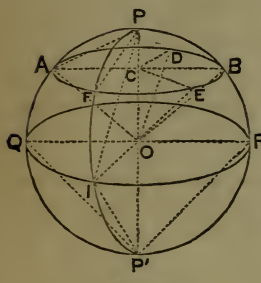

FIG. 313.

DEM.—Let AFEBD be a section of a sphere whose centre is O, made by a plane; then is it a circle.

For, let fall from the centre O a perpendicular upon the plane AFEBD, as OC, and draw CA, CD, CE, CB, etc., lines of the plane, from the foot of the perpendicular to any points in which the plane cuts the surface of the sphere. Join these points with the centre, O, of the sphere. Now, OA, OD, OB, OE, etc., being radii, are equal; whence, CA, CD, CB, CE, etc., are equal; *i. e.*, every point in the line of intersection of a plane and surface of a sphere is equally distant from a point in this plane. Hence, the intersection is a circle. Q. E. D.

534. DEF.—A circle made by a plane not passing through the centre is a *Small Circle;* one made by a plane passing through the centre is a *Great Circle.*

535. COR. 1.—*A perpendicular from the centre of a sphere, upon any small circle, pierces the circle at its centre;* and, conversely, *a perpendicular to a small circle at its centre passes through the centre of the sphere.*

536. DEF.—A diameter perpendicular to any circle of a sphere is called the *Axis* of that circle. The extremities of the axis are the *Poles* of the circle.

537. COR. 2.—*The pole of a circle is equally distant from every point in its circumference.*

The student should be able to give the reason.

538. COR. 3.—*Every circle of a sphere has two poles, which, in case of a great circle, are equally distant from every point in the circumference of the circle; but, in case of a small circle, one pole is nearer any point in the circumference than the other pole is.*

539. Cor. 4.—*A small circle is less as its distance from the centre of the sphere is greater.*

For, its diameter, being a chord of a great circle, is less as it is farther from the centre of the great circle, which is also the centre of the sphere.

540. Cor. 5.—*All great circles of the same sphere are equal,* their radii being the radius of the sphere.

PROPOSITION II.

541. *Theorem.*—*Any great circle divides the sphere into two equal parts called Hemispheres.*

Dem.—Conceive a sphere as divided by a great circle, *i. e.*, by a plane passing through its centre, and let the great circle be considered as the base of each portion. These bases being equal, reverse one of the portions and conceive its base placed in the base of the other, the convex surfaces being on the same side of the common base. Since the bases are equal circles, they will coincide, and since every point in the convex surface of each portion is equally distant from the centre of the common base, the convex surfaces will coincide. Therefore, the portions coincide throughout, and are consequently equal. Q. E. D.

PROPOSITION III.

542. *Theorem.*—*The intersection of any two great circles of a sphere is a diameter of the sphere.*

Dem.—The intersection of two planes is a straight line; and in the case of the two great circles, as they both pass through the centre of the sphere, this is one point of their intersection. Hence, the intersection of two great circles of a sphere is a straight line which passes through the centre. Q. E. D.

543. Cor.—*The intersections on the surface of a sphere of two circumferences of great circles are a semi-circumference, or* 180°, *apart, since they are at opposite extremities of a diameter.*

DISTANCES ON THE SURFACE OF A SPHERE.

544. Distances on the surface of a sphere are always to be understood as measured on the arc of a great circle, unless it is otherwise stated.

PROPOSITION IV.

545. Theorem.—*The distances, measured on the surface of a sphere, from a pole to all points in the circumference of a circle of which it is the pole, are equal.*

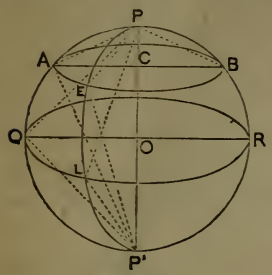

FIG. 314.

DEM.—Let P be a pole of the small circle AEB; then are the arcs PA, PE, PB, etc., which measure the distances on the surface of the sphere, from P to any points in the circumference of circle AEB, equal. For, by (***537***), the straight lines AP, PE, PB, etc., are equal, and these equal chords subtend equal arcs, as arc PA, arc PE, arc PB, etc., the great circles of which these lines are chords and arcs being equal (***540***). Thus, for like reasons, arc P'QA = arc P'LE = arc P'RB, etc.

546. COR.—*The distance from the pole of a great circle to any point in the circumference of the circle is a quadrant* (*a quarter of a circumference*).

Since the poles are 180° apart (being the extremities of a diameter), PAQP' = PELP' = a semicircumference. But, in case of a great circle, chord PL = chord P'L (= chord PQ = chord P'Q), whence arc PEL = arc P'L = arc PAQ = arc P'Q. Hence, each of these arcs is a quadrant.

FIG. 315.

547. SCH.—By means of the facts demonstrated in this proposition and corollary, we are enabled to draw arcs of small and great circles, in the surface of a sphere, with nearly the same facility as we draw arcs and lines in a plane. Thus, to draw the small circle AEB, we take an arc equal to PE, and placing one end of it at P, cause a pencil held at the other end to trace the arc AEB, etc. To describe the circumference of a great circle, a quadrant must be used for the arc. By bending a wire into an arc of the circle, and making a loop in each end, a wooden pin can be put through one loop and a crayon through the other, and an arc drawn as represented in the figure.

PROPOSITION V.

548. Problem.—*To pass a circumference of a great circle through any two points on the surface of a sphere.*

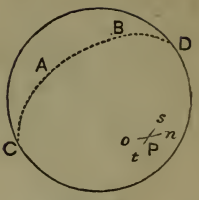

FIG. 316.

SOLUTION.—Let A and B be two points on the surface of a sphere, through which it is proposed to pass a circumference of a great circle. From B as a pole, with an arc equal to a quadrant, strike an arc *on*, as nearly where the pole of the circle passing through A and B lies, as may be determined by inspection. Then, from A, with the same arc, strike an arc *st* intersecting *on* at P. Now, P is the pole of the great circle passing through A and B. Hence, from P as a pole, with a quadrant arc draw a circle; it will pass through A and B, and will be a great circle, since its pole is a quadrant's distance from its circumference. [The student should make the construction on the spherical blackboard.]

549. COR. 1.—*Through any two points on the surface of a sphere, one great circle* can always be made to pass, and only one, except when the two points are at the extremities of the same diameter, in which case an infinite number of great circles can be passed through the two points.*

Since the arcs *on* and *st* are arcs of great circles, the circumferences of which they form parts will intersect also on the opposite side of the sphere, at a distance of a semicircumference from P. But these two points are poles of the *same* great circle. Now, as the two great circles can intersect at no other points, there can be only one great circle passed through A and B. But if the two given points were at the extremities of the same diameter, as at D and C, the arcs *st* and *on* would coincide, and *any* point in this circumference being taken as a pole, great circles can be drawn through D and C. [The student should trace the work on the spherical blackboard.]

550. SCH.—The truth of the corollary is also evident from the fact that three points not in the same straight line determine the position of a plane. Thus A, B, and the centre of the sphere, fix the position of one, and only one, great circle passing through A and B. Moreover, if the two given points are at the extremities of the same diameter, they are in the same straight line with the centre of the sphere, whence an infinite number of planes can be passed through them and the centre. The meridians on the earth's surface afford an example, the poles (of the equator) being the given points.

551. COR. 2.—*If two points in the circumference of a great circle of a sphere, not at the extremities of the same diameter, are at a quadrant's distance from a point on the surface, that point is the pole of the circle.*

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* The word circle may be understood to refer either to the circle proper, or to its circumference. The word is in constant use in the higher mathematics, in the latter sense.

PROPOSITION VI.

552. Theorem.—*The shortest distance on the surface of a sphere, between any two points in that surface, is measured on the arc less than a semicircumference of the great circle which joins them.*

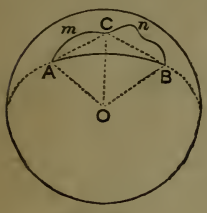

FIG. 317.

DEM.—Let A and B be any two points in the surface of a sphere, AB the arc of a great circle joining them, and A*m*C*n*B any other path in the surface between A and B; then is arc AB less than A*m*C*n*B.

Let C be any point in A*m*C*n*B, and pass the arcs of great circles through A and C, and B and C. Join A, B, and C with the centre of the sphere. The angles AOB, AOC, and COB form the facial angles of a triedral, of which angles the arcs AB, AC, and CB are the measures. Now, angle AOB < AOC + COB (*434*); whence arc AB < arc AC + arc CB, and the path from A to B is less on arc AB than on arcs AC, CB. In like manner, joining any point in A*m*C with A and C by arcs of great circles, their sum would be greater than AC. So, also, joining any point in C*n*B with C and B, the sum of the arcs would be greater than CB. As this process is indefinitely repeated, the path from A to B on the arcs of the great circles will continually increase, and also continually approximate the path A*m*C*n*B. Hence, arc AB is less than the path A*m*C*n*B. Q. E. D.

553. COR.—*The least arc of a circle of a sphere joining any two points in the surface, is the arc less than a semicircumference of the great circle passing through the points; and the greatest arc is the circumference minus this least arc.*

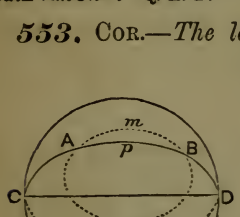

FIG. 318.

DEM.—Let A*m*B*n* be any small circle passing through A and B, and ABD*o*C the great circle. As shown above, A*p*B < A*m*B. Now, circumference ABD*o*C > circumference A*m*B*n* (*539*). Subtracting the former inequality from the latter, we have BD*o*CA > B*n*A. Q. E. D.

PROPOSITION VII.

554. Theorem.—*The shortest path on the surface of a hemisphere, from any point therein to the circumference of the great circle forming its base, is the arc less than a quadrant of a great circle perpendicular to the base, and the longest path, on any arc of a great circle, is the supplement of this shortest path.*

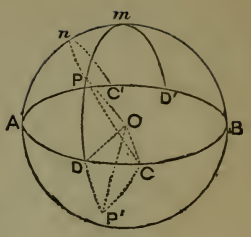

Fig. 319.

Dem.—Let P be a point in the surface of the hemisphere whose base is ACBC′, and DP*m*D′ an arc of a great circle passing through P and perpendicular to ADCBC′; then is PD the shortest path on the surface from P to circumference ADBC′, and P*m*D′ is the longest path from P to the circumference, measured on the arc of a great circle.

For, the shortest path from P to any point in circumference ADBC′ is measured on the arc of a great circle (*552*). Now, let PC be any oblique arc of a great circle. We will show that arc PD < arc PC. Produce PD until DP′ = PD; and pass a great circle through P′ and C. Draw the radii OP, OD, OC, and OP′. The triedrals O-PDC and O-P′DC have the facial angle POD = P′OD, they being measured by equal arcs, and the facial angle DOC common. Hence, as the included diedrals are equal, both being right, the triedrals are equal or symmetrical (*446*). In this case they are symmetrical, and the facial angle POC = P′OC; whence the arc PC = arc P′C. Finally, since PC + P′C > PP′, PC, the half of PC + P′C, is greater than PD, the half of PP′.

Secondly, P*m*D′ is the supplement of PD, and we are to show that it is greater than any other arc of a great circle from P to the circumference ADBC′. Let P*n*C′ be any arc of a great circle oblique to ADCBC′. Produce C′*n*P to C. Now CP*n*C′ is a semicircumference and consequently equal to DP*m*D′. But we have before shown that PD < PC, and subtracting these from the equals CP*n*C′ and DP*m*D′, we have P*m*D′ > P*n*C′.

555. Cor.—*From any point in the surface of a hemisphere there are two perpendiculars to the circumference of the great circle which forms the base of the hemisphere; one of which perpendiculars measures the least distance to that circumference, and the other the greatest, on the arc of any great circle of the sphere.*

Thus PD and P*m*D′ are two perpendiculars from P upon the circumference ADBC′.

SPHERICAL ANGLES.

556. The angle formed by two arcs of circles of a sphere is conceived as the same as the angle included by the tangents to the arcs at the common point.

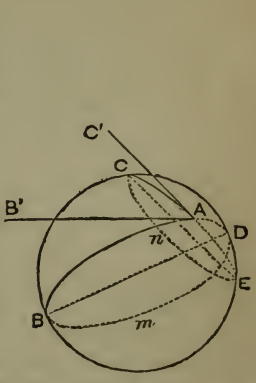

Fig. 320.

Ill.—Let AB and AC be two arcs of circles of the sphere, meeting at A; then the angle BAC is conceived as the same as the angle B′AC′, B′A being tangent to the circle BAD*m*, and C′A to the circle CAE*n*.

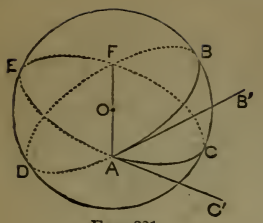

FIG. 321.

557. A Spherical Angle is the angle included by two arcs of *great circles.*

ILL.—BAC, *Fig.* 321, is a spherical angle, and is conceived as the same as the angle B'AC', B'A and C'A being tangents to the *great circles* BADF and CAEF. [The student should not confound such an angle as BAC, *Fig.* 320, with a *spherical angle.*]

PROPOSITION VIII.

558. Theorem.—*A spherical angle is equal to the measure of the diedral included by the great circles whose arcs form the sides of the angle.*

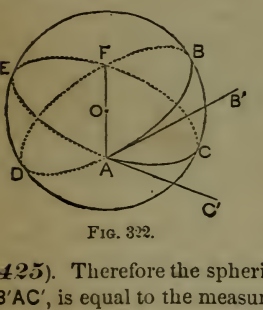

FIG. 322.

DEM.—Let BAC be any spherical angle, and BADF and CAEF the great circles whose arcs BA and CA include the angle; then is BAC equal to the measure of the diedral C-AF-B. For, since two great circles intersect in a diameter (***542***), AF is a diameter. Now B'A is a tangent to the circle BADF, that is, it lies in the same plane and is perpendicular to AO at A. In like manner C'A lies in the plane CAEF and is perpendicular to AO. Hence B'AC' is the measure of the diedral C-AF-B (***425***). Therefore the spherical angle BAC, which is the same as the plane angle B'AC', is equal to the measure of the diedral C-AF-B. Q. E. D.

559. COR. 1.—*If one of two great circles passes through the pole of the other, their circumferences intersect at right angles.*

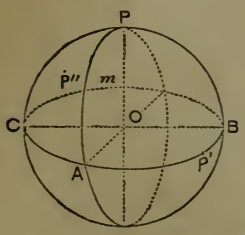

FIG. 323.

DEM.—Thus, P being the pole of the great circle CAB*m*, PO is its axis, and any plane passing through PO is perpendicular to the plane CAB*m* (***427***). Hence, the diedral B-AO-P is right, and the spherical angle PAB, which is equal to the measure of the diedral, is also right.

560. COR. 2.—*A spherical angle is measured by the arc of a great circle intercepted between its sides, and at a quadrant's distance from its vertex.*

Thus, the spherical angle CPA is measured by CA, PC and PA being quadrants. For, since PC is a quadrant, CO is perpendicular to PO, the edge of the diedral C-PO-A, and for a like reason AO is perpendicular to PO. Hence, COA is the measure of the diedral, and consequently CA, its measure, is the measure of the spherical angle CPA.

Note the arc CA subtends the spherical ∠ C-PO-A

561. COR. 3.—*The angle included by two arcs of small circles is the same as the angle included by two arcs of great circles passing through the vertex and having the same tangents.*

Thus BAC = B″AC″. For the angle BAC is, by definition, the same as B′AC′, B′A and C′A being tangents to BA and CA. Now, passing planes through C′A, B′A, and the centre of the sphere, we have the arcs B″A, C″A, and B′A, C′A tangents to them. Hence, B″AC″ is the same as B′AC′, and consequently the same as BAC.

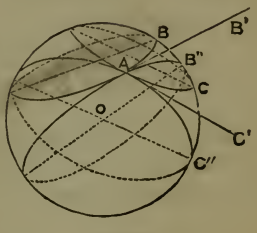

FIG. 324.

562. SCH.—*To draw an arc of a great circle which shall be perpendicular to another; or, what is the same thing, to construct a right spherical angle.* Let it be required to erect an arc of a great circle perpendicular to CAB at A, *Fig.* 323. Lay off from A, on the arc CAB, a quadrant's distance, as AP′, and from P′ as a pole, with a quadrant describe an arc passing through A. This will be the perpendicular required.

In a similar manner we may let fall a perpendicular from any point in the surface, upon any arc of a great circle. To let fall a perpendicular from P″ upon the arc CAB, from P″ as a pole, with a quadrant describe an arc cutting CAB, as at P′. Then from P′ as a pole, with a quadrant describe an arc passing through P″ and cutting CAB, and it will be perpendicular to CAB. [The student should have practice in making these constructions on the sphere.]

PROPOSITION IX.

563. Problem.—*To pass the circumference of a small circle through any three points on the surface of a sphere.*

SOLUTION.—Let A, B, and C be the three points in the surface of the sphere through which we propose to pass the circumference of a circle. Pass arcs of great circles through the points, forming the spherical triangle ABC. Thus, to pass an arc of a great circle through B and C, from B as a pole, with a quadrant strike an arc as near as may be to the pole of the required circle; and from C as a pole, with the quadrant strike an arc intersecting the former, as at P; then is P the pole of a great circle passing through B and C (?). Hence, from P as a pole, with a quadrant pass an arc through B and C, and it will be the arc required (***551***). In like manner pass arcs through A and C, A and B. Now, bisect two of these arcs, as BC and AC, by arcs of great

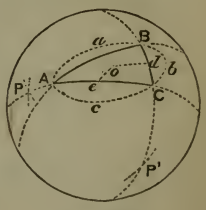

FIG. 325.

circles perpendicular to each. [The student will readily perceive how this is done.] The intersection of these perpendiculars, o, will be the pole of the small circle required (?). Then from o, as a pole, with an arc oB draw the circumference of a small circle: it will pass through A, B, and C (?), and hence is the circumference required.

OF TANGENT PLANES.

564. A Tangent Plane to a curved surface at a given point is the plane of two lines respectively tangent to two plane sections through the point.

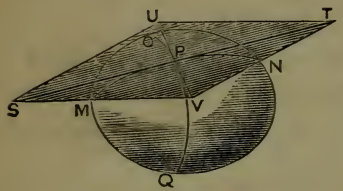

FIG. 326.

ILL.—Let P be a point in the curved surface at which we wish a tangent plane. Pass any two planes through the surface and the point P, and let OPQ and MPN represent the intersections of these planes with the curved surface. Draw UV and ST in the planes of the sections, and tangent to OPQ and MPN, at P. Then is the plane of UV and ST the tangent plane at P.

PROPOSITION X.

565. Theorem.—*A tangent plane to a sphere is perpendicular to the radius at the point of tangency.*

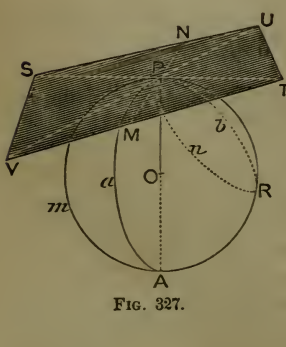

FIG. 327.

DEM.—Let P be any point in the surface of a sphere; pass two great circles, as PaA, etc., and PmAR, through P, and draw ST tangent to the arc mP, and UV tangent to the arc aP; then is the plane SVTU a tangent plane at P, and perpendicular to the radius OP. For, a tangent (as ST) to the arc mP is perpendicular to the radius of the circle, *i. e.*, to OP, and also a tangent (as VU) to the arc aP is perpendicular to the radius of *this* circle, *i. e.*, to OP. Hence, OP is perpendicular to two lines of the plane SVTU, and consequently to the plane of these lines (?). Q. E. D.

566. COR. 1.—*Every point in a tangent plane to a sphere, except the point of tangency, is without the sphere.*

For, OP, the perpendicular, is shorter than any line which can be drawn from O to any other point in the plane (?), hence any other point in the plane than P lies farther from the centre of the sphere than the length of the radius, and is, therefore, without the sphere.

567. Cor. 2.—A *tangent through* P *to* ANY *circle of the sphere passing through this point, lies in the tangent plane.*

Dem.—Thus MN, tangent to the small circle P*n*R*b* through P, lies in the tangent plane. For, conceive the plane of the small circle extended till it intersects the tangent plane. This intersection is tangent to the small circle, since it touches it at one point, but cannot cut it; otherwise the tangent plane would have another point than P common with the surface of the sphere. But there can be only one tangent to a circle at a given point. Hence this intersection is MN, which is consequently in the tangent plane.

OF SPHERICAL TRIANGLES.

568. A Spherical Triangle is a portion of the surface of a sphere bounded by three arcs of great circles. In the present treatise these arcs will be considered as each less than a semicircumference.

The terms scalene, isosceles, equilateral, right angled, and oblique angled, are applied to spherical triangles in the same manner as to plane triangles.

PROPOSITION XI.

569. Theorem.—*The sum of any two sides of a spherical triangle is greater than the third side, and their difference is less than the third side.*

Dem.—Let ABC be any spherical triangle; then is BC < BA + AC, and BC − AC < BA; and the same is true of the sides in any order. For, join the vertices A, B, and C, with the centre of the sphere, by drawing AO, BO, and CO. There is thus formed a triedral O-ABC, whose facial angles are measured by the sides of the triangle (*208*). Now, angle BOC < BOA + AOC (*434*), whence BC < BA + AC: and subtracting AC from both members, we have BC − AC < BA.

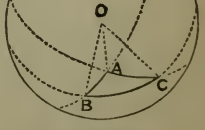

Fig. 328.

PROPOSITION XII.

570. Theorem.—*The sum of the sides of a spherical triangle may be anything between* 0 *and a circumference.*

DEM.—The sides of a spherical triangle are measures of the facial angles of a triedral whose vertex is at the centre of the sphere. Hence their sum may be anything between 0 and the measure of 4 right angles, as these are the limits of the sum of the facial angles of a triedral (***436***).

571. SCH.—As the sides of a spherical triangle are arcs, they can be measured in degrees. Hence, we speak of the side of a spherical triangle as 30°, 57°, 115° 10′, etc. In accordance with this, we say that the limit of the sum of the sides of a spherical triangle is 360°.

PROPOSITION XIII.

572. Theorem.—*The sum of the angles of a spherical triangle may be anything between two and six right angles.*

DEM.—The sum of the angles of a spherical triangle is the same as the sum of the measures of the diedrals of a triedral having its vertex at the centre of the sphere, as in (***569***). Now the limits of the sum of the measures of these diedrals are 2 and 6 right angles (***439***). Hence the sum of the angles of any spherical triangle may be anything between 2 and 6 right angles. Q. E. D.

573. SCH.—It will be observed, that the sum of the angles of a spherical triangle is not constant, as is the sum of the angles of a plane triangle. Thus, the sum of the angles of a spherical triangle may be 200°, 290°, 350°, 500°, anything between 180° and 540°.

574. DEF.—***Spherical Excess*** is the amount by which the sum of the angles of a spherical triangle exceeds the sum of the angles of a plane triangle; *i. e.*, it is the sum of the spherical angles $-180°$, or π.

ILL.—It is not difficult to observe the occasion of this *excess* in the case of the equilateral spherical triangle. Thus, let ABC be such a triangle. Conceive the plane triangle formed by the chords AB, AC, and CB. The sum of the angles of this plane triangle is 180°. But each angle of the spherical triangle is larger than the corresponding angle of the plane triangle. Thus, the spherical angle BAC is the same as the plane angle C′AB′, included between the tangents C′A and B′A, which are perpendicular to the edge of the diedral C-AO-B, and include its measuring angle. Now, CA and BA

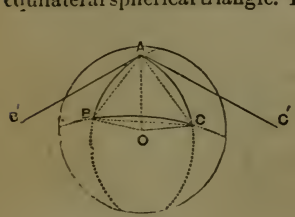

FIG. 329.

being different lines from C'A and B'A are oblique to the edge AO, and include an angle *less* than its measure, and consequently less than CAB. For a like reason the plane angle ACB < the spherical angle ACB, and plane angle ABC < spherical angle ABC. Moreover, it is easy to see that the inequality between any plane angle and the corresponding spherical angle increases as the chords BA and CA deviate more from the tangents. Whence we see why the sum of the angles of the spherical triangle is not a fixed quantity.

575. COR.—*A spherical triangle may have one, two, or even three right angles; and, in fact, it may have one, two, or three obtuse angles; since, in the latter case, the sum of the angles will not necessarily be greater than* 540°.

576. DEF.—***A Trirectangular Spherical Triangle*** is a spherical triangle which has three right angles.

PROPOSITION XIV.

577. Theorem.—*The trirectangular triangle is one-eighth of the surface of a sphere.*

DEM.—Pass three planes through the centre of a sphere, respectively perpendicular to each other. They will divide the surface into 8 trirectangular triangles, any one of which may be applied to any other. Thus, let ABA'B', ACA'C', and CBC'B' be the great circles formed by the three planes, mutually perpendicular to each other. The planes being perpendicular to each other the diedrals, as A-CO-B, C-BO-A, C-AO-B, etc., are right, and hence the angles of the 8 triangles formed are all right. Also, as AOB is a right angle, AB is a quadrant; as BOC is a right angle, CB is a quadrant, etc. Hence, each side of every triangle is a quadrant. Now any one triangle may be applied to any other. [Let the student make the application.] Hence the trirectangular triangle is one-eighth of the surface of a sphere. Q. E. D.

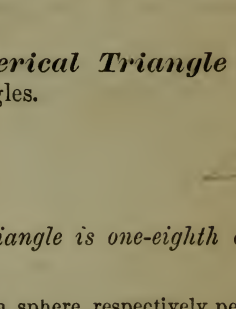

FIG. 330.

578. COR.—*The trirectangular triangle is equilateral and its sides are quadrants.*

PROPOSITION XV.

579. Theorem.—*In an isosceles spherical triangle the angles opposite the equal sides are equal;* and, conversely, *If two angles of a spherical triangle are equal, the triangle is isosceles.*

DEM.—Let ABC be an isosceles spherical triangle in which AB = AC; then angle ABC = ACB. For, draw the radii AO, CO, and BO, forming the edges of the triedral O-ABC. Now, since AB = AC, the facial angles AOC and AOB are equal, and the triedral is isosceles. Hence the diedrals A-OB-C and A-OC-B are equal (*442*), and consequently the spherical angles ABC and ACB are equal (*558*). Again, if angle ABC = angle ACB, side AC = side AB. For in the triedral O-ABC, the diedrals A-OB-C and A-OC-B are equal, whence the facial angles AOB and AOC are equal (*443*), and consequently the sides AB and AC which measure these angles.

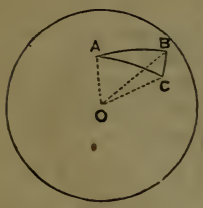

FIG. 331.

580. COR.—*An equilateral spherical triangle is also equiangular;* and, conversely, *If the angles of a spherical triangle are equal the triangle is equilateral.*

PROPOSITION XVI.

581. Theorem.—*On the same or on equal spheres two isosceles triangles having two sides and the included angle of the one equal to two sides and the included angle of the other, each to each, can be superimposed, and are consequently equal.*

DEM.—In the triangles ABC and AB′C′, let AB = AC, AB′ = AC′; and let AB = AB′, BC = B′C′, and angle ABC = AB′C′; then can the triangle AB′C′ be superimposed upon ABC. For, since the triangles are isosceles, we have angle ABC = ACB, AB′C′ = AC′B′, and, as by hypothesis ABC = AB′C′, these four angles are equal each to each. For a like reason AB = AC = AB′ = AC′. Now, applying AC′ to its equal AB, the extremity A at A and C′ at B, with the angle B′ on the same side of AB as C, the convexities of the arcs AC′ and AB being the same, and in the same direction, the arcs will coincide. Then, as angle AC′B′ = ABC, C′B′ will take the direction BC, and since these arcs are equal by hypothesis, B′ will fall at C. Hence B′A will fall in CA, as only one arc of a great circle can pass between C and A, and the triangle AB′C′ is superimposed upon ABC; wherefore they are equal. [Let the student give the application when other parts are assumed equal.]

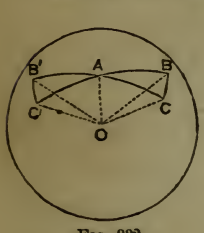

FIG. 332.

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582. Symmetrical Spherical Triangles are such as have the parts (sides and angles) of the one respectively equal to the parts of the other, but arranged in a different order, so that the triangles are not capable of superposition.

ILL.—In *Fig.* 333, ABC and A′B′C′ represent symmetrical spherical triangles. In these triangles A = A′, B = B′, C = C′, AC = A′C′, AB = A′B′, and BC = B′C′; nevertheless we cannot conceive one triangle superimposed upon the other. Thus, were we to make the attempt by placing A′B′ in its equal AB, A′ at A, and B′ at B, the angle C′ would fall on the opposite side of AB from C. Now, we cannot revolve A′C′B′ on AB (or its chord), and thus make the two coincide, for this would bring their convexities together. Nor can we make them coincide by reversing A′B′C′, and placing B′ at A, and A′ at B. For, although these two arcs will thus coincide, as the angle B′ is not equal to A, B′C′ will not fall in AC; and, again, if it did, C′ would not fall at C, since B′C′ and AC are not equal.

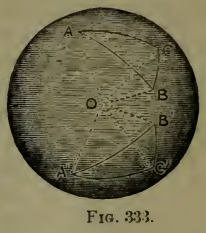

FIG. 333.

But, considering the triangles ABC and A′B′C′ in *Fig.* 334, in which A = A′, B = B′, C = C′, AC = A′C′, AB = A′B′, and BC = B′C′, we can readily conceive the latter as superimposed upon the former. [The student should make the application.] Now, the two triangles are equal in each case, as will subsequently appear of the former. Such triangles as those in *Fig.* 333 are called *symmetrically equal*, while the latter are said to be equal by *superposition*.

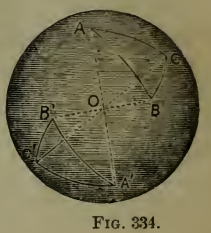

FIG. 334.

Fig. 335 represents the same triangles as *Fig.* 334, and exhibits a complete projection* of the semicircumferences of which the sides of the triangles are arcs. The student should become perfectly familiar with it, and be able to draw it readily. Thus, *a*AB*b* is the projection of the semicircumference of which AB is an arc, *a*AC*c* of the semicircumference of which AC is an arc, etc., etc.

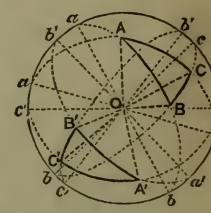

FIG. 335.

PROPOSITION XVII.

583. Theorem.—*Symmetrical spherical triangles are equivalent.*

* To understand what is meant by the projection of these lines, conceive a hemisphere with its base on the paper, and represented by the circle *abc*, and all the arcs raised up from the paper as they would be on the surface of such a hemisphere. Thus, considering the arc *a*AB*b*, the ends *a* and *b* would be in the paper just where they are, but the rest of the arc would be off the paper, as though you could take hold of B and raise it from the paper while *a* and *b* remain fixed. The lines in the figure are representations of lines on the surface of such a hemisphere, as they would appear to an eye situated in the axis of the circle *abc*, and at an infinite distance from it; that is, just as if each point in the lines dropped *perpendicularly* down upon the paper. Arcs of great circles perpendicular to the base are projected in straight lines passing through the centre, and oblique arcs are projected in ellipses. See *Spherical Trigonometry* (*97-109*).

DEM.—Let ABC and A′B′C′ be two symmetrical spherical triangles, with AB = A′B′, AC = A′C′, BC = B′C′, A = A′, B = B′, and C = C′; then are they equivalent.

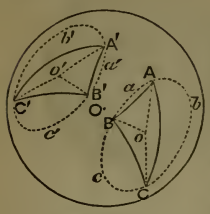

FIG. 336.

For, pass circumferences of small circles through the vertices A, B, C and A′, B′, C′, as *abc* and *a′b′c′*, of which *o* and *o′* are the poles. [The student should execute this on the spherical blackboard.] Now, by reason of the mutual equality of the sides, the *chord* AC = *chord* A′C′, *chord* AB = *chord* A′B′, and *chord* BC = *chord* B′C′, and as the small circles are circumscribed about the equal *plane* triangles ABC and A′B′C′, these circles are equal. Hence, *o*A = *o′*A′ = *o*B = *o′*B′ = *o*C = *o′*C′. The triangle A*o*B is therefore equal to A′*o′*B′, B*o*C = B′*o′*C′, and A*o*C = A′*o′*C′. [The student should make the application of these equal triangles.] Hence, ABC is equivalent to A′B′C′, as the two are composed of equal parts.

If the poles of the small circles fell without the given triangles, ABC would be equivalent to the sum of two of the partial triangles minus the third.

PROPOSITION XVIII.

584. Theorem.—*On the same or equal spheres, two spherical triangles having two sides and the included angle of the one equal to two sides and the included angle of the other, each to each, are equal, or symmetrical and equivalent.*

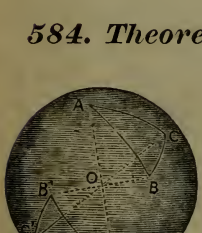

FIG. 337.

DEM.—Let ABC and A′B′C′, *Fig.* 337, be two spherical triangles having AB = A′B′, AC = A′C′, and A = A′. In this case, as the parts are similarly arranged, by placing AC in its equal A′C′, AB will fall in its equal A′B′ (as A = A′), and the two triangles will coincide. Hence, they are equal. Again, let the two triangles be ABC and A′B′C′, *Fig.* 338, in which AB = A′B′, AC = A′C′, and A = A′, the parts not being similarly arranged, so that the triangles are incapable of superposition. Thus, if AB is placed in its equal A′B′, A at A′, and B at B′, C and C′ will fall on *opposite* sides of AB. We may, however, construct ABC, *Fig.* 337, symmetrical with A′B′C′ in this figure, and apply ABC of *Fig.* 338 to it, and find that they coincide. Now, ABC, *Fig.* 337, and A′B′C′, *Fig.* 338, are equivalent (***583***); hence ABC, *Fig.* 338, is equivalent to A′B′C′, *Fig.* 338.

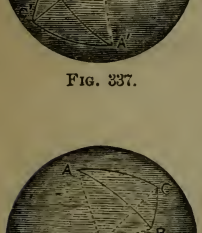

FIG. 338.

585. SCH.—This proposition is virtually the same as (***446***) concerning triedrals. Thus, in *Fig.* 337, drawing the radii AO, BO, CO, A′O, B′O, and C′O, two

triedrals are formed, having the facial angle AOB = A'OB, AOC = A'OC', the included diedrals equal, and the parts similarly disposed, whence the triedrals are equal. In like manner the triedral O-ABC, *Fig.* 338, is symmetrical and equivalent to O-A'B'C', *Fig.* 338. Hence, in either case, all the parts of one spherical triangle are equal to all the parts of the other, each to each.

PROPOSITION XIX.

586. Theorem.—*On the same, or on equal spheres, two spherical triangles having two angles and the included side of the one equal to two angles and the included side of the other, each to each, are equal, or symmetrical and equivalent.*

DEM.—Using the same triangles as in the preceding proposition, the student should be able to make the application directly, when the parts are similarly disposed; and when not similarly disposed, he should be able to show that ABC, of *Fig.* 338, can be applied to ABC, *Fig.* 337, symmetrical with A'B'C', *Fig.* 338.

587. SCH.—This proposition is also virtually the same as (***447***) concerning triedrals. Let the student point out the identity.

PROPOSITION XX.

588. Theorem.—*On the same, or on equal spheres, if two spherical triangles have two sides of the one equal to two sides of the other, each to each, and the included angles unequal, the third sides are unequal, and the greater third side belongs to the triangle having the greater included angle.* Conversely, *If the two sides are equal, each to each, and the third sides unequal, the angles included by the equal sides are unequal, and the greater belongs to the triangle having the greater third side.*

DEM.—In the triangles ABC and A'B'C', let AB = A'B', AC = A'C', and A > A'; then is BC > B'C'. For, join the vertices with the centre, forming the two triedrals O-ABC and O-A'B'C'. In these triedrals AOB = A'OB', AOC = A'OC', being measured by equal arcs; and C-AO-B > C'-A'O-B', having the same measures as A and A' (***558***). Hence COB > C'OB' (***449***). Therefore CB, the measure of COB, is greater than C'B', the measure of C'OB'.

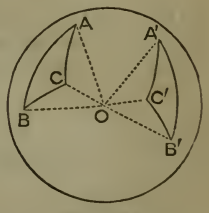

FIG. 339.

In like manner, the same sides of the triangles, and consequently the same facial angles of the triedrals, being granted equal, and

BC > B′C′, A > A′. For, BC being greater than B′C′, COB > C′OB′; whence B-AO-C > B′-A′O-C′ (*450*), or A is greater than A′.

PROPOSITION XXI.

589. Theorem.—*On the same, or on equal spheres, two spherical triangles having the sides of the one respectively equal to the sides of the other, or the angles of the one respectively equal to the angles of the other, are equal, or symmetrical and equivalent.*

Dem.—The sides of the triangles being equal, the facial angles of the triedrals at the centre are equal, whence the triedrals are equal or symmetrical (*451*). Consequently the angles of the triangles are equal, and the triangles are equal, or symmetrical and equivalent.

Again, the triangles being mutually equiangular, the triedrals have their diedrals mutually equal; whence the triedrals are equal or symmetrical (*452*). Therefore, the sides of the triangles are mutually equal, and the triangles are equal, or symmetrical and equivalent. (See *Figs.* 333, 334.)

PROPOSITION XXII.

590. Theorem.—*On spheres of different radii, mutually equiangular triangles are similar* (not equal).

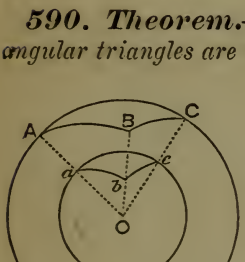

Fig. 340.

Dem.—Let O be the common centre of two unequal spheres; and let ABC be a spherical triangle on the surface of the outer. Draw the radii AO, BO, and CO, constructing the triedral O-ABC. Now, the intersections of these faces with the surface of the inner sphere will constitute a triangle which is mutually equiangular with ABC. Thus, A = *a*, B = *b*, and C = *c*, since in each case the corresponding diedrals are the same. From the similar sectors *a*O*b*, AOB, we have *ab* : AB :: *a*O : AO; and, in like manner, *ac* : AC :: *a*O : AO. Whence, *ab* : AB :: *ac* : AC. So, also, *ab* : AB :: *b*O : BO, and *bc* : BC :: *b*O : BO; whence, *ab* : AB :: *bc* : BC. Thus we see that ABC and *abc*, having their angles equal each to each, have also their sides proportional: therefore they are similar.

POLAR OR SUPPLEMENTAL TRIANGLES.

591. One triangle is polar to another when the vertices of one are the poles of the sides of the other. Such triangles are also

called *supplemental*, since the angles of one are the supplements of the sides opposite in the other, as will appear hereafter.

PROPOSITION XXIII.

592. Problem.—*Having a spherical triangle given, to draw its polar.*

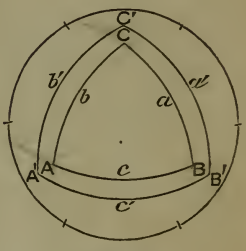

Fig. 341.

Solution.—Let ABC be the given triangle.* From A as a pole, with a quadrant strike an arc, as C'B'. From B as a pole, with a quadrant strike the arc C'A'; and from C, the arc A'B'. Then is A'B'C' polar to ABC.

593. Cor.—*If one triangle is polar to another*, conversely, *the latter is polar to the former;* i. e., *the relation is reciprocal.*

Thus, A'B'C' being polar to ABC; reciprocally, ABC is polar to A'B'C'; that is, A' is the pole of CB, B' of AC, and C' of AB. For every point in A'B' is at a quadrant's distance from C, and every point in A'C' is at a quadrant's distance from B. Hence, A' is at a quadrant's distance from the two points C and B of CB, and is therefore its pole. [In like manner the student should show that B' is the pole of AC, and C' of AB.]

594. Scil.—By producing each of the arcs struck from the vertices of the given triangles sufficiently, *four* new triangles will be formed, viz., A'B'C', QC'B', PC'A', and RA'B'. Only the first of these is called polar to the given triangle. It is easy to observe the relation of any of the parts of any one of the other three triangles to the parts of the polar. Thus, $QC' = 180° - b'$, $QB' = 180° - c'$, $QC'B' = 180° - B'C'A'$, $QB'C' = 180° - C'B'A'$, and $Q = A' = 180° - a$, as will appear hereafter.

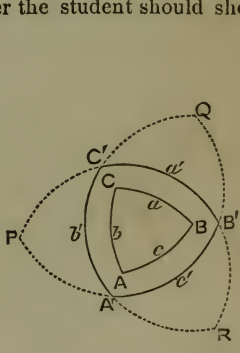

Fig. 342.

* This should be executed on a sphere. Few students get clear ideas of polar triangles without it. Care should be taken to construct a variety of triangles as the given triangle, since the polar triangle does not always lie in the position indicated in the figure here given. Let the given triangle have one side considerably greater than 90°, another somewhat less, and the third quite small. Also, let each of the sides of the given triangle be greater than 90°.

PROPOSITION XXIV.

595. Theorem.—*Any* ANGLE *of a spherical triangle is the supplement of the* SIDE *opposite in its polar triangle; and any* SIDE *is the supplement of the* ANGLE *opposite in the polar triangle.*

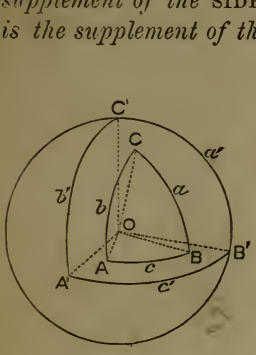

FIG. 343.

DEM.—Let ABC and A'B'C' be two spherical triangles polar to each other; and let the sides of each be designated as a, b, c, a', b', c', a being opposite A, a' opposite A', b opposite B, etc. Then $A = 180° - a'$, $B = 180° - b'$, $C = 180° - c'$, $a = 180° - A'$, $b = 180° - B'$, and $c = 180° - C'$.

For, join the vertices of the triangles with the centre of the sphere, thus forming the triedrals O-ABC, and O-A'B'C'. These triedrals are supplemental; for, A being the pole of C'B', AO is the axis of the great circle of which C'B' is an arc (?), hence is perpendicular to the plane C'OB', and consequently to OB' and OC' (?). In like manner, BO is perpendicular to the plane A'OC', and hence to OA' and OC'. So, also, CO is perpendicular to OA' and OB'. Now, these triedrals being supplementary, the diedral B-AO-C is the supplement of the facial angle C'OB' (***438***); or, since the diedral B-AO-C is the same as the spherical angle A, and the facial angle C'OB' is measured by a', A is the supplement of a', *i. e.*, $A = 180° - a'$. For like reasons, $B = 180° - b'$, and $C = 180° - c'$. [Let the student give them in full.] Again, the diedral B'-A'O-C' is the supplement of the facial angle COB (***438***); whence $A' = 180° - a$. In like manner $B' = 180° - b$, and $C' = 180° - c$.

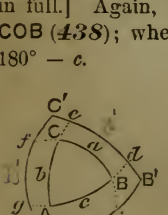

FIG. 344.

SECOND DEMONSTRATION.—Let ABC and A'B'C' be two polar triangles. Let CB, CA, and AB be represented by a, b, and c respectively, and C'B', C'A', and A'B' by a', b', and c'. To show that $A = 180° - a'$, produce b and c, if necessary, till they meet the side a', of the triangle polar to ABC, in e and d. Now A is measured by ed (***560***). But, since $C'd = 90°$, and $B'e = 90°$, $C'd + B'e$, or $C'B' + ed = 180°$; whence transposing, and putting a' for C'B', we have $ed = A = 180° - a'$. In like manner $C'g + A'f = C'A' + fg = 180°$; whence $fg = B = 180° - C'A'$, or $180° - b'$. So, also, $C = 180° - c'$. To show that $A' = 180° - a$, consider that A' being the pole of CB, fi is the measure of A'. Now $Bf = 90°$ (?), and $Ci = 90°$; whence $Bf + Ci = 180°$. But $Bf + Ci = fi + a$, wherefore $fi + a = 180°$, and transposing, and putting A' for fi, we have $A' = 180° - a$. In like manner we may show that $B' = 180° - b$, and $C' = 180° - c$. [The student should give the details.]

QUADRATURE OF THE SURFACE OF THE SPHERE.

596. The *Quadrature** of a surface is the same as finding its area. The term is applied under the conception that the process consists in finding a *square* which is equivalent to the given surface.

PROPOSITION XXV.

597. *Lemma.*—*The surface generated by the revolution of a regular semi-polygon of an even number of sides, about the diameter of the circumscribed circle as an axis, is equivalent to the circumference of the inscribed circle multiplied by the axis.***

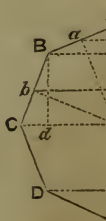

FIG. 345.

DEM.—Let ABCDE be one half of a regular octagon, AE being the diameter of the circumscribing circle. If the semi-perimeter ABCDE be revolved about AE as an axis, the surface generated will be $2\pi r \times AE$, r being the radius of the inscribed circle, as aO, or bO.

This surface is composed of the convex surfaces of cones and frustums of cones. Thus AB generates the surface of a cone, BC the frustum of a cone, etc. Let a and b be the middle points of AB and BC, and draw am, Bc, bn, and CO perpendicular to the axis, and Bd parallel to it. Also draw the radii of the inscribed circle, aO and bO. Indicate the surfaces generated by the sides, as *Surf.* AB, *Surf.* BC, etc. The areas of these surfaces are:

$$\textit{Surf.}\ AB = 2\pi \times am \times AB\ (\textbf{516}), \qquad (1)$$

$$\textit{Surf.}\ BC = 2\pi \times bn \times BC\ (\textbf{518}), \text{ etc.} \qquad (2)$$

Now, from the similar triangles Oam and BAc,

We have aO : AB :: am : Ac, or $2\pi \times a$O : AB :: $2\pi \times am$: Ac;

Whence $2\pi \times am \times AB = 2\pi r \times Ac$, putting r for aO.

Also, from the similar triangles Obn and CBd,

We have bO : BC :: bn : Bd (= cO), or $2\pi \times b$O : BC :: $2\pi \times bn$: cO;

Whence $2\pi \times bn \times BC = 2\pi r \times c$O, putting r for bO.

Substituting these values in (1) and (2), we obtain

$$\textit{Surf.}\ AB = 2\pi r \times Ac,$$

$$\textit{Surf.}\ BC = 2\pi r \times cO,$$

And, in like manner, $\textit{Surf.}\ CD = 2\pi r \times Op,$

And, $\textit{Surf.}\ DE = 2\pi r \times pE.$

Adding, $\textit{Surf.}\ ABCDE = 2\pi r\,(Ac + cO + Op + pE) = 2\pi r \times AE.$

Finally, since the same course of reasoning is applicable to the semi-polygons of 16, 32, 64, etc., sides, the truth of the proposition is established.

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* Latin *quadratus*, squared.

598. Sch.—This proposition is only a particular case of surfaces generated by *any* broken line revolving about an axis; and the general proposition can be established in a manner altogether similar to the method given above. But this case is all that we need for our present purpose.

PROPOSITION XXVI.

599. Theorem.—*The surface of a sphere is equivalent to four great circles; that is, to* $4\pi R^2$, R *being the radius of the sphere.*

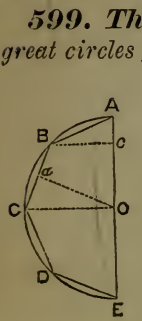

Fig. 346.

Dem.—Let the semicircumference ABCDE revolve upon the diameter AE, and thus generate the surface of a sphere. Conceive the half of a regular octagon inscribed in the semicircle. Call the radius of the inscribed circle, as aO, r, and let $AO = R$. The surface generated by the broken line ABCDE is, by the last proposition, $2\pi r \times 2R = 4\pi rR$. Now, conceive the arcs AB, BC, etc., bisected, and the chords drawn, and let r' be the radius of the circle inscribed in the regular polygon thus formed. The surface generated by this semi-polygon will be $4\pi r'R$. By repeating the bisections, the broken line approximates to the semicircumference, the radius of the inscribed circle to R, and the surface generated to the surface of the sphere, the three quantities reaching their limits at the same time. Hence *at* the limit we have

$$\textit{Surf.}\ \text{of sphere} = 2\pi R \times 2R = 4\pi R^2. \quad \text{Q. E. D.}$$

600. Cor. 1.—*The area of the surface of a sphere is equivalent to the circumference of a great circle multiplied by the diameter, that is,* $2\pi R \times 2R$, *as above.*

601. Cor. 2.—*The surfaces of spheres are to each other as the squares of their radii.*

Thus, if R and R′ are the radii of two spheres, the surfaces are $4\pi R^2$ and $4\pi R'^2$. Now, $4\pi R^2 : 4\pi R'^2 :: R^2 : R'^2$.

---

602. Def.—***A Zone*** is the portion of the surface of a sphere included between the circumferences of two parallel circles of a sphere. The *altitude* of a zone is the distance between the parallel circles forming its bases.

Ill.—The surface generated by arc CB, or any arc of the circle ABCDE, *Fig.* 346, in its revolution, conforms to the definition, and is a zone. Such a portion of the surface as is generated by AB is called a zone with one base, the circle whose circumference would form the upper base having become tangent to the sphere.

PROPOSITION XXVII.

603. Theorem.—*The area of a zone is to the area of the surface of the sphere as the altitude of the zone is to the diameter of the sphere; which gives for the area of a zone* $2\pi aR$, *a being the altitude of the zone, and* R *the radius of the sphere.*

Dem.—It is evident that in passing to the limit the surface generated by such a portion of the broken line as would lie between C and B, *Fig.* 346, would be measured by the circumference of the inscribed circle multiplied by cO. Hence, at the limit, the zone generated by arc BC is measured by $2\pi R \times cO$, that is, it is such a part of the surface of the sphere as cO is of AE, or 2R. Letting a represent the altitude cO, the fraction $\frac{a}{2R}$ represents the part of the surface of the sphere constituting the area of the zone. Hence, $4\pi R^2 \times \frac{a}{2R}$, which equals $2\pi aR$, is the area of the zone.

604. Cor.—*On the same or on equal spheres, zones are to each other as their altitudes.*

OF LUNES.

605. A Lune is a portion of the surface of a sphere included by two semicircumferences of great circles.

The surface $AmBn$ is a lune.

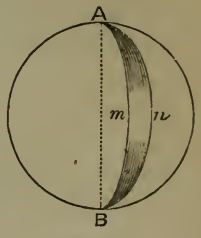

Fig. 347.

606. The *Angle of the Lune* is the angle included by the arcs which form its sides; or, what is the same thing, the measure of the diedral included between the great circles.

Thus, the spherical angle mAn, or the measure of the diedral m-AB-n, is the angle of the lune $AmBn$.

PROPOSITION XXVIII.

607. Theorem.—*The area of a lune is to the area of the surface of the sphere on which it is situated, as the angle of the lune is to four right angles.*

Dem.—Let ACEB be a lune whose angle is the spherical angle CAB, or what

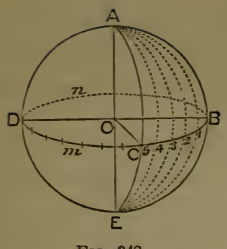

FIG. 348.

is the same thing, the plane angle BOC measured by the arc CB, of which A is the pole; then is

lune ACEB : *surface of sphere* :: CAB : 4 *right angles*.

For, suppose the arc CB commensurable with the circumference BC*m*D*n*, and suppose that they are to each other as 5 : 24. Dividing BC into 5 equal arcs, and the entire circumference BC*m*D*n* into 24 arcs of the same length, and passing arcs of great circles through A and these points of division, the lune will be divided into 5 equal lunes, and the entire surface into 24 equal lunes of the same size. That these lunes are equal to each other is evident from the fact that they are composed of equal isosceles triangles. Hence,

lune ACEB : *surface of sphere* :: 5 : 24.

Now, *angle* BOC : 4 *right angles* :: BC (= 5) : BC*m*D*n* (= 24).

Therefore, *lune* ACEB : *surface of sphere* :: BOC (or CAB) : 4 *right angles*, since the circumference measures 4 right angles.

If BC has no finite common measure with the circumference, we may divide it into any number of equal arcs, bisect these arcs, then bisect the last formed, and continue the process of bisection (in conception) to any required extent; and as, when any one of the arcs thus obtained is applied to the circumference, if it is not an exact measure, the remainder is less than the arc, we can continue the subdivision of BC (in conception) until this remainder is less than any assignable quantity. Hence, we may always consider the arc BC as commensurable with the circumference by making the measure infinitesimal.

608. COR.—*The sum of several lunes on the same sphere is equal to a lune whose angle is the sum of the angles of the lunes; and the difference of two lunes is a lune whose angle is the difference of their angles.*

609. SCH. 1.—The case in which the arc measuring the angle of the lune is incommensurable with the circumference, may be treated as in (***206***), by the method of reasoning called the *Reductio ad absurdum, i. e.*, by showing a thing to be true, since it would be absurd to suppose it untrue.

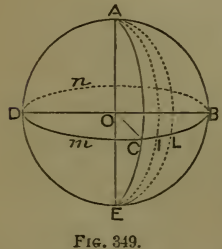

FIG. 349.

Thus, there is some arc to which the circumference bears the same ratio as the surface of the sphere does to the surface of the lune. If that arc be not BC let it be BL, an arc *less* than BC, so that

surface of sphere : *lune* ACEB :: BC*m*D*n* : BL. (1)

Conceive the circumference BC*m*D*n* divided into equal parts, each of which is less than CL, the assumed difference between BC and BL. Then conceive one of these equal parts applied to BC as a measure, beginning at B. Since the measure is less

than LC, one point of division, at least, will fall between L and C. Let I be such a point, and pass the arc of a great circle through A and I.

Now, *surface of sphere* : *lune* AIEB :: BC*m*D*n* : BI, (2)

since the arc BI is commensurable with the circumference. In (1) and (2), the antecedents being equal, the consequents should be proportional, hence we should have

lune ACEB : *lune* AIEB :: BL : BI.

But this is absurd, since *lune* ACEB > lune AIEB, whereas BL < BI. In a similar manner we can show that

surface of sphere is not to *lune* ACEB :: BC*m*D*n* : any arc greater than BC.

Hence, as the fourth term can neither be less nor greater than BC, it must be equal to BC, and we have

surface of sphere : *lune* ACEB :: BC*m*D*n* : BC,

i. e., as 4 right angles, to the angle of the lune.

610. Sch. 2.—*To obtain the area of a lune whose angle is known, on a given sphere*, find the area of the sphere, and multiply it by the ratio of the angle of the lune (in degrees) to 360°. Thus, R being the radius of the sphere, $4\pi R^2$ is the surface of the sphere; and the lune whose angle is 30° is $\frac{30}{360}$ or $\frac{1}{12}$ the surface of the sphere, *i. e.*, $\frac{1}{12}$ of $4\pi R^2 = \frac{1}{3}\pi R^2$.

PROPOSITION XXIX.

611. *Theorem.*—*If two semicircumferences of great circles intersect on the surface of a hemisphere, the sum of the two opposite triangles thus formed is equivalent to a lune whose angle is that included by the semicircumferences.*

Dem.—Let the semicircumferences CEB and DEA intersect at E on the surface of the hemisphere whose base is CABD; then the sum of the triangles CED and AEB is equivalent to a lune whose angle is AEB.

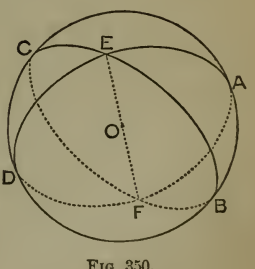

Fig. 350.

For, let the semicircumferences CEB and DEA be produced around the sphere, intersecting on the opposite hemisphere, at the extremity F of the diameter through E. Now, FBEA is a lune whose angle is AEB. Moreover, the triangle AFB is equivalent to the triangle DEC: since angle AFB = AEB = DEC, side AF = side ED, each being the supplement of AE; and BF = CE, each being the supplement of EB. Hence, the sum of the triangles CED and AEB is equivalent to the lune FBEA.

Q. E. D.

PROPOSITION XXX.

612. Theorem.—*The area of a spherical triangle is to the area of the surface of the hemisphere in which it is situated, as its spherical excess is to four right angles, or* 360°.

DEM.—Let ABC be a spherical triangle whose angles are represented by A, B, and C; then is

area ABC : *surf. of hemisphere* : : A + B + C − 180° : 4 *right angles*, or 360°.

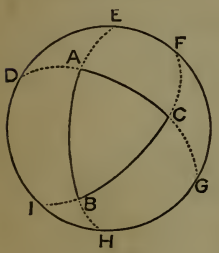

FIG. 351.

Let *lune* A represent the lune whose angle is the angle A of the triangle, *i. e.*, angle CAB, and in like manner understand *lune* B and *lune* C.

Now, triangle AHC + AED = *lune* A (***611***),

BHI + BEF = *lune* B,

CGF + CDI = *lune* C.

Adding, 2ABC + *hemisphere* = *lune* (A + B + C)*, (1) since the six triangles AHC, AED, BHI, BEF, CGF, and CDI, make the whole hemisphere and 2ABC besides, ABC being reckoned *three* times. From (1), we have by transposing and remembering that a hemisphere is a lune whose angle is 180°, and dividing by 2,

$$\text{ABC} = \tfrac{1}{2}\,\textit{lune}\,(\text{A} + \text{B} + \text{C} - 180°).$$

But, by (***607***),

$\frac{1}{2}$ *lune* (A + B + C − 180°) : *surf. of hemisph.* : : A + B + C − 180° : 4 *right angles.*

Therefore, ABC : *surf. of hemisph.* : : A + B + C − 180° : 4 *right angles.*

613. SCH. 1.—*To find the area of a spherical triangle* on a given sphere, the angles of the triangle being given, we have simply to multiply the area of the hemisphere, *i. e.*, $2\pi R^2$, by the ratio of the spherical excess to 360°. Thus, if the angles are A = 110°, B = 80°, and C = 50°, we have

$$\textit{area}\ \text{ABC} = 2\pi R^2 \times \frac{\text{A} + \text{B} + \text{C} - 180°}{360°} = 2\pi R^2 \times \frac{60}{360} = \tfrac{1}{3}\pi R^2.$$

614. SCH. 2.—This proposition is usually stated thus: *The area of a spherical triangle is equal to its spherical excess multiplied by the trirectangular triangle.* When so stated the spherical excess is to be estimated in terms of the right angle; *i. e.*, having subtracted 180° from the sum of its angles, we are to divide the remainder by 90°, thus getting the spherical excess in right angles. In the example in the preceding scholium, the spherical excess estimated in this way would be $\frac{110° + 80° + 50° - 180°}{90°} = \frac{2}{3}$; and the area of the triangle would

* This signifies the lune whose angle is A + B + C, which is of course the sum of the three lunes whose angles are A, B, and C.

be $\frac{2}{3}$ of the trirectangular triangle. Now, the trirectangular triangle being $\frac{1}{8}$ of the surface of the sphere (*577*) is $\frac{1}{8}$ of $4\pi R^2$, or $\frac{1}{2}\pi R^2$. This multiplied by $\frac{2}{3}$ gives $\frac{1}{3}\pi R^2$, the same as above.

The proportion,

$$ABC : \textit{surf. of hemisph.} :: A + B + C - 180^\circ : 360^\circ,$$

is readily put into a form which agrees with the enunciation as given in this scholium. Thus, *surf. of hemisph.* $= 2\pi R^2$, whence

$$ABC = 2\pi R^2 \times \frac{A+B+C-180^\circ}{360^\circ} = \tfrac{1}{2}\pi R^2 \times \frac{A+B+C-180^\circ}{90^\circ}.$$

VOLUME OF SPHERE.

PROPOSITION XXXI.

615. Theorem.—*The volume of a sphere is equal to the area of its surface multiplied by $\frac{1}{3}$ of the radius, that is, $\frac{4}{3}\pi R^3$, R being the radius.*

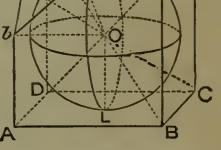

Fig. 352.

Dem.—Let OL = R be the radius of a sphere. Conceive a circumscribed cube, that is, a cube whose faces are tangent planes to the sphere. Draw lines from the vertices of each of the polyedral angles of the cube, to the centre of the sphere, as BO, CO, DO, AO, etc. These lines are the edges of six pyramids, having for their bases the faces of the cube, and for a common altitude the radius of the sphere (?). Hence the volume of the circumscribed cube is equal to its surface multiplied by $\frac{1}{3}$R.

Again, conceive each of the polyedral angles of the cube truncated by planes *tangent to the sphere.* A new circumscribed solid will thus be formed, whose volume will be nearer that of the sphere than is that of the circumscribed cube. Let *abc* represent one of these tangent planes. Draw from the polyedral angles of this new solid, lines to the centre of the sphere, as *a*O, *b*O, and *c*O, etc.; these lines will form the edges of a set of pyramids whose bases constitute the surface of the solid, and whose common altitude is the radius of the sphere (?). Hence the volume of this solid is equal to the product of its surface (the sum of the bases of the pyramids) into $\frac{1}{3}$R.

Now, this process of truncating the angles by tangent planes may be conceived as continued indefinitely; and, to whatever extent it is carried, it will *always* be true that the volume of the solid is equal to its surface multiplied by $\frac{1}{3}$R. Therefore, as the sphere is the limit of this circumscribed solid, we have the volume of the sphere equal to the surface of the sphere, which is $4\pi R^2$, multiplied by $\frac{1}{3}$R, *i. e.*, to $\frac{4}{3}\pi R^3$. Q. E. D.

616. COR.—*The surface of the sphere may be conceived as consisting of an infinite number of infinitely small plane faces, and the volume as composed of an infinite number of pyramids having these faces for their bases, and their vertices at the centre of the sphere, the common altitude of the pyramids being the radius of the sphere.*

617. A Spherical Sector is a portion of a sphere generated by the revolution of a circular sector about the diameter around which the semicircle which generates the sphere is conceived to revolve. It has a zone for its base; and it may have as its other surfaces one, or two, conical surfaces, or one conical and one plane surface.

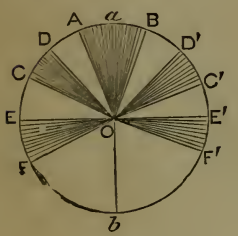

FIG. 353.

ILL.—Thus let *ab* be the diameter around which the semicircle *a*C*b* revolves to generate the sphere. The solid generated by the circular sector AO*a* will be a spherical sector having a zone (AB) for its base; and for its other surface, the conical surface generated by AO. The spherical sector generated by COD, has the zone generated by CD for its base; and for its other surfaces, the concave conical surface generated by DO, and the convex conical surface generated by CO. The spherical sector generated by EOF, has the zone generated by EF for its base, the plane generated by EO for one surface, and the concave conical surface generated by FO for the other.

618. A Spherical Segment is a portion of the sphere included by two parallel planes, it being understood that one of the planes may become a tangent plane. In the latter case, the segment has but one base; in other cases, it has two. A spherical segment is bounded by a zone and one, or two, plane surfaces.

PROPOSITION XXXII.

619. Theorem.—*The volume of a spherical sector is equal to the product of the zone which forms its base into one-third the radius of the sphere.*

DEM.—A spherical sector, like the sphere itself, may be conceived as consisting of an infinite number of pyramids whose bases make up its surface, and whose common altitude is the radius of the sphere. Hence, the volume of the sector is equal to the sum of the bases of these pyramids, that is, the spherical surface of the sector, multiplied by one-third their common altitude, which is one-third the radius of the sphere. Q. E. D.

620. Cor.—*The volumes of spherical sectors of the same or equal spheres are to each other as the zones which form their bases; and, since these zones are to each other as their altitudes (**604**), the sectors are to each other as the altitudes of the zones which form their bases.*

PROPOSITION XXXIII.

621. Theorem.—*The volume of a spherical segment of one base is* $\pi A^2(R - \frac{1}{3}A)$, A *being the altitude of the segment, and* R *the radius of the sphere.*

Dem.—Let CO = R, and CD = A; then is the volume of the spherical segment generated by the revolution of CAD about CO equal to $\pi A^2(R - \frac{1}{3}A)$.

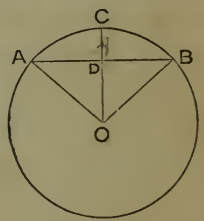

Fig. 354.

For, the volume of the spherical sector generated by AOC is the zone generated by AC, multiplied by $\frac{1}{3}R$, or $2\pi AR \times \frac{1}{3}R = \frac{2}{3}\pi AR^2$. From this we must subtract the cone, the radius of whose base is AD, and whose altitude is DO. To obtain this, we have DO $= R - A$: whence, from the right angled triangle ADO, $AD = \sqrt{R^2 - (R - A)^2} = \sqrt{2AR - A^2}$. Now, the volume of this cone is

$$\frac{1}{3}OD \times \pi\overline{AD}^2, \text{ or } \frac{1}{3}\pi(R - A)\,(2AR - A^2) = \frac{1}{3}\pi(2AR^2 - 3A^2R + A^3).$$

Subtracting this from the volume of the spherical sector, we have

$$\frac{2}{3}\pi AR^2 - \frac{1}{3}\pi(2AR^2 - 3A^2R + A^3) =$$
$$\pi(A^2R - \frac{1}{3}A^3) = \pi A^2(R - \frac{1}{3}A). \quad \text{Q. E. D.}$$

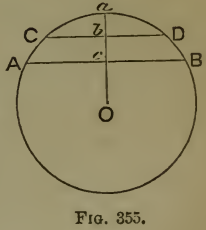

Fig. 355.

622. Sch.—The volume of a spherical segment with two bases is readily obtained by taking the difference between two segments of one base each. Thus, to obtain the volumes of the segment generated by the revolution of *b*CA*c* about *a*O, take the difference of the segments whose altitudes are *ac* and *ab*.

EXERCISES.

1. What is the circumference of a small circle of a sphere whose diameter is 10, the circle being at 3 from the centre?

Ans., 25.1328.

2. Construct on the spherical blackboard a spherical angle of 60°. Of 45°. Of 90°. Of 120°. Of 250°.

SUG's.—Let P be the point where the vertex of the required angle is to be situated. With a quadrant strike an arc from P, which shall represent one side of the required angle. From P as a pole, with a quadrant, strike an arc from the side before drawn, which shall measure the required angle. On this last arc lay off from the first side the measure of the required angle,* as 60°, 45°, etc. Through the extremity of this arc and P pass a great circle (*548*). [The student should not fail to give the reasons, as well as *do* the work.]

3. On the spherical blackboard construct a spherical triangle ABC, having AB = 100°, AC = 80°, and A = 58°.

4. Construct as above a spherical triangle ABC, having AB = 75°, A = 110°, and B = 87°.

5. Construct as above, having AB = 150°, BC = 80°, and AC = 100°. Also having AB = 160°, AC = 50°, and BC = 85°.

6. Construct as above, having A = 52°, AC = 47°, and CB = 40°.

SUG's.—Construct the angle A as before taught, and lay off AC from A equal to 47°, with the tape. This determines the vertex C. From C, as a pole, with an arc of 40°, describe an arc of a small circle; in this case this arc will cut the opposite side of the angle A in two places. Call these points B and B′. Pass circumferences of great circles through C, and B, and B′. There are two triangles, ACB and ACB′.

NOTE.—The teacher can multiply examples like the three preceding at pleasure. This exercise should be continued till the pupil can draw a spherical triangle as readily as a plane triangle.

7. What is the area of a spherical triangle on the surface of a sphere whose radius is 10, the angles of the triangle being 85°, 120°, and 150°? *Ans.*, 305.4 +.

8. What is the area of a spherical triangle on a sphere whose diameter is 12, the angles of the triangle being 82°, 98°, and 100°?

9. A sphere is cut by 5 parallel planes at 7 from each other. What are the relative areas of the zones? What of the segments?

10. Considering the earth as a sphere, its radius would be 3958 miles, and the altitudes of the zones, North torrid = 1578, North temperate = 2052, and North frigid = 328 miles. What are the relative areas of the several zones?

SUG.—The student should be careful to discriminate between the *width* of a zone, and its altitude. The altitudes are found from their widths, as usually given in degrees, by means of trigonometry.

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* For this purpose a tape equal in length to a semicircumference of a great circle of the sphere used, and marked off into 180 equal parts, will be convenient. A strip of paper may be used.

11. The earth being regarded as a sphere whose radius is 3958 miles, what is the area of a spherical triangle on its surface, the angles being 120°, 130°, and 150°? What is the area of a trirectangular triangle on the earth's surface?

12. Construct on the spherical blackboard a spherical triangle ABC, having $A = 59°$, $AC = 120°$, and $AB = 88°$. Then construct the triangle polar to ABC.

13. Construct triangles polar to each of those in Examples 3, 4, and 5.

14. In the spherical triangle ABC given $A = 58°$, $B = 67°$, and $AC = 81°$; what can you affirm of the polar triangle?

15. What is the volume of a globe which is 2 feet in diameter? What of a segment of the same globe included by two parallel planes, one at 3 and the other at 9 inches from the centre?

16. Compare the convex surfaces of a sphere and its circumscribed cylinder and cone, the vertical angle of the cone being 60°.

17. Compare the volumes of a sphere and its circumscribed cube, cylinder, and cone, the vertical angle of the cone being 60°.

18. If a and b represent the distances from the centre of a sphere whose radius is r, to the bases of a spherical segment, show that the volume of the segment is $\pi[r^2(b - a) - \frac{1}{3}(b^3 - a^3)]$. See (**621, 623**).

PART III.

AN ADVANCED COURSE IN GEOMETRY.

CHAPTER I.

EXERCISES IN GEOMETRICAL INVENTION.

SECTION I.

THEOREMS IN SPECIAL OR ELEMENTARY GEOMETRY.

623. This chapter will afford a review of Parts I. and II., while it will greatly extend the student's knowledge of geometrical facts. Great pains should be taken to secure good habits as to neatness of execution in the construction of figures, orderly and proper arrangement of thought, and in style of expression. The practice of constructing every figure upon geometrical principles—guessing at nothing—cannot be too strongly commended. As to the *form* of a geometrical argument, observe the following order:

1st. The enunciation of the theorem or problem in general terms.

2d. The elucidation of the general statement, by reference to the particular figure which it is proposed to use.

3d. A description of the figure, with reference to any auxiliary construction which is used in the demonstration or solution.

4th. The demonstration proper.

624. If two adjacent sides of a quadrilateral are equal each to each, and the other two adjacent sides equal each to each, the diagonals intersect at right angles.

SUG'S.—1st. Draw a quadrilateral having such sides as the data require, and draw its diagonals. 2d. State the proposition with reference to the figure.

3d. [In this case the regular third step is not required, as no auxiliary lines are necessary.] 4th. Prove that the diagonals are at right angles to each other. The demonstration is based upon a corollary in Section I., Part II., Chapter I.

625. Cor.—One of the diagonals is *bisected*. [State which one, and show why.]

---

626. If a parallelogram has one oblique angle, all its angles are oblique; and if it has one right angle, all its angles are right angles.

Sug's.—Let the student be careful to follow the order as heretofore given. No auxiliary construction is needed. The demonstration is based upon the doctrine of parallels.

---

627. The sum of three straight lines drawn from any point within a triangle to the vertices is less than the sum, and greater than the half sum of the three sides of the triangle.

Sug's.—The first statement is proved from (***276***) and the second from (***274***.)

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628. A line drawn from any angle of a triangle to the middle of the opposite side, is less than the half sum of the adjacent sides, and greater than the difference between this half sum and half the third side.

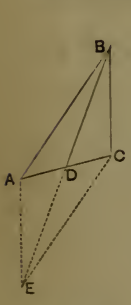

Fig. 356.

Sug's.—1st. Draw a triangle, as ABC, bisect one side, as AC, and draw BD. 2d. Make the statement with reference to the figure. 3d. Produce BD until DE = BD, and draw AE and EC. 4th. The first step in the proof is to show the triangle ADE equal to CBD, and ADB equal to DCE; whence AE = BC, and EC = AB.

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629. If lines be drawn from the extremities of either of the non-parallel sides of a trapezoid to the middle of the opposite side, the triangle thus formed is half the trapezoid.

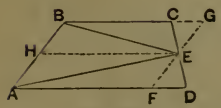

Fig. 357.

Sug's.—The third step, or construction, consists in drawing HE parallel to AD and hence to BC (?), and FG through E parallel to AB.

---

630. Any line drawn through the centre of the diagonal of a parallelogram bisects the figure.

631. Prove that the sum of the angles of a triangle is two right angles, by producing two of the sides about an angle and through this angle drawing a line parallel to the third side.

Prove the same by producing one side of the triangle and drawing a line through the exterior angle parallel to the non-adjacent side.

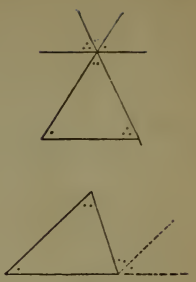

Fig. 358.

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632. If any point, not the centre, be taken in a diameter of a circle, of all the chords which can pass through that point, that one is the least which is at right angles to the diameter.

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633. If from any point there extend two lines tangent to a circumference, the angle contained by the tangents is double the angle contained by the line joining the points of tangency and the radius extending to one of them.

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634. The angle included by two lines drawn from any angle of a triangle, the one bisecting the angle and the other perpendicular to the opposite side, is half the difference of the other two angles of the triangle.

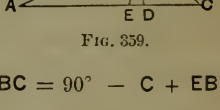

Fig. 359.

Sug's. ABD = 90° − A, whence ABD − EBD = 90° − A − EBD. Also, DBC = 90° − C, whence EBC = 90° − C + EBD = 90° − A − EBD, etc.

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635. If three lines be drawn from the acute angles of a right angled triangle—two bisecting these angles, and a third a perpendicular to one of the bisecting lines—the triangle included by these lines will be isosceles.

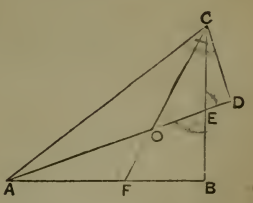

Fig. 360.

Sug's.—It is to be proved that OD = CD. COD = OAC + ACO = 45°, etc.

---

636. If one circumference be described on the radius of another as a diameter, any straight line extending from their point of contact to the outer circumference is bisected by the inner.

Sug.—The demonstration is based upon (***159***, ***211***).

637. Prove that the sum of the angles of a regular five point star (Fig. 101) is two right angles. Show, also, that the figure formed by the intercepted portions of the lines is a regular pentagon.

638. If the sides of a regular hexagon are produced till they meet, show that the exterior figures will be equilateral triangles.

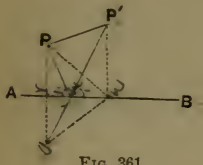

Fig. 361.

639. If from two given points on the same side of a given line, two lines be drawn meeting in the line, their sum is least when they make equal angles with the line.

640. If from two given points without a circumference, two lines be drawn meeting in the circumference, their sum is least when they make equal angles with a tangent at the common point, the points being on the opposite side of the tangent from the circle.

641. The side of an equilateral triangle inscribed in a circle is equal to the diagonal of a rhombus, whose other diagonal and each of whose sides are equal to the radius.

642. If two circumferences intersect each other, and from either point of intersection a diameter be drawn in each, the other extremities of these diameters and the other point of intersection are in the same straight line.

643. If any straight line joining two parallels be bisected, any other line through the point of bisection and included by the parallels, is bisected at the same point.

644. If the sides of any quadrilateral are bisected, the quadrilateral formed by joining the adjacent points of bisection is a parallelogram.

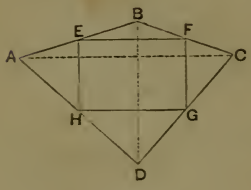

Fig. 362.

Sug's.—1st. Draw a quadrilateral, bisect its sides, and join the adjacent points of bisection. 2d. State the proposition, with reference to the figure. 3d. Draw the diagonals. 4th. Give the proof. It is based on the similarity of triangles.

645. Cor. 1.—The parallelogram is one-half the trapezium. Prove it. What figure is formed by joining the centres of EF, FC, and FC, HC, etc. ?

646. Cor. 2.—Lines joining the middle points of the opposite sides of any trapezium bisect each other (?).

---

647. If two straight lines join the alternate ends of two parallels, the line joining their centres is half the difference of the parallels.

Sug's.—We are to prove that EF $= \frac{1}{2}$ (CD − AB). CH = EF $= \frac{1}{2}$ (CD − AB).

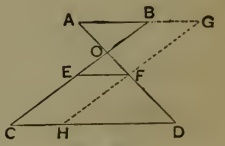

Fig. 363.

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648. In any right-angled triangle the line drawn from the right angle to the middle of the hypotenuse is equal to one-half the hypotenuse.

---

649. The perpendiculars which bisect the three sides of a triangle meet in a common point.

Sug's.—First show that the intersection of two of the perpendiculars is equally distant from the three vertices of the triangle. Then that a line drawn from this point to the middle of the third side is perpendicular to it.

---

650. The three perpendiculars drawn from the angles of a triangle upon the opposite sides intersect in a common point.

Sug's.—Draw through the vertices of the triangle lines parallel to the opposite sides. The proposition may then be brought under the preceding.

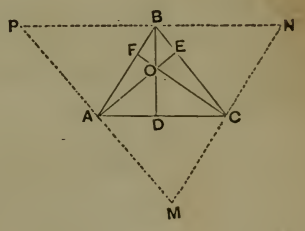

Fig. 364.

651. Cor.—The following triangles are similar—viz., BOE, BDC, AOD, and AEC, each to each; also BOF, BDA, DOC, and CFA. Prove it.

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652. If from a point without a circle two secants be drawn, making equal angles with a third secant passing through the same point and the centre of the circle, the intercepted chords of the first two are equal.

Sug.—Prove by revolving one part of the figure.

653. The sum of the alternate angles of *any* hexagon inscribed in a circle is four right angles.

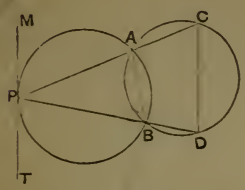

Fig. 365.

654. If two circles intersect in A and B, and from P, any point in one circumference, the chords PA and PB be drawn to cut the other in C and D, CD is parallel to a tangent at P.

---

655. If two lines intersect, two lines which bisect the opposite angles are perpendicular to each other.

---

656. The angle included by two lines drawn from a point within a triangle to the vertices of two of the angles, is greater than the third angle.

Sug's.—The demonstration may be founded on (***219***) or (***231***).

---

657. In a triangle whose angles are 90°, 60°, and 30°, show that the longest side is twice the shortest.

---

658. Lines which bisect the adjacent angles of a parallelogram are mutually perpendicular.

---

659. If from any point in the base of an isosceles triangle lines are drawn parallel to the sides, a parallelogram is formed whose perimeter is constant and equal to the sum of the two equal sides of the triangle.

Fig. 366.

660. If from any point in the base of an isosceles triangle perpendiculars be drawn to the sides of the triangle, their sum is *constant* and equal to the perpendicular from one of the equal angles of the triangle upon the opposite side.

Fig. 367.

661. If from any point within an equilateral triangle, three perpendiculars be let fall upon the sides, their sum is constant and equal to the altitude of the triangle.

662. If from a fixed point without a circle two tangents be drawn terminating in the circumference, the triangle formed by them and any tangent to the included arc has a constant perimeter equal to the sum of the first two tangents.

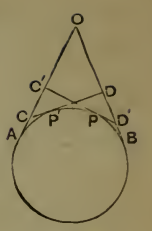

Fig. 368.

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663. The sum of two opposite sides of a quadrilateral circumscribed about a circle, is equal to the sum of the other two.

---

664. If two opposite angles of a quadrilateral are supplementary, it may be circumscribed by a circumference.

---

665. The square described on the sum of two lines is equivalent to the sum of the squares on the lines, *plus* twice the rectangle of the lines.

Sug's.—Be careful to give the construction fully, and show that the parts are rectangles, etc.

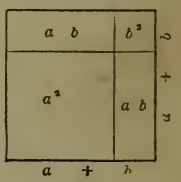

Fig. 369.

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666. The square described on the difference of two lines is equivalent to the sum of the squares on the lines, *minus* twice the rectangle of the lines.

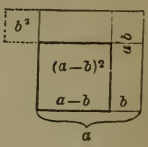

Fig. 370.

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667. The rectangle of the sum and difference of two lines is equivalent to the difference of the squares described on the lines.

Sch.—The three preceding propositions are but geometrical conceptions and demonstrations of the algebraic *formulæ*, $(a + b)^2 = a^2 + 2ab + b^2$, $(a - b)^2 = a^2 - 2ab + b^2$, and $(a + b)(a - b) = a^2 - b^2$.

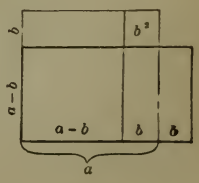


Fig. 371.

VARIOUS DEMONSTRATIONS OF THE PYTHAGOREAN PROPOSITION.

668. The square described on the hypotenuse of a right-angled triangle is equivalent to the sum of the squares described on the other two sides.

Fig. 372.

1st Method.—Let ABC be the given triangle, and ACED the square described on the hypotenuse. Complete the construction. Show that the four triangles are equal. The square HF is $(AB - BC)^2$. The student can complete the demonstration.

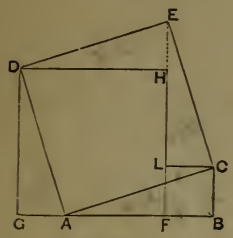


Fig. 373.

2d Method.—Let ACED be the square on the hypotenuse. Let fall the perpendiculars EF, DG, etc. Show that the three triangles are equal, and that FD and LB are the squares of the two sides AB and BC.

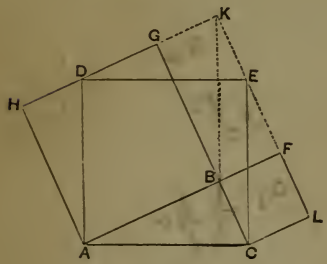


Fig. 374.

3d Method.—Let BL and BH be the squares on the sides. Produce FL and HG till they meet in K. Draw DA and EC perpendicular to AC, and draw DE and KB. Prove that ACED is a square, and also that the triangles ABC, CLE, BFK, KBG, DKE, and AHD are all equal to each other. The demonstration is then readily made.

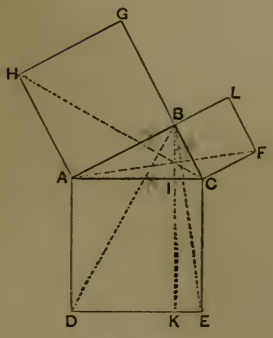
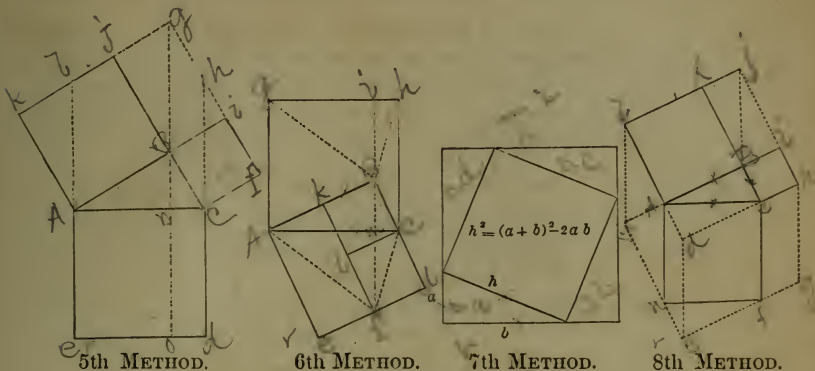


Fig. 375.

4th Method.—This is the demonstration usually given in our text-books. Drawing the squares on the three sides, let fall BI perpendicular to AC and produce it to K. Draw BD, BE, HC and AF. Show that the triangle HAC = BAD, and that the former is half the square AG, and the latter half the rectangle AK. Hence AG = AK. In like manner show that LC = CK.

We will now give a few other figures by means of which the demonstration can be effected, and leave the student to his own resources in effecting it.



5th METHOD. 6th METHOD. 7th METHOD. 8th METHOD.

FIG. 376.

9th METHOD.—The truth of the theorem appears also as a direct consequence of (***360***).

FIG. 377.

669. In an oblique angled triangle the square of a side opposite an acute angle is equivalent to the sum of the squares of the other two sides diminished by twice the rectangle of the base, and the distance from the acute angle to the foot of the perpendicular let fall upon the base from the angle opposite.

SUG'S.—It is to be shown that $\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 - 2AC \times DC$. Observe that $\overline{AD}^2 = (AC - DC)^2 = \overline{AC}^2 + \overline{DC}^2 - 2AC \times DC$. Whence, by a simple application of the preceding theorem, the truth of this becomes apparent.

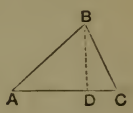


FIG. 378.

670. In an obtuse angled triangle the square of the side opposite the obtuse angle is equivalent to the sum of the squares on the other two sides, increased by twice the rectangle contained by the base and the distance from the obtuse angle to the foot of the perpendicular let fall from the angle opposite upon the base produced.

SUG.—The demonstration is analogous to the preceding, C being made obtuse in this case; whence $AD = AC + DC$, etc.

671. The following is an outline of a general demonstration covering the three preceding propositions:

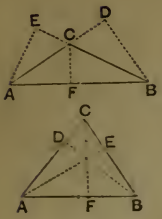


Fig. 379.

Letting AE, BD, and CF be the three perpendiculars from the angles upon the opposite sides, and observing that a circumference described on any side as a diameter passes through the feet of two of the perpendiculars, (***356***) and (***355***) readily give the following:

$$AB \times AF = AC \times AD = \overline{AC}^2 \pm AC \times CD,$$

$$\text{and } AB \times BF = BC \times BE = \overline{BC}^2 \pm BC \times CE;$$

adding, $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2 \pm 2AC \times CD$ (or $2BC \times CE$), the + sign being taken when C is obtuse, and the − sign when C is acute. If C is right CE and CD become 0, whence $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$.

672. Def.—The line drawn from any angle of a triangle to the middle of the opposite side is called a *medial line.*

673. The sum of the squares of any two sides of a triangle is equivalent to twice the square of the medial line drawn from their included angle, plus twice the square of half the third side.

Sug.—Proved by applying (***669, 670***).

674. The three medial lines of a triangle mutually trisect each other, and hence intersect in a common point.

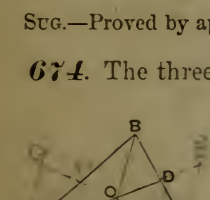


Fig. 380.

Sug's.—To prove that $OE = \frac{1}{3}BE$, draw FC parallel to AD until it meets BE produced. Then the triangles AEO and FEC are equal (?); whence EF = OE. Also, BO = OF (?).

Having shown that $OE = \frac{1}{3}BE$, by a similar construction we can show that $OD = \frac{1}{3}AD$.

Finally, we may show that the medial line from C to AB cuts off $\frac{1}{3}$ of BE, and hence cuts BE at the same point as does AD.

Another Dem.—Lines through O parallel to the sides trisect the sides, etc.

675. In any quadrilateral the sum of the squares of the sides is equivalent to the sum of the squares of the diagonals, plus four times the square of the line joining the centres of the diagonals.

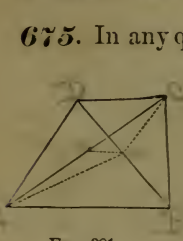


Fig. 381.

676. Cor.—The sum of the squares of the sides of a parallelogram is equivalent to the sum of the squares of the diagonals.

677. In any quadrilateral which may be inscribed in a circle, the product of the diagonals is equal to the sum of the products of the opposite sides.

---

678. In any triangle the rectangle of two sides is equivalent to the rectangle of the perpendicular let fall from their included angle upon the third side, into the diameter of the circumscribed circle.

FIG. 382.

SUG.—This proposition is an immediate consequence of the similarity of two triangles in the figure.

679. COR.—The area of a triangle is equivalent to the product of its sides divided by twice the diameter of the circumscribed circle.

---

680. If there be an isosceles and an equilateral triangle on the same base, and if the vertex of the inner triangle is equally distant from the vertex of the outer one and from the ends of the base, then, according as the isosceles triangle is the inner or the outer one, its base angle will be $\frac{1}{4}$ of, or $2\frac{1}{2}$ times the vertical angle.

---

681. Of all triangles on the same base, and having the same vertical angle, the isosceles has the greatest area.

SUG'S.—Describe a segment on the given base, which shall contain the given angle. The triangle on this base and having its vertex in the arc of the segment is the triangle to be considered.

---

682. Two triangles are similar, when two sides of one are proportional to two sides of the other, and the angle opposite to that side which is equal to or greater than the other given side in one, is equal to the homologous angle in the other.

ALGEBRAIC DEMONSTRATIONS.

683. The difference of the squares on any side of a regular pentagon and any side of regular decagon, inscribed in the same circle, is equivalent to the square of the radius.

SUG'S.—We will give the outline of what may be termed an *Algebraic Demonstration* of this proposition. This method is often the most convenient and ex-

peditious. Letting p represent a side of the pentagon, d a side of the decagon, and r the radius, the student should be able to discover the following relations:

(1) $r : d :: d : r - d$, or $r^2 - dr = d^2$;

(2) $\sqrt{d^2 - \frac{1}{4}p^2} + \sqrt{r^2 - \frac{1}{4}p^2} = r$.

From (2), $2r\sqrt{r^2 - \frac{1}{4}p^2} = 2r^2 - d^2 = r^2 + dr$, by substituting for d^2 its value from (1). Hence $4r^2 - p^2 = r^2 + 2dr + d^2$, or $3r^2 - 2dr = p^2 + d^2$. In this, substituting the value of dr as found in (1), we readily obtain $r^2 = p^2 - d^2$.

Q. E. D.

---

684. Demonstrate *algebraically* that the square on the sum of two lines, together with the square on the difference, is double the sum of the squares on the lines separately.

---

685. The sum of the squares of the three medial lines of a triangle is three-fourths of the sum of the squares of the sides.

---

686. The square of any side of a triangle is equivalent to twice the sum of the squares of the segments of the medial lines adjacent to its extremities, minus the square of the non-adjacent segment of the third medial line.

Deduced algebraically from the preceding.

---

687. The sum of the squares of the three greater segments of the medial lines of a triangle is equivalent to one-third the sum of the squares of the sides of a triangle.

Deduced algebraically from the preceding.

---

688. The lines from the vertices of a triangle to the points of tangency of the inscribed circle intersect in a common point.

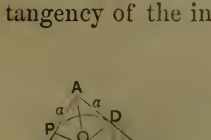


FIG. 383.

SUG's. DG is parallel to AF, $BD = BF = b$, $CF = CP = c$, $AD = AP = a$, $DG = d$, $FG = e$. $OF = \frac{dc}{c + e}$, $d = \frac{AF \times b}{a + b}$, $e = \frac{ab}{a + b}$. $\therefore OF = AF \times \frac{bc}{a(b + c) + bc}$. In like manner we may find where PB intersects AF, by drawing through P a parallel to AF. This distance is found to be $OF = AF \times \frac{bc}{a(b + c) + bc}$, a result which might have been anticipated, since b and c are similarly involved.

689. The area of a triangle, as expressed in terms of its sides, is *Area = the square root of the continued product of half the sum of the sides into this half sum minus each side separately.*

Sug's.—We will give the outlines of both the Geometric and Algebraic demonstrations:

1st. Geometric Demonstration. CD = CB, CE = CA, and through F, the centre of DA, HG is drawn parallel to AB. With F as a centre, and FH as a radius, a circumference passes through G (?). CN is perpendicular to AE and passes through H (?).

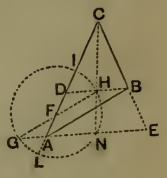


Fig. 384.

Now $CF = \frac{1}{2}(AC + CB)$, and $FL = \frac{1}{2}AB$ (?).

Hence $CL = \frac{1}{2}(AC + CB + AB) = \frac{1}{2}S$, S being the sum of the sides;

Hence, also, $DL = AI = \frac{1}{2}S - CB$, $CI = \frac{1}{2}S - AB$, and $AL = \frac{1}{2}S - AC$.

Again, $CN \times AN =$ area ACE;

and $HN \times AN =$ area ABE;

Subtracting, $AN(CN - HN) = AN \times CH =$ area ACB (1).

Once more, $CH \times DH =$ area CDB;

and $HN \times DH =$ area ADB.

Adding, $DH(CH + HN) = GA \times CN =$ area ACB (2).

Multiplying (1) and (2), we have

$$GA \times AN \times CH \times CN = (\text{area ACB})^2.$$

But $CN \times CH = CL \times CI$, and $GA \times AN = AL \times AI = AL \times DL$.

Therefore, *area* $ACB = \sqrt{CL \times CI \times AL \times DL} =$

$$\sqrt{\tfrac{1}{2}S(\tfrac{1}{2}S - AB)(\tfrac{1}{2}S - AC)(\tfrac{1}{2}S - CB)}.$$

2d. Algebraic Demonstration. From the right angled triangles BCD and ACD, we find $m = \dfrac{a^2 - b^2 + c^2}{2c}$.

Whence $p = \sqrt{a^2 - \left(\dfrac{a^2 - b^2 + c^2}{2c}\right)^2}$, and

Fig. 385.

$$\text{area ABC} = \frac{c}{2}\sqrt{a^2 - \left(\frac{a^2 - b^2 + c^2}{2c}\right)^2}$$

$$= \tfrac{1}{4}\sqrt{-a^4 + 2a^2b^2 + 2a^2c^2 - b^4 + 2b^2c^2 - c^4}$$

$$= \tfrac{1}{4}\sqrt{(a + b + c)(a + b - c)(a - b + c)(-a + b + c)}$$

$$= \sqrt{\tfrac{1}{2}s(\tfrac{1}{2}s - c)(\tfrac{1}{2}s - b)(\tfrac{1}{2}s - a)}.$$

690. From any point in the plane of a circle the greatest and least distances to the circumference are measured on the line passing through the centre.

Sug's.—There are three cases :—1st. When the point is without the circle. 2d. When the point is within. 3d. When the point is in the circumference.

Fig. 386.

691. From any point except the centre of a circle, two, and only two, equal lines can be drawn to the circumference.

Sug.—This is a direct consequence of (***181***, ***182***).

692. If two opposite sides of a parallelogram be bisected, straight lines from the points of bisection to the opposite vertices will trisect the diagonal.

693. The feet of two perpendiculars let fall from two given points upon a given line are equally distant from the middle of the line joining the points.

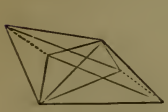


Fig. 387.

694. Two quadrilaterals are equivalent when their diagonals are respectively equal, and form equal angles.

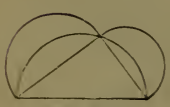


Fig. 388.

695. If, on the hypotenuse and sides of a right angled triangle, semicircumferences be described, that upon the hypotenuse passing through the vertex, the sum of the crescents thus formed will be equal to the area of the triangle.

696. The bisectors of any two exterior angles of a triangle meet in a point which is the centre of a circle, to which one side of the triangle and the other two produced are tangents.

These circles are called the *escribed circles.*

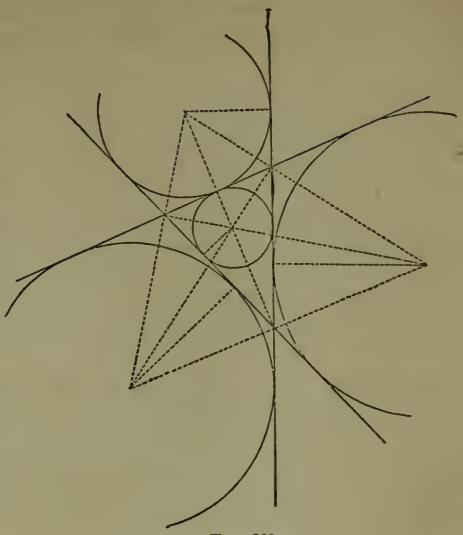


FIG. 389.

THE NINE POINTS CIRCLE.

697. In any triangle the centres of the THREE sides, the feet of the THREE perpendiculars from the vertices upon the opposite sides or sides produced, and the THREE middle points of the distances from the vertices to the common intersection of the perpendiculars, are NINE points in the circumference of one and the same circle; the centre of this circle is at the middle of the line joining the centre of the circumscribed circle and the common intersection of the perpendiculars; and the radius is half the radius of the circumscribed circle.

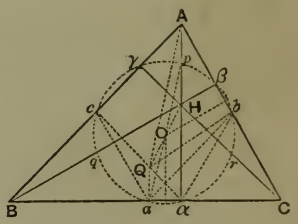


FIG. 390.

SUG'S.—The student will do well to confine his attention in the first instance to the first figure, and after he sees the demonstration in this case—*i. e.*, when the perpendiculars fall within, to trace it in the case of an obtuse angled triangle, in which the perpendiculars fall on the sides produced

1st. To show that the circle which passes through *a*, *b*, and *c*, also passes through α, we show the following relations among the angles: *cab* = *cAb* =

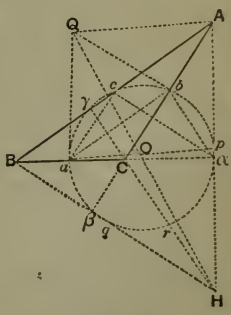


FIG. 391.

$cA\alpha + \alpha Ab = c\alpha b$. Hence, the vertices a and α are in the same circumference. In like manner we show that β and γ are in the circumference passing through a, b, and c.

2d. Considering one of the partial triangles as BHC, α, β, and γ are the feet of the three perpendiculars from *its* vertices upon one of its sides and the prolongation of the other two. Therefore, by the first part r and q are the middle points of CH and BH. Considering either of the other partial triangles we find p the centre of AH.

3d. $a\alpha$ and $b\beta$ being chords of the nine points circle, O is its centre, and letting Q be the centre of the circumscribed circle, we may readily show that O is *in* QH, and also is at its middle point.

4th. Drawing aO, and producing it, we may show that it intersects AH in p, and hence pH = Ap = Qa, and AQap is a parallelogram. Therefore Op = $\frac{1}{2}pa$ = $\frac{1}{2}$QA.

698. Cor.—The middle points of the three lines joining the centres, two and two, of the escribed circles of a triangle, and the middle points of the three lines joining the centres of the escribed circles with the centre of the inscribed circle, are six points in the circumference of the circle circumscribed about the same triangle.

---

699. If one triangle be inscribed in another, the circumferences circumscribing the three exterior triangles thus formed intersect in a common point.

Sug.—The demonstration is founded on the property of the opposite angles of an inscribed quadrilateral. The construction lines extend from the vertices of the inscribed triangle to the intersection to be examined.

---

700. The difference between the hypotenuse and the sum of the other two sides of a right angled triangle is equal to the diameter of the inscribed circle.

Fig. 392.

701. If from the extremities of any side of a triangle two lines be drawn, one bisecting an interior and the other an exterior angle, these lines will meet if sufficiently produced, and their included angle will be half the third angle of the triangle.

---

702. An inscribed equilateral triangle is one-fourth the circumscribed equilateral triangle about the same circle.

---

703. The three altitudes of a triangle are to each other inversely as the sides upon which they fall.

704. The bisectors of the angles included by the opposite sides of an inscribed quadrilateral intersect at right angles.

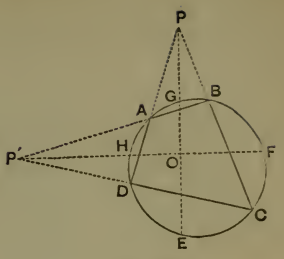


Fig. 393.

Sug.—By means of (***214***) show that FC + CE + HA + AG = 180°. Whence FOE = 90°.

---

705. Two triangles which have an angle in each equal, are to each other as the rectangle of the sides including the equal angle.

Sug's. A and D being equal, we are to show that ABC : DEF :: AB × AC : DE × DF. Take AE′ = DE, AF′ = DF, and draw E′F′. Now from the facts that the triangles AE′F′ and DEF are equal, and that triangles of the same altitudes are to each other as their bases, the proposition is proved.

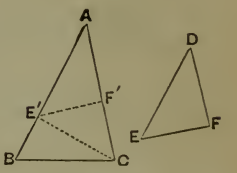


Fig. 394

---

706. If of the four triangles into which the diagonals divide a quadrilateral, two opposite ones are equivalent, the figure is a trapezoid.

---

707. The difference between the angles which a bisector in a triangle makes with the side to which it is drawn, is equal to the difference of the angles of the triangle including this side.

---

708. If any number of equal right lines radiate from a common point, making equal consecutive angles, and any line be drawn through the common point, the sum of the perpendiculars upon this line from the extremities of the radiating lines on one side, is equal to the sum of those on the other side.

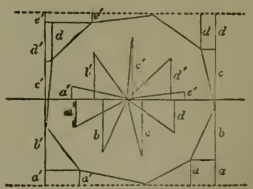


Fig. 395.

709. Cor.—In any regular polygon, the sum of the perpendiculars let fall from the vertices on the one side of any line passing through the centre, is equal to the sum of those let fall from the vertices on the other side.

710. If the sum of two opposite sides of a quadrilateral is equal to the sum of the other two opposite sides, a circle may be inscribed in it.

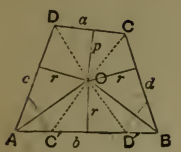


FIG. 396.

SUG'S.—Bisect any two adjacent angles, as A and B. Then are the perpendiculars r, r, r, equal (?); and it remains to be shown that the perpendicular $p = r$. Take AD′ = AD, and BC′ = BC, and draw OD′ and OC′. Since $a + b = c + d$, and C′D′ $= b - (b - c) - (b - d)$, C′D′ $= a$, and the triangles DOC and D′OC′ are equal. Hence $p = r$.

---

711. If two planes are parallel, any right line which pierces one, pierces the other also.

SUG.—Proof based on (***410***).

---

712. If two planes are parallel, any plane which intersects one, intersects the other also, and the lines of intersection are parallel.

713. COR.—Two planes which are parallel to a third, are parallel to each other.

---

714. A plane which is perpendicular to a line of another plane, or to a line parallel to that plane, is itself perpendicular to the latter plane.

---

715. If a straight line is perpendicular to a plane, any line parallel to the plane is perpendicular to the first line.

SUG.—Two lines in space which are not in the same plane, are said to make the same angle with each other as two lines respectively parallel to them and both lying in one plane.

---

716. In order that a straight line be perpendicular to a plane, it is sufficient that it be perpendicular to two lines not parallel to each other, and situated in the plane or parallel to it.

---

717. If two right lines in space are perpendicular to each other (not necessarily intersecting), their projections on a plane parallel to either line are perpendicular to each other.

SUG.—The *Projection* here referred to is that which is called the *Orthographic Projection.* The proposition is not generally true of the *Perspective Projection*, *i. e.*, the spaces which the lines (considered as material) would appear to occupy if they were placed between the eye and the plane. (See Ex. 8, page 174, PART II.)

718. The angle of inclination (***392***, PART II.) of a line oblique to a plane, is less than the angle included between this line and any line of the plane, except its projection, which passes through the point in which the first line pierces the plane.

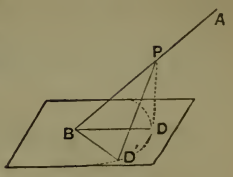


FIG. 397.

SUG. BD being the projection of AB, ABD < ABD', BD' being any line other than BD, passing through B.

---

719. Between any two lines not in the same plane, one line, and only one, can be drawn, which shall be perpendicular to both the given lines.

SUG'S.—Pass a plane through one of the lines parallel to the other; and through the other line pass a plane perpendicular to the first plane.

---

720. In a warped quadrilateral, *i. e.*, one whose sides do not all lie in the same plane, the middle points of the sides are in one plane, and are the vertices of the angles of a parallelogram.

SUG.—Conceive the planes of two opposite angles of the quadrilateral, the intersection of which will be a diagonal of the given quadrilateral.

---

721. A line being given in a plane, one plane can be drawn including the given line and perpendicular to the first plane, and only one. Hence all right diedrals are equal.

SUG.—Demonstration similar to (***390***).

---

722. The plane angle formed by drawing two lines in the faces of a diedral, from a common point in the edge, is less than the measure of the diedral if the angle is acute, and lines lie on the same side of the plane of the measure and are equally inclined to it, and greater if they lie on opposite sides.

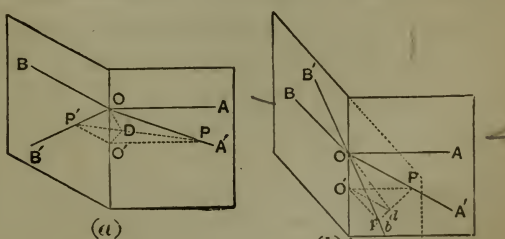


FIG. 398.

In general, if the diedral is acute, the limits of the varying angle are 0 and 90° when both the lines lie

on the same side of the plane of the measuring angle, and 180° when they lie on opposite sides. If the diedral is obtuse the limits are 0 and the angle itself when the lines lie on the same side of the measure, and 90° and 180° when they lie on opposite sides.

---

723. If the projections of a line in the two faces of a diedral are straight, the line is a straight line.

SUG.—Proof based on (*386*).

---

724. If from the vertex of a triedral a line be drawn at pleasure within the triedral, the sum of the plane angles formed by this line and any two edges is less than the sum of the facial angles formed by the other edge and these two.

---

725. If through a point in space two lines be drawn parallel to a given plane, and through the same point two planes be passed respectively perpendicular to the two lines, the intersection of these two planes will be perpendicular to the given plane.

---

726. The three planes which bisect the three diedrals of a triedral intersect in a common line.

---

727. In any convex polyedral, the sum of the diedrals is greater than the sum of the angles of a polygon having the same number of sides that the polyedral has faces.

SUG.—Proof based upon (*722*).

---

728. DEF.—*A Polyedron* is a solid bounded by plane surfaces. A *Regular Convex Polyedron* is a polyedron whose faces are all equal regular polygons, and each of whose solid angles is convex outward, and is enclosed by the same number of faces.

---

729. There are five and only five regular convex polyedrons—viz.: *The Tetraedron*, whose faces are four equal equilateral triangles; *The Hexaedron*, or *Cube*, whose faces are six equal squares; *The Octaedron*, whose faces are eight equal equilateral triangles; *The Dodecaedron*, whose faces are twelve equal regular pentagons; and *The Icosaedron*, whose faces are twenty equal equilateral triangles.

DEM.—We demonstrate this proposition by showing—1st, that such solids can be constructed ; and 2d, that no others are possible.

The Regular Tetraedron.—Taking three equal equilateral triangles, as ASB, ASC. and BSC, it is possible to enclose a solid angle, as S, with them, since the sum of the three facial angles is (what?) (PART II., ***436***). Then, since AC = AB = CB (?), considering ACB the fourth face, we have a regular polyedron whose four faces are equilateral triangles.

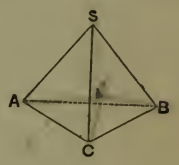


FIG. 399.

The Regular Hexaedron or *Cube.*—This is a familiar solid, but for purposes of uniformity and completeness we may conceive it constructed as follows: Taking three equal squares, as ASCB, CSED, and ASEF, we can enclose a solid angle, as S, with them (?). Now, conceive the planes of CB and CD, AB and AF, EF and ED produced. The plane of CB and CD being parallel to ASEF (?) will intersect the plane of EF and ED in HD parallel to FE (?). In like manner FH can be shown parallel to ED, BH to CD, and HD to BC. Hence the solid has for its faces six equal squares.

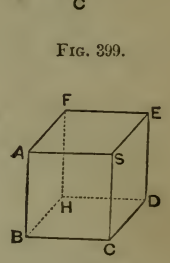


FIG. 400.

The Regular Octaedron.—At the intersection P, of the diagonals of a square, ABCD, erect a perpendicular SP to the plane of the square, and making SP = AP (half of one of the diagonals) draw SA, SD, SC, and SB. Making a similar construction on the other side of the plane ABCD, we have a solid having for faces eight equal equilateral triangles.

FIG. 401.

The Regular Dodecaedron.—Taking twelve equal regular pentagons, it is evident that we may group them in two sets of six each, as in the figure. Thus, around O we may place five, forming 5 triedrals at the vertices of O. These triedrals are possible, since the sum of the facial angles enclosing each is $3\frac{3}{5}$ right angles (?) —*i. e.*, between 0 and 4 right angles (PART II., ***436***). In like manner the other 6 may be grouped by placing 5 of them about O'. Now, conceiving the *convexity* of the group O in front and the *concavity* of group O', we may place the two together so as to inclose a solid. Thus, placing A at *b*, the three faces 5, 6, 1, will enclose a triedral, since the diedral included by 5 and 1 is the diedral of such a triedral. Then will vertex B fall at *c*, and a like triedral will be formed at that point, and so of all the other vertices. Hence we have a polyedron having for faces 12 equal regular pentagons.

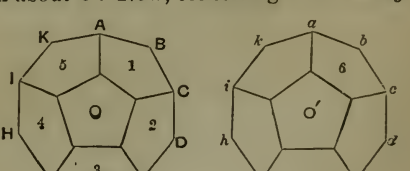


FIG. 402.

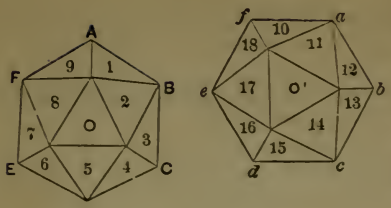


Fig. 403.

The Regular Icosaedron.—Taking 20 equal equilateral triangles, they can be grouped in two sets, as in the figure, in a manner altogether similar to the preceding case. The solid angles in this case are included by 5 facial angles whose sum is $3\frac{1}{3}$ right angles (?), which is a possible case (Part II., *436*). As before, conceiving the *convexity* of group O in front, and the *concavity* of O', we can place them together by placing A at *a*, thus enclosing a solid angle with 5 faces, whence B will fall at *b*, etc. Thus we obtain a solid with 20 equal equilateral triangles for its faces.

That there can be no other regular polyedrons than these 5 is evident, since we can form no other convex solid angles by means of regular polygons. Thus, with equilateral triangles (the simplest polygon) we have formed solid angles with 3 faces (the least number possible), as in the tetraedron; with 4, as in the octaedron; and with 5, as in the icosaedron. Six such facial angles cannot enclose a solid angle, since their sum is four right angles (?), and much less any greater number. Again, with squares (the next most simple polygon) we have formed solid angles with 3 faces as in the hexaedron, and can form no other, for the same reason as above. With regular pentagons we can only enclose a triedral, as in the dodecaedron, for a like reason. With regular hexagons we cannot enclose a solid angle (?), and much less with any regular polygon of more than six sides.

Sch.—Models of the regular polyedrons are easily formed by cutting the following figures from cardboard, cutting half-way through the board in the dotted lines, and bringing the edges together as the forms will readily suggest.

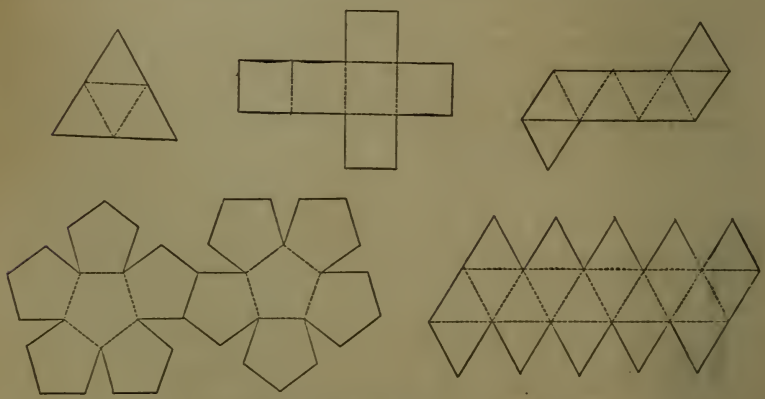


Fig. 404.

730. Any regular polyedron is inscriptible and circumscriptible by a sphere.

Sug's.—From the centres of any two adjacent faces, as c and c', let fall perpendiculars upon the common edge, and they will meet it in the same point o (?). The plane of these lines will be perpendicular to this edge (?), and perpendiculars to these faces from their centres, as cS, c'S, will lie in this plane (?), and hence will intersect at a point equally distant from these faces.

Fig. 405.

In like manner c''S = c'S, and the point S can be shown to be equally distant from all of the faces, and is therefore the centre of the inscribed sphere.

Joining S with the vertices, we can readily show that S is also the centre of the circumscribed sphere.

---

731. Show that a, being the edge of a regular tetraedron, its volume is $\frac{a^3\sqrt{2}}{12}$.

---

732. Def.—***A Truncated Prism*** is one whose upper and lower bases are not parallel.

733. The volume of a truncated triangular prism is equal to the sum of the volumes of three pyramids whose common base is the lower base of the prism, and whose vertices are the angles of the upper base.

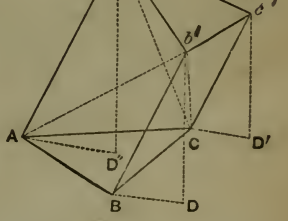


Fig. 406.

Sug's.—Let b'D, c'D', and aD'' be perpendicular to the lower base. Volume of b'-ABC is $\frac{1}{3}$ b'D × ABC. Volume a'-b'Cc' : volume b'-ABC :: $c'b'$C : b'BC :: c'C : b'B :: c'D' : b'D. ∴ Volume a-b'Cc' = $\frac{1}{3}c'$D' × ABC. In a similar manner volume b'-aAC = $\frac{1}{3}a'$D'' × ABC.

734. Cor.—The volume of a prism, one of whose bases is a right section and the other an oblique section, is the product of the right section into the arithmetical mean of its edges.

Sug's.—The volume of abc-ABC is as shown above ABC $\left(\frac{b\mathrm{D} + c\mathrm{D}' + a\mathrm{D}''}{3}\right)$. But if ABC is a right section, bD = bB, cD' = cC, and aD'' = aA. Hence the volume is ABC $\left(\frac{b\mathrm{B} + c\mathrm{C} + a\mathrm{A}}{3}\right)$.

735. The volume of any polyedron having for its bases any two polygons whatever, situated in parallel planes, and for lateral faces trapezoids, is the product of $\frac{1}{6}$ the distance between the bases into the sum of the two bases plus 4 times a section midway between the bases; or $V = \frac{H}{6}(B + B' + 4B'')$, in which H is the distance between the bases, B and B′ the bases, and B″ a section midway between the bases.

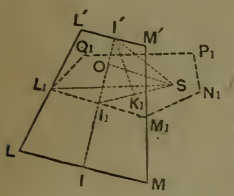


Fig. 407.

Dem.—Let $L_1M_1N_1P_1Q_1$ be the section of such a polyedron midway between its bases, and S any point in this section. Joining S with the vertices of the polyedron, we divide the solid into as many pyramids as it has faces. The volumes of the two which have B and B′ for their bases are evidently $\frac{1}{6}H \times B$, and $\frac{1}{6}H \times B'$. It remains to find the volume of the others.

Let LML′M′ be a lateral face corresponding to L_1M_1 and SO a perpendicular from S upon this face. Draw I′I through O perpendicular to LM, and consequently to L′M′. Take $I'K_1$ perpendicular to the plane section, whence $I'K_1 = \frac{1}{2}H$. Now the volume of the pyramid having L′M′LM for its base and S for its vertex is $L_1M_1 \times 2I'I_1 \times \frac{1}{3}SO$. But $I'I_1 \times SO = SI_1 \times I'K_1$ (?); whence the volume of this pyramid is $\frac{2}{3}L_1M_1 \times SI_1 \times I'K_1 = \frac{2}{3} \times 2SL_1M_1 \times I'K_1 = \frac{1}{3}I'K_1 \times 4SL_1M_1 = \frac{1}{6}H \times 4SL_1M_1$. In like manner the volume of the pyramid having for its base the face in which M_1N_1 is situated, can be shown to be $\frac{1}{6}H \times 4SM_1N_1$ and similarly of all the others. Whence the whole volume is $\frac{1}{6}H(B + B' + 4B'')$.

736. Cor.—The proposition is equally true when some or all of the lateral faces are triangles; *i. e.*, when one base has more sides than the other.

Sch.—The preceding propositions are of much value in calculating earth-work.

---

737. If we cut a pyramid by a plane parallel to its base, a second pyramid is formed similar to the first.

---

738. Two triangular pyramids are similar whenever they have an equal diedral angle contained between faces, similar each to each, and similarly placed.

---

739. Two polyedrons composed of the same number of tetraedrons, similar each to each, and similarly disposed, are similar.

740. All regular polyedrons of the same number of faces are similar solids.

---

741. The intersection of the surfaces of two spheres is the circumference of a circle whose plane is perpendicular to the line which joins their centres.

---

742. Through any four points not in the same plane one sphere may be made to pass, and only one.

Sug's.—The four points may be considered as the vertices of a tetraedron. Conceive perpendiculars drawn to the triangular faces from the intersections of lines drawn in these faces perpendicular to the sides at their middle points. These perpendiculars will meet at a common point (?), which is the centre of the circumscribed sphere (?).

[The student should show why *only* one sphere can be circumscribed.]

743. Cor. 1.—The four perpendiculars erected at the centres of the circles circumscribing the faces of a tetraedron intersect at a common point.

744. Cor. 2.—The six planes, perpendicular to the six edges of a tetraedron at their middle points, intersect at the centre of the circumscribed sphere.

---

745. One sphere and only one may be inscribed in any tetraedron.

Sug.—Bisect the diedrals with planes.

---

746. The angle included by any two curves intersecting on the surface of a sphere, is equal to the angle included by the arcs of two great circles passing through the point of intersection, and whose planes produced include the tangents to the curves at their intersection.

SECTION II.

PROBLEMS IN SPECIAL OR ELEMENTARY GEOMETRY.

747. To bisect the angle formed by two lines whose intersection is inaccessible.

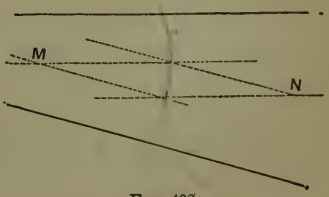


Fig. 408.

Sug.—M and N are points in the bisector.

748. To pass a circumference through three points, not in the same straight line, when the radius is so long as to render the ordinary method impracticable.

Sug.—Let A, B, and C be the three points; then are M and N other points in the same circumference.

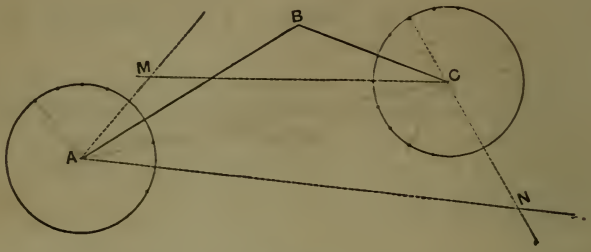


Fig. 409.

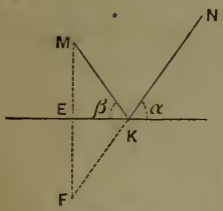


Fig. 410.

749. From two given points on the same side of a line given in position, to draw two lines which shall meet in that line and make equal angles with it.

Sug.—If α and β are equal, what is the relation of ME to EF?

---

750. To construct an isosceles triangle with a given base and vertical angle.

Sug.—See Prob. 4, p. 102.

---

751. To trisect a right angle.

Sug.—What is the value of an angle of an equilateral triangle?

---

752. Given the perpendicular of an equilateral triangle, to construct the triangle.

---

753. Given the diagonal of a square, to construct it.

---

754. To construct an isosceles triangle, so that the base shall be a given line, and the vertical angle a right angle.

755. Given the sum of the diagonal and a side of the square, to construct it.

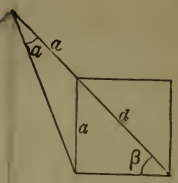


Fig. 411.

Sug.—What are the values of α and β respectively?

---

756. To construct a triangle when the altitude, the vertical angle, and one of the sides are given.

---

757. To construct a triangle when the sum of the three sides and the angles at the base are given.

Sug's.—MN being the sum of a, b, and c, what are the angles M and N as compared with the given angles α and β?

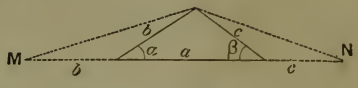


Fig. 412.

758. In a right angled triangle the perimeter, and the perpendicular from the right angle upon the hypotenuse being given, to construct the triangle.

Sug's.—DE is equal to the perimeter, DBE is an angle of 135°, and FE is the perpendicular on the hypotenuse. ABC is the required triangle. Let the student give the solution in full, and the proof.

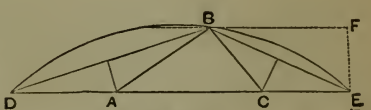


Fig. 413.

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759. From two given points on the same side of a given line, to draw two equal straight lines which shall meet in the same point of the line.

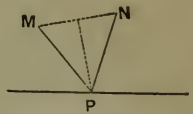

Fig. 414.

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760. To pass a circumference through two given points, which shall have its centre in a given line.

---

761. To construct a quadrilateral when three sides, one angle, and the sum of two other angles are given.

Sug's.—What is the fourth angle? When two sides and their included angle are known, there will be two cases, according as the two angles whose sum is known are adjacent to each other or opposite. In the latter case we have to describe a segment on a diagonal, which will contain the fourth angle. For the third case see Ex. 13, page 136.

Fig. 415.

762. To construct a quadrilateral when three angles and two opposite sides are given.

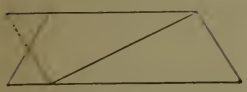

Fig. 416.

763. To bisect a trapezoid by a line drawn from one of its angles.

---

764. In a given circle, to inscribe a triangle equiangular with a given triangle.

Sug.—How does an angle at the centre compare with one inscribed in the same segment?

---

765. To describe three circles of equal diameters which shall touch each other.

---

766. In an equilateral triangle, to inscribe three equal circles which shall touch each other and the three sides of the triangle.

---

767. To describe a circle of given radius touching the two sides of a given angle.

Sug.—How far is the centre from each line?

---

768. To describe a circumference which shall be embraced between two parallels and pass through a given point within the parallels.

Sug.—In what line is the centre? How far from the given point?

---

769. To describe a circle with a given radius, which shall pass through a given point and be tangent to a given line.

---

770. To find in one side of a triangle the centre of a circle which shall touch the other two sides.

---

771. Through a given point on a circumference, and another

given point without, to describe a circle touching the given circumference.

SUG.—Consider in what two lines the centre must lie.

---

772. In the diameter of a circle produced, to determine the point from which a tangent drawn to the circumference shall be equal to the diameter.

SUG.—What is the relation between the radius, the required tangent, and the distance from the centre to the intersection of the produced diameter and the required tangent?

---

773. To describe a circle of given radius, touching two given circles.

---

774. In a given circle, to inscribe a right angle, one side of which is given.

---

775. In a given circle, to construct an inscribed triangle of given altitude and vertical angle.

---

776. To inscribe a square in a given right-angled isosceles triangle, one side being in the hypotenuse.

FIG. 417.

---

777. To inscribe a square in a given quadrant of a circle, the vertex of an angle being at the centre.

---

778. To find the centre of a circle in which two given lines meeting in a point shall be a tangent and a chord.

---

779. To describe a circumference which shall pass through a given point and be tangent to a given line at a given point.

---

780. To bisect a quadrilateral by a line drawn from one of its angles.

SUG.—The demonstration is based upon the principle that triangles having equal bases and equal altitudes are equivalent.

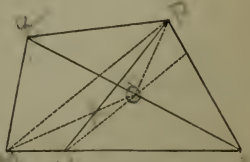

FIG. 418.

781. Through a given point situated between the sides of an angle, to draw a line terminating at the sides of the angle, and in such a manner as to be bisected at the point.

Sug.—Conceive the point as situated in the third side of a triangle of which the two given lines are the other two.

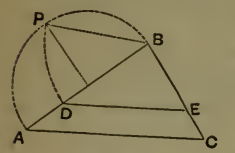

Fig. 419.

782. To draw a line parallel to the base of a triangle so as to divide the triangle into two equivalent parts.

Sug. $\overline{PB}^2 = \overline{DB}^2 = \frac{1}{2}\overline{AB}^2$. See (***344, 362***).

Fig. 420.

783. To construct a square when the difference between the diagonal and a side is given.

Sug.—Consider the angles.

---

784. To determine the point in the circumference of a circle from which chords drawn to two given points shall have a given ratio.

Sug.—Draw a chord dividing the chord joining the given points in the required ratio, and bisecting one of the subtended arcs.

---

785. To bisect a given triangle by a line drawn from one of its angles.

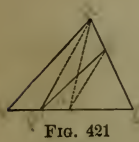

Fig. 421

786. To bisect a given triangle by a line drawn from a given point in one of its sides.

---

787. In the base of a triangle find the point from which lines extending to the sides, and parallel to them, will be equal.

---

788. To construct a parallelogram having the diagonals and one side given.

---

789. To construct a triangle when the three altitudes are given.

SUG.—What is the relation of the perpendiculars to the sides upon which they fall? If a triangle can be formed with the perpendiculars as sides, how will it compare with the first triangle? How proceed when the perpendiculars will not form a triangle?

---

790. What is the area of the sector whose arc is 50°, and whose radius is 10 inches?

---

791. To construct a square equivalent to the sum, or to the difference of two given squares.

---

792. To divide a given straight line in the ratio of the areas of two given squares.

---

793. To construct a triangle, when the altitude, the line bisecting the vertical angle, and the line from the vertex to the middle of the base are given.

SUG.—The centre of the circle circumscribing the required triangle is in the perpendicular to the base at its middle point; and the intersection of this perpendicular and the bisectrix is a point in this circumference.

Show that the bisector always lies between the perpendicular and the medial line.

---

794. Through a given point, draw a line such that the parts of it, between the given point and perpendiculars let fall on it from two other given points, shall be equal.

What would be the result, if the first point were in the straight line joining the other two?

---

795. From a point without two given lines, to draw a line such that the part intercepted between the given lines shall be equal to the part between the given point and the nearest line.

SUG.—Produce the lines till they meet, if necessary. Draw a line through the given point parallel to one of the lines, and produce it till it meets the other.

---

796. Given one angle, a side adjacent to it, and the difference of the other two sides, to construct the triangle.

QUERIES.—How if $b > a$? How if B is obtuse?

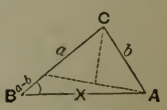

FIG. 422.

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797. To pass a circumference through two given points, having its centre in a given line.

798. To draw a line parallel to a given line and tangent to a given circumference.

Sug.—Draw a diameter perpendicular to the given line.

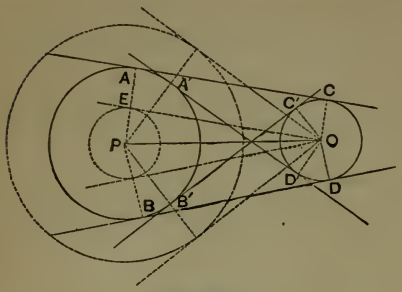

Fig. 423.

799. To draw a common tangent to two given circles.

Sug.—1st Method.—There are two sets of tangents, AC, BD, and A′D′, B′C′. For the first, observe that if PE = AP − OC, OE is parallel to AC, etc.

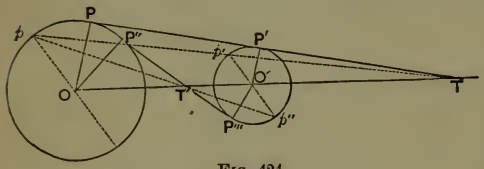

Fig. 424.

2d Method. pO, $p'O'$ being parallel to each other, $pp'T$ gives the intersection of the tangent with the line passing through the centres, since

$$pO : p'O' :: OT : O'T, \text{ or } pO - p'O' : p'O' :: OO' : O'T.$$

Also, $PO - P'O' : P'O' :: OO' : O'T$.

Hence $O'T$ is constant for all positions of the parallel radii. Prove that if the parallel radii are on different sides of the line joining the centres, T′ is the point where the internal tangent cuts OO′.

Queries.—How many tangents can be drawn—1st. When the circles are external one to the other; 2d. When they are tangent externally; 3d. When they intersect each other; 4th. When tangent internally; 5th. When one lies within the other?

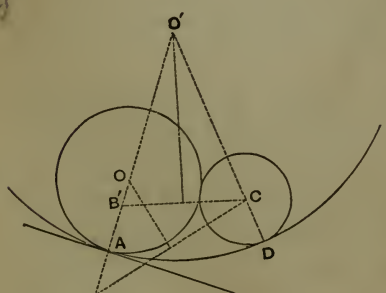

Fig. 425.

800. To describe a circle tangent to a given circumference and also to a given line at a given point.

Sug's.—There may be two cases—1st. When the given circle is exterior to the one sought; and 2d. When it is interior. In either case the centre of the required circle is in the perpendicular AO′. In the former case, O, the centre of the required circle, is at $r + r'$ from C; and in the latter O′ is at $r - r'$ from C. $AO = r$, and $CD = AB = AB' = r'$.

801. To construct a trapezoid when the four sides are given.

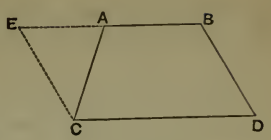

Fig. 426.

Sug.—Knowing the difference between the two parallel sides, we may construct the triangle AEC, and hence the trapezoid.

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802. On a given line, to construct a polygon similar to a given polygon.

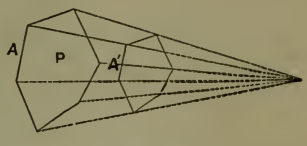

Fig. 427.

Sug's.—One method may be learned from (***90***). Ex. 8, page 152, furnishes another method. The following is an elegant method: To construct on A′ homologous with A, a polygon similar to P. Place A′ parallel to A, and the figure will suggest the construction.

---

803. To pass a plane through a given line and tangent to a given sphere.

Sug's.—Pass a plane through the centre of the sphere and perpendicular to the given line. Through the point of intersection and in this secant plane draw tangents to the great circle in which the secant plane intersects the surface of the sphere. The points of tangency will be the points of tangency of the required planes (?), of which there are thus seen to be two.

---

804. Def.—***A Tangent Plane*** to a cylindrical or conical surface is a plane which contains an element of the surface, but does not cut the surface. The element which is common to the surface and the plane is called the *Element of Contact.*

---

805. To pass a plane through a given point and tangent to a given cylinder of revolution.

Sug's.—1st. When the point is in the surface of the cylinder. Through the point draw an element of the cylinder, by passing a line parallel to the axis, or to any given element. Through the same point pass a plane perpendicular to this element, making a right section (a circle). To this circle draw a tangent. The plane of the element and tangent is the tangent plane required. [The student should prove that any point in the plane affirmed to be tangent, not in the element passing through the given point, is without the cylinder.]

2d. When the given point is without the cylinder. Pass a plane through the given point perpendicular to the axis of the cylinder, thus making a right section of the cylinder (a circle). In this secant plane draw tangents to the section. Through the points of contact of these tangents draw elements of the cylinder. These elements are the elements of contact of the tangent planes. Hence planes passing through them and the given point are the tan-

gent planes required. [The student should remember that this is but an *outline*, and be careful to fill it up, giving the proof.]

806. To pass a plane through a given point and tangent to a conical surface of revolution.

807. To find, with the compasses and ruler, the radius of a material sphere whose centre is inaccessible.

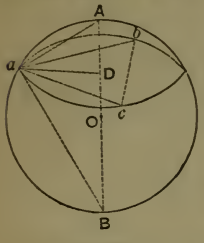

FIG. 428.

SUG's.—With one point of the compasses at any point in the surface, as A, trace a circle of the sphere, as *acb*. The chord A*a* is measured by the distance between the compass points. In like manner measure three other chords, as *ac*, *ab*, and *bc*. Draw a plane triangle having these chords for its sides, and circumscribe a circle about it. Thus *a*D is found. Knowing *a*A, and *a*D, and remembering that A*a*B is right angled at *a*, the triangle A*a*B can be drawn in a plane (?), whence AO becomes known.

SECTION III.

APPLICATIONS OF ALGEBRA TO GEOMETRY.

808. The mathematical method which is called technically *Applications of Algebra to Geometry* consists in finding, by means of equations, the numerical values of the unknown parts of a geometrical figure, when a sufficient number of the parts are given numerically.

809. By reference to the COMPLETE SCHOOL ALGEBRA, page 238, it will be seen that the algebraic solution of a problem consists of two parts: 1st. The *Statement*, which is the expressing by one or more equations of the conditions of the problem, *i. e.*, the relations between the known and unknown quantities (parts of the figure) to be compared; and 2d. The *Solution* of these equations, so as to find the values of the unknown quantities in known ones.

810. In applying the equation for the solution of such problems as are now proposed, we have to depend upon our previously acquired knowledge of the properties of geometrical figures for the relations between the known and unknown quantities, which will enable us to form the necessary equations, *i. e.*, to make the *State-*

ment. The resolution of the equations thus arising is effected in the ordinary ways. [See NOTE, page 239 of THE COMPLETE SCHOOL ALGEBRA.]

811. The details of this method will be most readily obtained from a careful study of examples.

EXAMPLES.

812. In a right angled triangle, given the hypotenuse and the sum of the other two sides, to find these sides separately.

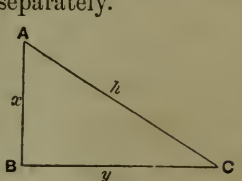

FIG. 429.

SOLUTION.—Let ABC be a triangle, right angled at B. Let the *known* hypotenuse be h, the *unknown* base, y; the *unknown* altitude, x; and the *known* sum of the base and altitude, s.

We have here two unknown quantities, and hence must have two equations, in order to find their values. One of these equations is furnished directly by the statement of the problem, which says that the sum of the base and perpendicular is to be given. Hence—

Equation 1 is $$x + y = s.$$

A second relation between x and y and the known quantity h is furnished by the relation given in PART II. (***346***). Whence—

Equation 2 is $$x^2 + y^2 = h^2.$$

Solving these equations we find—

$$y = \tfrac{1}{2}s \pm \tfrac{1}{2}\sqrt{2h^2 - s^2}, \text{ and } x = \tfrac{1}{2}s \mp \tfrac{1}{2}\sqrt{2h^2 - s^2}.$$

If $h = 10$ and $s = 14$, we find $x = 6$, and $y = 8$; or $x = 8$, and $y = 6$.

GEOMETRICAL SOLUTION.—It is exceedingly interesting and instructive to compare the algebraic solution of such problems with their geometrical solution, when the problem can be solved in both ways. The geometrical solution of this problem is as follows:

Take DC $= s$, the sum of the two sides, and make ODC $= 45°$. From C as a centre, with a radius h, the hypotenuse, describe an arc cutting DO, as in A and A′. Draw AC and the perpendicular AB, also A′C and the perpendicular A′B′. Both the triangles ABC and A′B′C fulfil the conditions. For AB $=$ DB (?), whence AB + BC $= s$, and AC $= h$, by construction. So, also, A′B′ $=$ DB′ (?), whence A′B′ + B′C $= s$, and A′C $= h$, by construction.

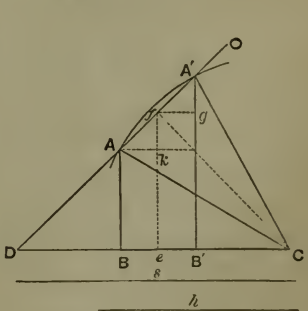

FIG. 430.

COMPARISONS OF THESE SOLUTIONS.—1st. We find in the *algebraic* solution, that, in

general, y may have two values—viz., $\frac{1}{2}s + \frac{1}{2}\sqrt{2h^2 - s^2}$, and $\frac{1}{2}s - \frac{1}{2}\sqrt{2h^2 - s^2}$; and that when $y = \frac{1}{2}s + \frac{1}{2}\sqrt{2h^2 - s^2}$, $x = \frac{1}{2}s - \frac{1}{2}\sqrt{2h^2 - s^2}$; but, when $y = \frac{1}{2}s - \frac{1}{2}\sqrt{2h^2 - s^2}$, $x = \frac{1}{2}s + \frac{1}{2}\sqrt{2h^2 - s^2}$. Correspondingly, we find in the *geometrical* solution that the base (y) may have two values—viz., BC, and B′C; and that when the base is BC, the altitude (x) is AB; but, when the base is B′C, the altitude is A′B′.

2d. From the algebraic solution, we observe that the base $y = \frac{1}{2}s \pm \frac{1}{2}\sqrt{2h^2 - s^2}$, may be considered as made up of two parts—viz., a rational part, $\frac{1}{2}s$, and a radical part, $\frac{1}{2}\sqrt{2h^2 - s^2}$; and that the altitude, $x = \frac{1}{2}s \mp \frac{1}{2}\sqrt{2h^2 - s^2}$, is made up of the *same* parts, only observing that, if the base is considered as the *sum* of these parts—viz., $\frac{1}{2}s + \frac{1}{2}\sqrt{2h^2 - s^2}$, the altitude is their *difference*—viz., $\frac{1}{2}s - \frac{1}{2}\sqrt{2h^2 - s^2}$. If, however, the base is $\frac{1}{2}s - \frac{1}{2}\sqrt{2h^2 - s^2}$, the altitude is $\frac{1}{2}s + \frac{1}{2}\sqrt{2h^2 - s^2}$. Now, we can discover exactly the same things *geometrically*, and can show exactly what is the geometrical meaning of each of the parts of the values of y and x. To do this, draw Cf* bisecting AA′; let fall the perpendicular fe, and draw Ak and fg parallel to DC. Cf is perpendicular to DO (?), and hence equal to Df (?). Also, $De = eC = fe = \frac{1}{2}s$ (?). From the right angled isosceles triangle DfC, $fC = \frac{1}{\sqrt{2}}DC = \frac{s}{\sqrt{2}}$ (?). Hence, from AfC, $Af = \sqrt{AC^2 - fC^2} = \sqrt{h^2 - \frac{1}{2}s^2} = \frac{1}{\sqrt{2}}\sqrt{2h^2 - s^2}$. Again, from the right angled isosceles triangle Akf, we have $Ak = \frac{1}{\sqrt{2}}Af$ (?) $= \frac{1}{2}\sqrt{2h^2 - s^2}$. But $Ak = fk = fg = A'g = Be = eB'$. Hence we see that the rational part of the value of y ($\frac{1}{2}s$) is eC, and that the radical part ($\frac{1}{2}\sqrt{2h^2 - s^2}$) is Be, or eB'. In the triangle ABC the *sum* of these parts is the base; and in the triangle A′B′C, their *difference* is the base. In like manner fe represents the rational part of the value of x, and $fk = A'g$, the radical part.

3d. From the algebraic solution we see that if $s^2 = 2h^2$, $y = \frac{1}{2}s$, and $x = \frac{1}{2}s$. The same thing is seen in the geometrical solution, for if $s^2 = 2h^2$, $h = \frac{1}{\sqrt{2}}s$, or fC; whence the arc struck from C as a centre, with h as a radius, would be tangent to DO, instead of intersecting it in *two* points. Again, if $s^2 > 2h^2$, the quantity under the radical sign is negative, and the radical becomes *imaginary*. This means, that *no triangle* can be formed under these circumstances. This case appears in the geometrical solution also, for then $h < \frac{1}{\sqrt{2}}s$, or less than fC, and consequently the arc struck from C as a centre, with radius h, will not touch DO, and we get no triangle.

---

* This part of the construction should not be allowed on the figure till it is wanted—*i.e.*, till this stage of the discussion.

813. SCH.—This problem is discussed thus at length as an illustration of what *may* be done by such methods. Of course, all problems are not equally fruitful; but the student should not rest satisfied with a mere determination of the values of the unknown parts in known terms, when anything farther is revealed either by the process or result of the algebraic solution. Especially should he desire to become expert in seeing what geometrical relations are indicated by the *form* of the answer obtained.

---

814. Given the lengths of the medial lines from the acute angles of a right angled triangle, to determine the triangle, *i. e.*, to find the base and perpendicular.

SUG'S.—Let AD $= a$, CE $= b$, AB $= 2x$, and CB $= 2y$; then $4x^2 + y^2 = a^2$, and $4y^2 + x^2 = b^2$ (?). $\therefore 2x = \text{AB} = 2\sqrt{\frac{4a^2 - b^2}{15}}$, and $2y = \text{CB} = 2\sqrt{\frac{4b^2 - a^2}{15}}$.

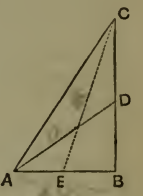

FIG. 431.

The *form* of these results indicates that CB sustains the same relation to CE and AD that AB does to AD and CE—a fact which is evident from the nature of the case.

Again, if $4a^2 < b^2$, $2x$ is imaginary; and if $4b^2 < a^2$, $2y$ is imaginary. In either case the triangle cannot exist. So also if $4a^2 = b^2$, $2x = 0$; and if $4b^2 = a^2$, $2y = 0$, and there can be no triangle. This may be seen from the figure by conceiving AB, for example, to diminish. As A approaches B, AD approaches equality with DB, and CE with CB. Hence the *limit* is AD $= \frac{1}{2}$CE.

Thus we see *that either medial line must be more than half the other*,—a proposition which is proved by this solution.

---

815. The hypotenuse and radius of the inscribed circle of a right angled triangle being given, to determine the triangle.

Results.—Calling the hypotenuse h, the radius r, the base x, and the perpendicular y, we have, $x = \frac{2r + h \pm \sqrt{h^2 - 4hr - 4r^2}}{2}$, and

$$y = \frac{2r + h \mp \sqrt{h^2 - 4hr - 4r^2}}{2}.$$

The results being the same in other respects, the double sign before the radical indicates that the base and perpendicular are interchangeable—a fact which is evident from the nature of the case.

If the radical is 0, *i. e.*, if $h^2 - 4hr - 4r^2 = 0$, $x = r + \frac{1}{2}h$, and $y = r + \frac{1}{2}h$, and the base and perpendicular are equal. Let the student show the same thing geometrically (from a figure).

Also, if $h^2 - 4hr - 4r^2 = 0$, $h = 2r(1 \pm \sqrt{2})$. In this result the negative

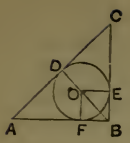

FIG. 432.

sign is to be rejected, since it would make h negative, as $\sqrt{2} > 1$.

The value $h = 2r(1 + \sqrt{2})$ is readily seen from the figure when AB = CB. Thus $AC = 2DB = 2(DO + OB) = 2(r + r\sqrt{2}) = 2r(1 + \sqrt{2})$ (?).

---

816. A tree of known height standing perpendicular on a horizontal plane, breaks so that its top strikes the ground at a given distance from the foot, while the other end hangs on the stump. How high is the stump? That is, given the base and the sum of the perpendicular and hypotenuse of a right angled triangle, to determine the perpendicular.

Result.—Let a be the height of the tree, b the distance from the foot to the point where the top strikes, and x the height of the stump; then $x = \frac{a^2 - b^2}{2a}$.

Since $\frac{a^2 - b^2}{2a} = \frac{1}{2}a - \frac{b^2}{2a}$, $\frac{b^2}{2a}$ is the distance below the middle, at which the tree breaks.

---

817. In a rectangle, knowing the diagonal and perimeter, to find the sides.

---

818. Knowing the base, b, and altitude, a, of any triangle, to find the side of the inscribed square, x.

Result, $x = \frac{ab}{a + b}$.

---

819. In an equilateral triangle, given the lengths, a, b, c, of the three perpendiculars from a point within upon the sides, to determine the sides.

Sug's.—Find an expression for the altitude in terms of the sides; and then get two expressions for the area of the whole triangle. Equate these.

Result, each side $= \frac{2(a + b + c)}{\sqrt{3}}$.

---

820. In a right angled triangle whose hypotenuse is h, and difference between the base and perpendicular d, to find these sides.

Results, $x = \frac{-d \pm \sqrt{2h^2 - d^2}}{2}$, $x + d = \frac{d \pm \sqrt{2h^2 - d^2}}{2}$.

QUERIES.—Why must the minus sign of the radical be omitted in the geometrical interpretation of these results? What is the least possible value of d? What are the sides of the triangle for these values? What is the superior limit of the value of d? What do the sides become at this limit?

---

821. In an equilateral triangle given the lines a, b, c, drawn to its three vertices from a point within or without, to find the sides.

Result.—Each side $=$

$$\left\{\frac{a^2+b^2+c^2 \pm \sqrt{6\,(a^2b^2+b^2c^2+c^2a^2)-3\,(a^4+b^4+c^4)}}{2}\right\}^{\frac{1}{2}}$$

The radical is $+$ when the point is within, and $-$ when it is without.

---

822. The perimeter of a right angled triangle and the perpendicular from the right angle upon the hypotenuse being given, to determine the triangle.

SUG'S.—Let s be the perimeter, p the perpendicular upon the hypotenuse, and $x+y, x-y$ the two sides about the right angle. Then the hypotenuse $= s - 2x$, and we readily form the two equations $p\,(s-2x) = x^2 - y^2$, and $(x+y)^2 + (x-y)^2 = (s-2x)^2$ (?). Hence $x = \frac{s(s+2p)}{4(s+p)}$, and this value substituted in either equation will give y.

---

823. The base of a plane triangle is b and its altitude a, required the distance from the vertex at which a parallel to the base must cut the altitude in order to bisect the triangle.

Result, $\dfrac{a}{\sqrt{2}}$.

QUERY.—What does the fact that b does not appear in the result show?

---

824. Having given the area of a rectangle inscribed in a triangle, can the triangle be determined? Can it, if the rectangle is a square? If the rectangle is a square and the triangle right angled? If the rectangle is a square and the triangle equilateral?

---

825. The sides of a triangle being a, b, c, to find the perpendicular upon c from the opposite angle.

$$\textit{Result, } p = \frac{1}{2c}\sqrt{2c^2\,(a^2+b^2)+2a^2b^2-a^4-b^4-c^4}.$$

SUG'S.—Observe that a and b are similarly involved in the result, but c is differently involved from either. This is evidently as it should be, since a and b are the sides about the angle from which p is let fall; and are thus similarly related to p. But c, the side on which p falls, is differently related to p from

either of the others. The student should be able to write the value of the perpendiculars upon each of the other sides, from this one. Thus, that on a is

$$\frac{1}{2a}\sqrt{2a^2(c^2+b^2)+2c^2b^2-a^4-b^4-c^4}.$$

---

826. The sides of a triangle are a, b, c, to find the side of an inscribed square one of whose sides falls in c.

Sug's.—The altitude may be found from the preceding, hence may be assumed as known. Call it p. Then the side of the required square is $\frac{cp}{c+p}$. What is the side of the square standing on a? On b?

Query.—Will the square be the same on whichever side it stands? Observe that though the values here found are apparently different, they *may* not be so really, since p is different in each case. But let the student decide.

---

827. Having the area of a rectangle inscribed in a given triangle and standing on a specified side, to determine the sides of the rectangle.

Result, b being the base on which the rectangle stands, p the altitude from this base, and s the given area, we have for the sides

$$x=\frac{b}{2}\pm\sqrt{\frac{b^2}{4}-\frac{sb}{p}},\text{ and }y=\frac{p}{2}\mp\sqrt{\frac{p^2}{4}-\frac{sp}{b}}.$$

Sug's.—The $\pm$ and $\mp$ signs indicate that, in general, there can be two equal rectangles inscribed standing on the same base. The student will do well to illustrate it with definite numerical values, as $p=10$, $b=6$, $s=10$.

Again, $\frac{b^2}{4}$ must be greater than $\frac{sb}{p}$, and $\frac{p^2}{4}>\frac{sp}{b}$, *i. e.*, s must be less than $\frac{1}{4}pb$.

That is, the greatest rectangle is half the area of the triangle, since $\frac{1}{2}pb$ is the area of the triangle.

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828. The Algebraic solution of a problem often enables us to effect a geometrical construction. We will give a few examples.

Through a given point within a circle, to draw a chord of a given length.

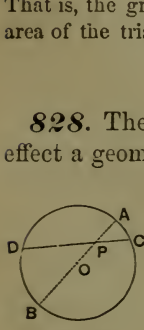

Fig. 433.

Solution.—Let s be the length of the required chord, and P the given point. Since P is a known point, call AP $= a$, PB $= b$, AB being the diameter through P. Let CD represent the required chord, and calling CP, x, PD $= s - x$. Then $sx - x^2 = ab$; whence $x = \frac{1}{2}s \pm \sqrt{\frac{1}{4}s - ab}$.

To effect the geometrical construction, let s be the length of the given chord, and P the point in the given circle. Draw the diameter through P, and erect PE perpendicular to it. Make $EH = \frac{1}{2}s$; then since $\overline{PE}^2 = ab$, $PH = \sqrt{\frac{1}{4}s^2 - ab}$. Now take $HI = \frac{1}{2}s$, and from P as a centre, with a radius $PI = \frac{1}{2}s + \sqrt{\frac{1}{4}s^2 - ab}$, strike the arc DI intersecting the circumference. DPC is the chord required.

FIG. 434.

From the radical $\sqrt{\frac{1}{4}s^2 - ab}$ we see that, if $ab > \frac{1}{4}s^2$, x is imaginary, as we say in algebra. In such a case the problem is geometrically impossible, as will appear from the construction, for then PE is greater than EH, which makes HP, the representative of $\sqrt{\frac{1}{4}s^2 - ab}$, impossible. If $\frac{1}{4}s^2 = ab$, x has but one value, and the segments are equal.

---

829. To find a point in a tangent to a circle from which, if a secant be drawn to the extremity of the diameter passing through the point of tangency, the external segment shall have a given length.

SOLUTION.—Let $AB = d$ be the diameter of the given circle, $DX = a$ the external segment of the required secant, and the whole secant $BX = x$. Then

$$x^2 - ax = d^2, \text{ and } x = \tfrac{1}{2}a \pm \sqrt{d^2 + \tfrac{1}{4}a^2}.$$

To effect the geometrical construction, construct the radical by taking $AC = \frac{1}{2}a$; whence $BC = \sqrt{d^2 + \frac{1}{4}a^2}$. Now make $CY = \frac{1}{2}a$, and with B as a centre, and BY as a radius, strike an arc cutting the tangent, as in X. Then is

$$BX = x = \tfrac{1}{2}a + \sqrt{d^2 + \tfrac{1}{4}a^2}.$$

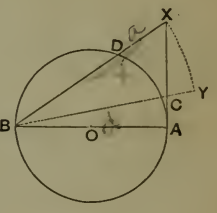

FIG. 435.

The negative value of the radical is inapplicable in this elementary, geometrical sense, since as $\sqrt{d^2 + \frac{1}{4}a^2} > \frac{1}{2}a$, this would make x a negative quantity. Again we see that no real value of a can render x imaginary.

We can observe the same things from the geometrical construction. Thus, if the negative value of the radical were taken, x would be *numerically* less than BC, by $\frac{1}{2}a$, or AC. But $BC - AC < BA$. Hence an arc struck from B with the required radius would not cut the tangent. We see also that a may have any value between 0 and ∞.

---

830. Given the hypotenuse and area of a right angled triangle, to construct the triangle.

SUG'S.—Let h be the hypotenuse, s^2 the area, and x the perpendicular from the right angle upon the hypotenuse. Then $hx = 2s^2$, or $\frac{1}{2}h : s :: s : x$, and $h : 2s :: 2s : 2x$.

The figure will suggest the construction.

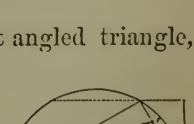

FIG. 436.

831. Through a point between two lines which intersect, to draw a line which shall cut off a triangle of given area.

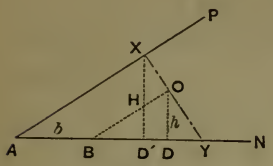

FIG. 437.

SUG'S.—Let AY $= x$, and the required area $= s^2$.

We have $h : \mathsf{H} :: x - b : x$. $\therefore \mathsf{H} = \frac{hx}{x-b}$.

And $\mathsf{H}x = 2s^2$. $\therefore \mathsf{H} = \frac{2s^2}{x}$. Thus

$$x = \frac{s^2}{h} \pm \sqrt{\frac{s^2}{h}\left(\frac{s^2}{h} - 2b\right)}.$$

To construct this, find $c = \frac{s^2}{h}$, *i. e.*, construct a third proportional to h and s. Then construct $\sqrt{c(c-2b)}$, *i. e.*, find a mean proportional between c and $c - 2b$; let this be m. Whence $x = c \pm m$. In general, there may be two solutions, if any, since there are two values of x. [This should also be observed from the figure.] But if $2b > \frac{s^2}{h}$ there is no solution. If $\frac{s^2}{h} = 2b$, there is but one solution. In the latter case where is the given point O? What is the geometrical difficulty when $2b > \frac{s^2}{h}$? Can m be numerically greater than c?

---

832. To construct the four forms of the affected or complete quadratic equation, viz., (1.) $x^2 + px - q = 0$, (2.) $x^2 - px - q = 0$, (3.) $x^2 - px + q = 0$, (4.) $x^2 + px + q = 0$, without solving the equations.

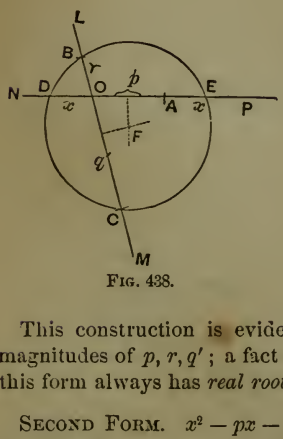

FIG. 438.

FIRST FORM. $x^2 + px - q = 0$.—Draw any two lines as LM, NP, intersecting in some point O. Resolve q of the equation into two factors, as r and q', so that we have $x^2 + px - r \times q' = 0$. Take OA $= p$, OB $= r$, OC $= q'$. Bisect CB and AO by perpendiculars, and from their intersection F as a centre, with a radius FB, draw a circle. Then DO, or AE, is x, the positive root. For $x(x + p) = rq'$, or $x^2 + px - rq' = 0$. The negative root is OE. Thus, let OE $= (-x)$. Then DO $=$ AE $= (-x - p)$. Hence $(-x)(-x - p) = x^2 + px = rq'$, or $x^2 + px - rq' = 0$.

This construction is evidently always possible irrespective of the relative magnitudes of p, r, q'; a fact which agrees with the statement in algebra that this form always has *real roots.*

SECOND FORM. $x^2 - px - rq' = 0$.—The construction is the same as for the first form; only, in this case OE is the positive, and DO the negative root. Thus for OE $= x$ (positive), we have DO $\times$ OE $= (x - p)x = rq'$, or $x^2 - px -$

$rq' = 0$. For $DO = (-x)$, we have $DO \times OE = DO\,(OA + AE) = DO\,(OA + DO)$ $= (-x)\,(p - x) = rq'$, or $x^2 - px - rq' = 0$.

Observe that in the first case the negative root is numerically greater than the positive; while it is the reverse in this form. This agrees with the conclusions of algebra (See COMPLETE SCHOOL ALGEBRA, ***104***).

THIRD FORM. $x^2 - px + rq' = 0$.—Draw any two lines, as OM, OP, meeting at O. Take $OA = p$, $OB = r$ or q', and $OC = q'$ or r. Erect perpendiculars at the middle points of OA, and BC; and from their intersection F as a centre, with a radius FB, strike a circumference. Then OE and OD are the values of x. For $OE = x$, $OE \times OD = OE \times EA = OE\,(OA - OE) = x\,(p - x)$ $= rq'$, or $x^2 - px + rq' = 0$. For $OD = x$, $OD \times OE = OD\,(OA - AE) = OD\,(OA - OD) = x\,(p - x) = rq'$, or $x^2 - px + rq' = 0$.

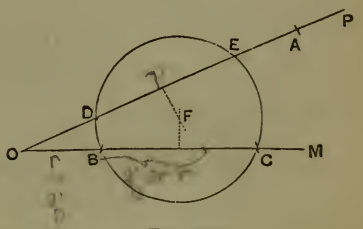

FIG. 439.

Observe that the former value of x is greater than the latter, but that neither is negative.

So also, we may readily see that the roots may become equal, and also, imaginary. Thus if the circle were tangent to OA, the roots would be equal, and if it did not touch OA they would *both* be imaginary. (See Algebra, as above.)

FOURTH FORM. $x^2 + px + rq' = 0$.—The construction is the same as the last, only both values of x are negative. Thus, $(-x)\,[p - (-x)] = (-x)$ $(p + x) = rq'$, $-px - x^2 - rq' = 0$, or $x^2 + px + rq' = 0$.

SCH.—Thus we see that we can construct any equation of the second degree containing but one unknown quantity, which has real roots. Hence, if the algebraic solution of a geometrical problem requires only the resolution of such an equation, the algebraic solution will lead to the geometrical construction.

---

833. We have now given sufficient illustrations of this most interesting and important subject, so that the student should have caught the spirit of this method of using algebra to subserve the purposes of geometrical investigation. We shall simply append a list of problems, upon which the student can put in exercise both his algebraic and geometric knowledge. But we cannot refrain from repeating the advice, that the learner should not rest satisfied with the mere algebraic resolution of the problem. He should be ambitious to trace, *as fully as possible*, the wonderful relations which exist between the abstract operations of algebra, and the more concrete relations of geometry.

EXAMPLES.

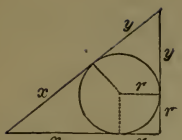

FIG. 440.

834. Given the perimeter of a right angled triangle and the radius of the inscribed circle, to determine the triangle.

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835. Given the hypotenuse of a right angled triangle and the side of the inscribed square, to determine the triangle.

---

836. In a right angled triangle, given the radius of the inscribed circle, and the side of the inscribed square, the right angle of the triangle constituting one angle of the square, to determine the triangle.

SUG'S.—Letting x and y be the sides, z the hypotenuse, r the radius of the inscribed circle, and s the side of the inscribed square, we have $s = \frac{xy}{x+y}$, $xy = r(x+y+z)$, and $x+y = z+2r$. Whence $z = 2r\left(\frac{s-r}{2r-s}\right)$, etc

---

837. In any triangle whose sides are a, b, c, to find the radius of the inscribed circle.

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838. Show that the area of a regular dodecagon inscribed in a circle whose radius is 1, is 3.

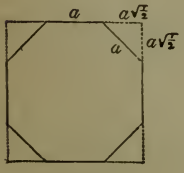

FIG. 441.

839. Find the area of a regular octagon whose side is a.

Result, $2(\sqrt{2}+1)a^2$.

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840. Find the radii of three equal circles described in a given circle, tangent to the given circle and to each other.

---

841. The space between three equal circles tangent to each other is a; what is the radius?

---

842. In a triangle, given the ratio of two sides, and the segments of the third side made by a perpendicular let fall from the angle opposite.

843. In a triangle, given the base and altitude, and the ratio of the other sides, to determine the triangle.

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844. Given the base, the medial line, and the sum of the other sides of a triangle, to determine the triangle.

---

845. To determine a right angled triangle, knowing the perimeter and area.

SUG's. $x^2 + y^2 = z^2$, $x + y + z = 2p$, and $xy = 2s^2$, give $y + x = 2p - z$, $x^2 + 2xy + y^2 = 4p^2 - 4pz + z^2$, $z^2 + 4s^2 = 4p^2 - 4pz + z^2$; whence $z = \frac{p^2 - s^2}{p}$. Now use $y + x = 2p - z = \frac{p^2 + s^2}{p}$, and $xy = 2s^2$.

---

846. To determine a right angled triangle, knowing the perimeter, and the sum of the hypotenuse, and the perpendicular upon the hypotenuse from the right angle.

SUG's. $x^2 + y^2 = z^2$, $x + y + z = 2p$, $z + u = a$, $xy = zu$. Then $x^2 + 2xy + y^2 = 4p^2 - 4pz + z^2$; whence $2xy = 4p^2 - 4pz$, and hence $2z(a - z) = 4p - 4pz$, etc.

---

847. The volume, the altitude, and a side of one of the bases of the frustum of a square pyramid being known, to determine a side of the other base.

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848. To determine a right angled triangle, knowing the perimeter, and the perpendicular let fall from the right angle upon the hypotenuse.

---

849. To determine a triangle, knowing the base, the altitude, and the difference of the other sides.

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850. To determine a triangle, knowing the base, the altitude, and the rectangle of the other sides.

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851. To determine a right angled triangle, knowing the hypotenuse and the difference between the lines drawn from the acute angles to the centre of the inscribed circle.

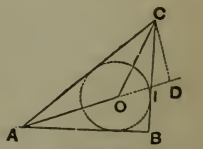

FIG. 442.

SUG's.—Let fall CD a perpendicular upon AO produced. Now, since the the angles BAC and ACB are bisected, and COD = OAC + OCA, and ICD = IAB, they being complements of the equal angles CID, IAB, we have, COD = OCD, and CD = OD = $\sqrt{\frac{1}{2}}$ CO. Hence, putting AC = h, CO = x, and AO = $x + d$, we have

$(x + d + \sqrt{\frac{1}{2}}\, x)^2 + (\sqrt{\frac{1}{2}}\, x)^2 = h^2$. From this x is readily found. The student should then be able to complete the solution.

852. Given two sides of a triangle and the bisector of their included angle, to determine the triangle.

853. Given the three medial lines, to determine a triangle.

854. Given the three sides of a triangle, to determine the radius of the circumscribed circle.

855. Four equal balls whose radius is r are placed on a plane so that each is tangent to the other three, thus forming a pyramid; what is its altitude?

856. Given the base of a triangle, the bisector of the opposite angle, and the radius of the circumscribing circle, to determine the triangle.

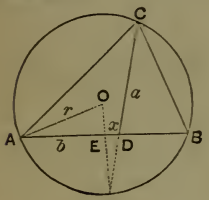

FIG. 443.

SUG'S.—First to find ED $= x$. Since EM $= r - \sqrt{r^2 - b^2}$, it may be considered known and put equal to c. We then have DM $= \sqrt{c^2 + x^2}$; and also, DM $\times\ a =$ AD $\times$ DB $= b^2 - x^2$, or DM $= \dfrac{b^2 - x^2}{a}$.

Whence $\sqrt{c^2 + x^2} = \dfrac{b^2 - x^2}{a}$, and x is readily found. Calling ED $= s$, the student will have no difficulty in proceeding with the solution.

CHAPTER II.

*INTRODUCTION TO MODERN GEOMETRY.**

SECTION I.

OF LOCI.

857. The term ***Locus***†, as used in geometry, is nearly synonymous with *geometrical figure*, yet having a latitude in its use which the other does not possess. *The locus of a point* is the line (geo-

* With strict propriety only the latter sections of this chapter belong to the *Modern Geometry*, technically so called. But, as the entire chapter is composed of matter which has not hitherto found place in our common text-books, and the relative importance of which is becoming more fully appreciated in modern times, the author has ventured to embrace the whole under this title.

† The word *Locus* is the Latin for *place*.

metrical figure) generated by the motion of the point according to some given law.

In the same manner, a surface is conceived as the locus of a line moving in some determinate manner.

ILL'S.—The locus of a point in a plane, which point is always equidistant from the extremities of a given right line, is a straight line perpendicular to the given line at its middle point. Thus, suppose AB a fixed line, and *the locus of a point* equidistant from its extremities is required; that point may be anywhere in a perpendicular to AB at its middle point, *and cannot be anywhere else* in this plane.

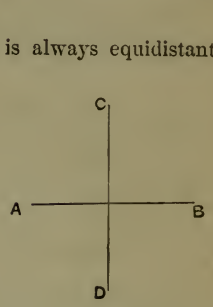

FIG. 444.

This perpendicular is the locus (place) of a point subject to the given law.

Again, a boy on the green is required to keep at just 20 feet from a certain stake; where may he be found? *i. e.*, what is his locus (place)? Evidently, the circumference of a circle whose radius is 20 feet. Thus, the locus of a point in a plane, equidistant from a given point, is the circumference of a circle. This is the *place* of such a point.

What is the locus in space of a point equidistant from a given point?

What is the locus of a point in space equidistant from the extremities of a given line? A plane.

What is the locus of a line moving so that each point in it traces a right line? In general, a plane; if it move in the direction of its length, a straight line.

What is the locus of a right line parallel to and equidistant from a given line?

What is the locus of a right line intersecting a given line at a constant angle? * A conical surface of revolution.

What is the locus of a semicircle revolving on its diameter?

PROPOSITIONS AND PROBLEMS IN DETERMINING PLANE LOCI.

[NOTE.—The student should be required to give every demonstration *in form*, and in detail. Frequent exercise in writing out demonstrations, is almost the only method of securing a good, independent style in demonstration.]

858. Theo.—*The locus of a point in a plane, equidistant from the extremities of a given line, is a perpendicular to that line at its middle point.*

SUG.—To prove this we have simply to show that every point in such a perpendicular is equidistant from the extremities of the given line, and that no other point has this property (PART II., ***129***).

* That is, an angle which remains of the same size.

859. Prob.—*Find the locus of a point at any constant distance* m *from a straight line.* Of what proposition in PART II. is this the converse?

SUG'S.—To prove the proposition which the answer to this question asserts, it will be necessary to show that every point in the affirmed locus is at the same distance from the given line and that *no other* point is at that distance. We affirm that *the locus is two right lines parallel to the given line and at a distance* m *therefrom.* The formal demonstration is as follows: Let AB be the given line, and OE, OE′, perpendiculars thereto, each equal to *m*. Through E and E′ draw CD and C′D′ parallel to AB; then is CD, C′D′, the locus required.* For, by Part II. (***156***), *every* point in CD, C′D′, is at the distance *m* from AB; and we may readily show that any other point, as P or P′, is at a distance greater or less than *m* from AB. Hence CD, C′D′, is the locus required.

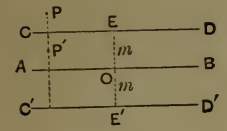

FIG. 445.

860. Theo.—*In a circle, the locus of the centre of a chord parallel to a given line is a diameter.*

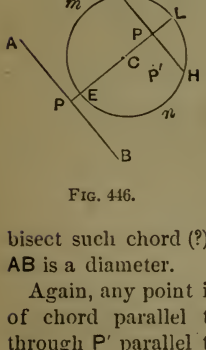

FIG. 446.

DEM.—Let *mn* be any circle, and AB a given line. Then is the locus of the centre of a chord parallel to AB, a diameter of the circle.

For, let DH be *any* chord parallel to AB. Through the centre of the circle C, and P, the middle point of DH, draw EL. Now EL is *perpendicular* to DH (?), and consequently to AB (?). Then will EL be perpendicular to *any* and every chord parallel to DH (?), and hence will bisect such chord (?). Therefore the locus of the centre of a chord parallel to AB is a diameter.

Again, any point in the circle and out of the line EL is not the middle point of chord parallel to AB. Thus, letting P′ be such a point, draw a chord through P′ parallel to AB. As there can be but one such chord (?), and as EL bisects it (?), P′ is without the diameter (?).

861. Theo.—*The locus of the centre of a circumference passing through two given points is a straight line.*

SUG.—Consult PART II. (***159, 163, 197***). The student should put the argument in form.

862. Theo.—*The locus of the centre of a circle which is tan-*

* It is important that the student think of these two lines as *one* locus, or as *parts* of *one and the same* locus, if this will aid the conception. A locus may consist of any number of detached parts; all that is necessary being that the given conditions be fulfilled. In this respect the word *locus* has a more enlarged meaning than the term *geometrical figure.*

gent to a given circle at a given point, is a straight line passing through the centre of the given circle.

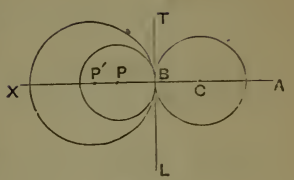

FIG. 447.

DEM.—Let C be the centre of the given circle, and B the point in the circumference to which the circle * shall be tangent, the locus of whose centre is required. Through B draw TL tangent to the given circle. Now, a circle passing through B, and tangent to the given circle, will have TL for its tangent (?), and as a radius is perpendicular to a tangent at its extremity, and only one perpendicular can be drawn to TL through B, the centre of a circle tangent to the given circle at B must be in this straight line. Moreover, as the given circle is tangent to the right line TL at B, its centre is in the perpendicular AX. Hence AX is the locus required.

---

863. Theo.—*The locus of the centre of a circle of given radius* R, *and tangent to a given straight line, is two parallels to this line at a distance* R *therefrom, on each side.* Give proof in form.

---

864. Prob.—*Find the locus of the centre of a circle of given radius* R, *whose circumference passes through a given point.* Give proof in form.

---

865. Theo.—*The locus of the centre of a line of constant length, having its extremities in two fixed lines which cut each other at right angles, is the circumference of a circle.*

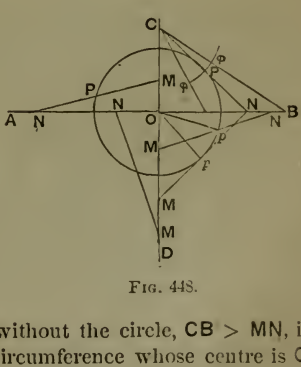

FIG. 448.

SUG'S.—Let MN be the length of the given line, and CD, and AB, the two lines intersecting at right angles, in which the extremities of MN are to remain. Now, in whatever position MN may be placed, its middle point, P, is at the same distance (½MN) from O (?). To show that any point not in this circumference, as φ, is not the middle point of a line equal to MN passing through it, and limited by the fixed lines, from φ as a centre, with a radius ½MN cut CD, as in C; and from C as a centre with the same radius strike the arc φP. If φ is without the circle, CB > MN, if within, less. Hence, the required locus is a circumference whose centre is O, and whose radius is ½MN.

---

* Observe the form of expression. We say "the circle," and not "the circles," using the term in a *generic sense*, as including all which have the required property, *i. e.*, all which are tangent to the given circle at B.

866. Prob.—*Find the locus of the centre of a chord of constant length, in a given circle.*

SUG.—We say, once again, *always give the proof in form.*

---

867. Prob.—*Find the locus of the vertex of the right angle of a right angled triangle of a constant hypotenuse.*

---

868. Prob.—*Find the locus of the middle point of the chord intercepted on a line through a given point, by a given circumference, when the given point is without the circumference, when it is in, and when it is within the circumference.*

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869. Prob.—*Find the locus of a point the sum of whose distances from two fixed intersecting lines is constant,* i. e., *is equal to a given line.*

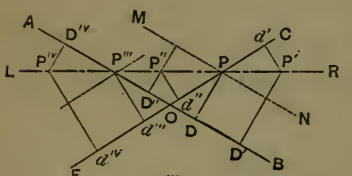

FIG. 449.

SOLUTION.—Let AB and CE be the fixed lines, and m the constant distance. Draw MN parallel to AB, and at a distance m from it. Bisect the angle CPN. Then is LR (a part of) the locus required. For [the student will here show that the sum of the distances from any point in LR to AB and CE, as PD, P''D'' + P''d'', P'D' + P'd', P'''d''', P''''D'''' + P''''d'''', is constant and equal to m], observe that when one of the perpendiculars measuring the distance from a point in the locus, changes from one side to the other of the line on which it is let fall, its *sign* changes. Thus P''D'', P''d'' being considered +, P'd' and P''''D'''' are to be considered −. This is a general principal in mathematics. See PART II. (***215***), and foot note.

Finally, LR is only a part of the locus, since there is another line on the opposite side of AB, obtained by drawing the auxiliary MN on that side, which fulfills the same condition. The student should show what the result is when we draw the auxiliary parallel to CE, and on either side of it, also that any point not in one of these lines cannot fulfill the required condition. The complete locus is four indefinite right lines intersecting each other at right angles, so as to inclose a rectangle.

---

870. Prob.—*Find the locus of a point such that the sum of the squares of its distances from two fixed points shall be equivalent to the square of the distance between the fixed points.*

---

871. Prob.—*Find the locus of the intersection of two secants*

drawn through the extremities of a fixed diameter in a given circle, one of the secants being always perpendicular to a tangent to the circle at the point where the other cuts it.

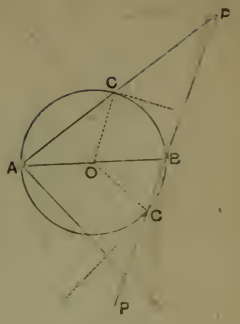

FIG. 450.

SUG's. P being the point, show that PB = AB, for any position of AP and BP. Hence, any point in the circumference having B for its centre, and AB for its radius, fulfills the conditions. Show that any point out of the circumference does not fulfill the conditions.

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872. Prob.—*Find the locus of the intersection of two lines drawn from the acute angles of a right angled triangle, through the points where the perpendicular to the hypotenuse cuts the opposite sides, or sides produced.*

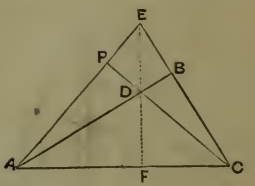

FIG. 451.

SUG's.—The locus of P is required. Prove that APC is always a right angled triangle, wherever the perpendicular EF to the hypotenuse AC is drawn.

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873. Prob.—*Find the locus of a point which divides a line drawn from a fixed point to a fixed line in a fixed ratio.*

SUG's.—Most problems in finding loci such as are treated in Elementary Plane Geometry, viz., right lines and circles, are readily solved by constructing a few points according to the given conditions, whence we can determine by inspection whether the required locus is a right line or the circumference of a circle; and, having discovered this fact by inspection, it will remain to show *why it should be so.* Thus, in the present problem, O being the fixed point, and AB the fixed line, drawing a few lines, OC, according to the requirements, and dividing them in the same ratio (in the figure 3 : 2), we find a few points P in the locus. We then discover at once that the locus is a right line parallel to AB, and can easily see *why* it should be so.

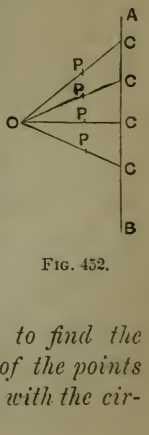

FIG. 452.

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874. Prob.—*Two fixed circumferences intersect: to find the locus of the middle point of the line drawn through one of the points of intersection and terminated by its other intersections with the circumferences.*

Sug's.—We will first give an example of the course which the mind of the student might take in his efforts to discover the solution. He would naturally draw two *unequal* * circles, as M and N, and, through one of the points of intersection, as A, draw BC, and bisect it at P. It is the locus of P that is desired. Now, suppose the line BC to revolve about A, B passing towards B′, and C towards A. It is the path of the middle point that he seeks. When C reaches A, the line becomes tangent to N, and P is the middle point of the chord AB′. In a similar manner, he sees that the middle point of the chord AC′, tangent to M at A, is also a point in the locus.

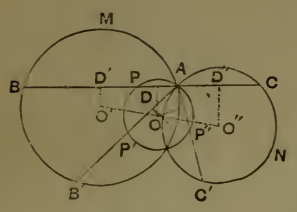

Fig. 453.

Again, he observes that as B moves towards M, and C towards N, P moves towards A, and when AC = AB, P is at A. It now appears probable that the locus of P is a circumference. Proceeding on this hypothesis, he reasons, that, if this is true, AP′ and AP″ are chords of the locus, and, bisecting them with perpendiculars, he will have the centre of the locus. Locating O thus, he observes that it *appears* to be in the line joining the centres O′ and O″, and about midway between them. This leads him to see, whether, by assuming the middle point of O O″ as the centre of a circle, and OA as a radius, he can prove that any such line as BC drawn through A is bisected by this circumference, as at P. This he can readily prove by means of the perpendiculars OD, O′D′, and O″D″, which bisect the chords AP, AB, and AC. For, since these perpendiculars are parallel, and O′O = OO″, D′D = DD″; whence D′P = AD″, and, adding AP to each, D′A = PD″, or D′B = PD″. Adding to BD′, D′P, and D″C (= AD″ = D P), there results BP = PC, and the hypothesis is true.

But, *in giving this problem as a recitation*, the student will proceed as follows: Letting M and N be the two fixed circumferences, intersecting at A, join their centres O′ and O″, and bisect O′O″ as at O. With O as a centre and OA as a radius, describe a circle. Then is this circumference the locus required. For, let BC be any secant line passing through A, we may show that P is the middle point of BC. [Having done this, as above, and shown that any point not in this circumference is not the middle of the secant line passing through A, his solution is complete.]

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875. ***Prob.***—*If the line* AB *is divided at* C, *find the locus of* P, *so that angle* APC = *angle* BPC.

Sug's.—*In seeking for the solution*, the following would be a natural process. Drawing *any* line, as AB, in the lower part of the figure, taking C, any point in it, and *conceiving* BP and AP drawn so as to make *equal angles* with PC, we would naturally discover that, if a circle were circumscribed about BPA, PC produced would bisect the arc below AB. Thus we discover a ready method of locating P; *i. e.*, in the main figure, bisect AB by a perpendicular, as ED, and

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* *Equal* circles would probably have *special* relations.

with any point on ED as a centre, pass a circumference through A and B. Through D and C draw a line, and P is a point in the locus (?). Any number of points can be found in this way; and, having found a few, as P, P′, P″, P‴, etc., the situation of these will suggest that, *probably*, the locus is the circumference of a circle whose centre is in AB produced. *If this should be the fact,* CP is a chord of that circle, and, erecting a perpendicular at the middle point of CP, its intersection with AB produced, as O, will be the centre of the locus (?). We will now endeavor to prove that *any* point in this circumference, as P′, is so situated that BP′C = CP′A, and that no point out of this circumference has this property. We can readily show that the angle OPB = OCP − BPC = OCP − CPA (?). But PAC = OCP − CPA (?). ∴ Triangle OPB is similar to OPA (?), and OA : OP :: OP : OB, or OA : OC :: OC : OB. Now, for *any* point in this circumference, as P′, we shall have OA : OP′ :: OP′ : OB, since OP′ = OC, and OA, OC, and OB are constant. Hence, wherever P′ is taken (in this circumference), the triangle OPB is similar to OPA, angle OP′B = P′AB, and BP′C = CP′A. Finally, that no point out of this circumference possesses this property is evident, since the distance of such a point from O would not equal OC, and the angle OP₁B (P₁ being such a point) would not equal P₁AB.

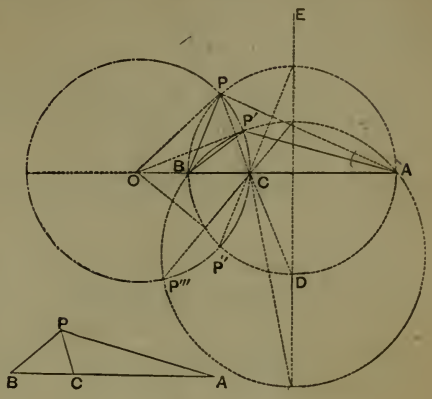

Fig. 454.

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876. Prob.—*In a fixed circle, any two chords intersect at right angles in a fixed point; find the locus of the centre of the chord joining their extremities.* Give the proof.

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877. Prob.—*Find the locus of the point in space equidistant from three given points.* Give the proof.

---

878. Prob.—*Find the locus of the point in space equidistant from two given points.* Give the proof.

---

879. Prob.—*Find the locus of the point in a plane such that the difference of the squares of the distances from it to two fixed points without the plane shall be constant.*

Sug's.—Conceive the two points without the plane joined by a right line, and a perpendicular to this line drawn from either extremity of it; the point where this perpendicular pierces the plane is a point in the locus (?). The required locus is two parallel straight lines.

880. Prob.—*Find the locus of the middle point of a straight line of constant length, whose extremities remain in two lines at right angles to each other, but which are not in the same plane.* Give the proof.

---

881. Prob.—*Find the locus of the point equidistant from two fixed planes.* Give proof.

SUG'S.—Consider, 1st, When the fixed planes are parallel; and 2d, when they intersect.

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882. Prob.—*What locus is the intersection of a plane and the surface of a sphere?* Give proof.

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883. Prob.—*What locus is the intersection of the surfaces of two given spheres?*

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884. Prob.—*Find the locus of the point in space such that the ratio of its distance from a given right line to its distance from a fixed point in that line is constant.*

SECTION II.

OF SYMMETRY.

885. DEF.—Two points are said to be *symmetrical with respect to a third,* when the right line joining the two points is bisected by the point of reference, called the *Centre of Symmetry.*

886. DEF.—Two loci, or two parts of the same locus, are *symmetrical with respect to a point,* when every point in one has its symmetrical point in the other.

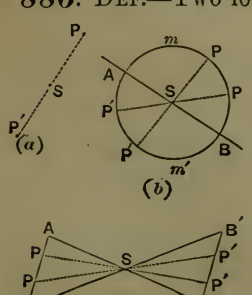

FIG. 455.

ILL'S.—In (*a*) P′ is symmetrical with P in respect to S, if S is the middle point of PP′. In (*b*) we observe that the semi-circumference A*m*′B is symmetrical with the semi-circumference A*m*B, in respect to the centre S; for any point P in the latter has a symmetrical point P′ in the former. In (*c*) the triangle A′SB′ is symmetrical with ASB in respect to S (?).

887. Theo.—*The symmetrical of a straight line, with respect to a point, is an equal straight line.*

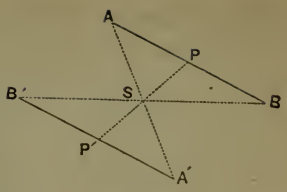

Fig. 456.

Sug's.—We commence by assuming A′S = AS, and B′S = BS, and drawing B′A′. We then have to show that *any* point in AB, as P, has its symmetrical point P′ in B′A′.

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888. Theo.—*The symmetrical of an angle, with respect to a point, is an equal angle.*

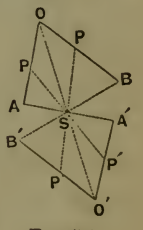

Fig. 457.

Sug's.—To show the symmetrical of AOB, with respect to S, take A′S = AS, B′S = BS, O′S = OS, and draw O′B′, O′A′. Then show that any point in OA has its symmetrical in O′A′, and any point in OB has its symmetrical in O′B′. Hence, A′O′B′ is the symmetrical of AOB, with respect to S.

Then apply AOB to A′O′B′ and show that these symmetricals are equal.

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889. Prob.—*Having given a polygon, to draw its symmetrical with respect to a given point.*

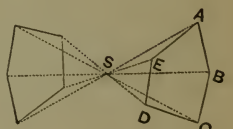

Fig. 458.

890. Theo.—*Any polygon is equal to its symmetrical with respect to a given point.*

Sug's.—Proof by revolving the figure about S, keeping it in its own plane, each line, as AS, ES, etc., passing through 180°.

891. Def.—The lines AS, ES, BS, etc., are *radii of symmetry.*

892. Def.—Two points are *symmetrical with respect to a straight line* in the same plane, when the straight line which joins them is bisected at right angles by the line of reference, called the *Axis of Symmetry.*

893. Def.—Two loci, or two parts of the same locus, are *symmetrical with respect to a straight line* when every point in the one has its symmetrical point in the other.

Ill's.—Thus in (*a*) P and P′ are symmetrical points with respect to the right line X′X, P′*s* = P*s* and X′X is perpendicular to PP′. So the part of (*b*) above X′X is symmetrical with the part below, *i. e.*, the curve is symmetrical with respect to X′X.

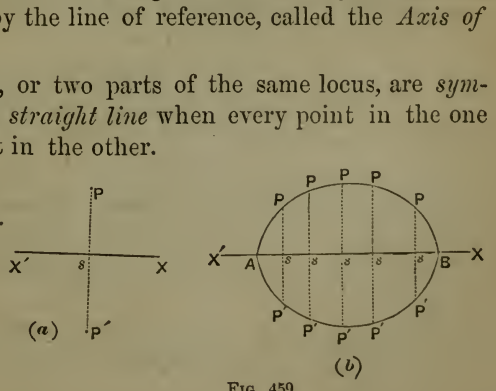

Fig. 459.

894. Theo.—*The symmetrical of a straight line, with respect to a rectilinear axis of symmetry, is an equal right line.*

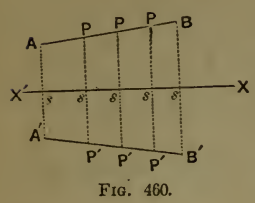

Fig. 460.

Dem.—Let AB be any straight line, and X'X the axis. Let fall the perpendiculars A*s* and B*s*, and produce them till A'*s* = A*s*, and B'*s* = B*s*. Then is AB = A'B', the symmetrical of AB. For, taking P, *any* point in AB, letting fall P*s*, and producing it to P', the point P' is symmetrical with P; since revolving *s*A'B'*s* upon X'X, A'B' will coincide with AB, and P' will fall at P. Hence A'B' = AB, and every point in AB has its symmetrical point in A'B' (***893***).

895. Cor. 1.—*If two straight lines intersect, their symmetricals intersect, and the points of intersection are symmetrical.*

The student should show how this follows from the proposition.

896. Cor. 2.—*Two rectilinear symmetricals meet the axis in the same point, and make equal angles therewith.*

Student give proof.

897. Def.—A *Trapezoid* like ABB'A', having its non-parallel sides equal, is called *Isosceles.*

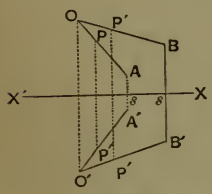

Fig. 461.

898. Theo.—*The symmetrical of an angle, with respect to a rectilinear axis of symmetry, is an equal angle.*

Sug's.—The student should be able to give the demonstration from the figure, in a manner altogether similar to the preceding; or, drawing AB and A'B', he can base it upon the preceding.

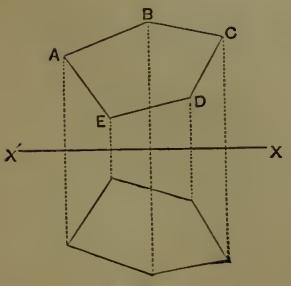

Fig. 462.

899. Prob.—*Having given a polygon, to draw its symmetrical with respect to a given axis.*

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900. Theo.—*Any plane figure is equal to its symmetrical, with reference to a rectilinear axis.*

Proof by applying one to the other by revolution.

901. DEF.—Two loci in space, or two parts of the same locus (planes or solids), are symmetrical with respect to a point, when every point in one has a corresponding point in the other, such that the line joining them is bisected by the point called the centre of symmetry.

ILL'S.—Symmetrical triedrals (PART II., ***432***) afford an illustration of solids symmetrical with respect to a point—the vertex. The two hemispheres into which a great circle divides a sphere are symmetrical parts of the solid (sphere) with reference to the centre.

902. DEF.—Two points in space are symmetrical with respect to a plane called the *Plane of Symmetry*, when the line joining the points is perpendicular to the plane and bisected by it.

903. DEF.—Two loci in space (planes or solids), or two parts of the same locus, are symmetrical with respect to a plane when every point in one has its symmetrical point in the other.

904. DEF.—The corresponding (symmetrical) parts of symmetrical figures are called *Homologous* parts.

---

905. Theo.—*The symmetrical of a right line, with respect to a plane, is an equal right line.*

DEM.—Let AB be any right line, and MN the plane of symmetry. Let fall the perpendiculars Bb, Aa, upon the plane, produce them, making B′b = Bb, A′a = Aa, and join A′ and B′. Then A′B′ = AB, and is its symmetrical. For ABB′A′ being a plane figure (?) and ab, the intersection of this plane with MN, being a right line bisecting AA′ and BB′ at right angles (?), we may revolve abB′A′ upon ab and bring A′B′ into coincidence with AB. Hence A′B′ = AB. Again, P being any point in AB, draw PP′ perpendicular to ab, and upon revolution P′ will fall in P, and P′s = Ps. Hence, every point in AB has its symmetrical in A′B′, and the latter line is the symmetrical of the former.

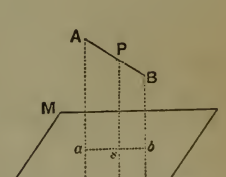

FIG. 463.

906. COR. 1.—*A right line and its symmetrical, with respect to a plane, pierce the plane at the same point.*

Student give proof.

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907. Theo.—*The symmetrical of a plane angle, with respect to a plane, is an equal plane angle.*

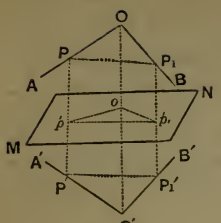

Fig. 464.

Dem.—Let AOB be any plane angle, and MN the plane of symmetry. Let P be any point in AO, and P_1 any point in OB. Let O′ be the symmetrical of O, P′ of P, and P_1' of P_1; then is A′O′ the symmetrical of AO, O′B′ of OB, and angle A′O′B′ of AOB. Now by the preceding proposition the two triangles POP_1, and $P'O'P_1'$, are mutually equilateral, whence AOB = its symmetrical A′O′B′.

Query.—When will the triangle *pop′* exist, and when not?

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908. Theo.—*Any plane polygon has for its symmetrical, with reference to a plane, an equal plane polygon.*

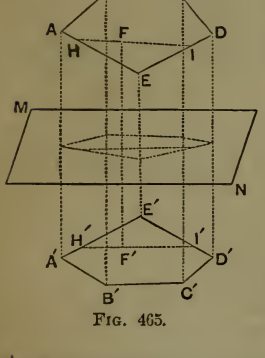

Fig. 465.

Sug's.—ABCDE being any plane polygon, and MN the plane of symmetry, by constructing A′,B′,C′,D′,E′ symmetrical with A, B, C, D, E, we have by the preceding propositions A′B′C′D′E′ equilateral and equiangular with ABCDE; whence it only remains to show that A′B′C′D′E′ is a plane (not a warped) surface. Let F be any point in the angle AED, draw HI, and let H′ and I′ be the symmetricals of H and I (***895***). Draw H′I′. Then is the symmetrical of F in H′I′ (?), as at F′. Now, every point in HF within the angle BAE has its symmetrical in H′F′ (***905***). Thus, by taking three points, not in a straight line, in the angle BAE, we can show that their symmetricals are in the plane B′A′E′, and also in A′E′D′. In like manner, all the angles of A′B′C′D′E′ can be shown to be in the same plane.

909. Cor.—*If two planes intersect, their symmetricals intersect, and the two intersections are symmetrical right lines.*

The student should show how this grows out of the proposition.

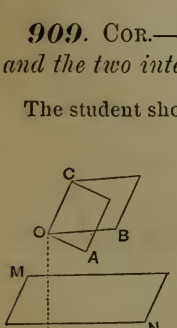

Fig. 466.

910. Theo.—*The symmetrical of a diedral is an equal diedral.*

Dem. AOB being the measure of the diedral A-OC-B, and A′-O′C′-B′ the symmetrical diedral, and O′ the symmetrical of O, the symmetrical of AO being A′O′, the angle A′O′C′ is right, and in like manner B′O′ being the symmetrical of BO, B′O′C′ is right. But BOA = B′O′A′ (?), whence the diedrals are equal.

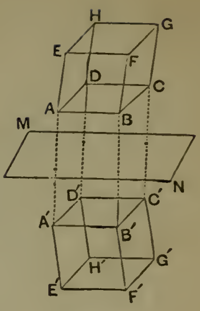

FIG. 467.

911. Theo.—*Two polyedrons, symmetrical with respect to a plane, have their faces equal, each to each, and their homologous solid angles symmetrical.*

SUG'S.—This is an immediate consequence of preceding propositions. Thus E′ being the symmetrical solid homologous with E, the homologous plane faces including them are equal (***908***). Again, the facial angles being equal, but not similarly disposed, the solid angles are symmetrical.

912. COR.—*Two symmetrical polyedrons can be decomposed into the same number of tetraedrons, symmetrical each to each.*

For we can decompose one of the polyedrons into tetraedrons having for their common vertex one of the vertices of this polyedron, and each of these tetraedrons will have its symmetrical in the other.

913. Theo.—*Two symmetrical polyedrons are equivalent.*

DEM.—From the last corollary it will appear that it is sufficient for the demonstration of this proposition to show that two symmetrical tetraedrons are equivalent (?). Let S-ABC, and S′-ABC be two tetraedrons symmetrical with respect to their common base. They have a common base and equal altitudes (?), hence they are equivalent.

914. GENERAL SCHOLIUM.—We may speak of two loci, or two parts of the same locus, as symmetrical with respect to a line or plane, whenever all the points in one have symmetrical points in the other, even though the line joining the symmetrical points be not *perpendicular* to the axis, or the plane, of symmetry; observing, however, that this line is always bisected by the axis or plane. Thus, the ellipse in the figure is symmetrically divided by the line X′X, since every point in one portion has a symmetrical point in the other, as P*s* = P′*s*, for every point in the curve. In such a case the parts cannot be brought into coincidence by simple revolution: one part must be reversed.

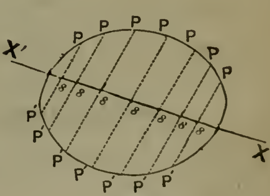

FIG. 468.

SECTION III.

OF MAXIMA AND MINIMA.

915. DEF.—***A Maximum*** value of a magnitude conceived to vary continuously in some specified way, is a value which is greater than the preceding and succeeding values of the magnitude.

FIG. 469.

ILL'S.—Thus, suppose in a given circle, a chord passing through a fixed point, P, revolves so as to take successively the positions 1*a*, 2*b*, Ac, 3*d*, 4*e*, etc. It is at a maximum when it passes through the centre, as Ac. The chord is the magnitude which is conceived to vary in the way specified, and Ac is a value greater than the preceding and the succeeding values. Again, conceive a circle to be compressed or extended, as in the direction *mn*, so as to take the forms indicated by the dotted lines, its *area* will be diminished, the perimeter remaining the same. That is, of all figures of a given perimeter, the circle has the maximum area.

FIG. 470.

916. DEF.—***A Minimum*** value of a magnitude conceived to vary continuously in some specified way, is a value which is less than the preceding and succeeding values of the magnitude.

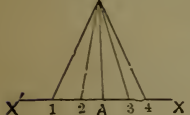

FIG. 471.

ILL'S.—Thus, conceive the varying magnitude to be a straight line from the fixed point P to the fixed line X'X; that is, suppose such a line to start from some position P1, and move through the successive positions P2, PA, P3, P4. PA is a minimum, since it is less than the preceding and succeeding values.

PROPOSITIONS CONCERNING MAXIMA AND MINIMA.

917. ***Axiom.***—*The minimum distance between two points is a straight line.*

---

918. ***Theo.***—*The minimum distance from a point to a line is a straight line perpendicular to the given line.*

Student give proof.

---

919. ***Theo.***—*The maximum line which can be inscribed in a given circle is a diameter.*

Proof based on the fact that the hypotenuse of a right angled triangle is the greatest side.

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920. Theo.—*The sum of the distances from two points on the same side of a line, to a point in the line, all being in the same plane, is a minimum when the lines measuring the distances make equal angles with the given line.*

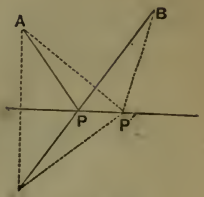

FIG. 472.

Student prove $AP + BP < AP' + BP'$.

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921. Theo.—*If a triangle have a constant base and altitude, its vertical angle is a maximum when the triangle is isosceles.*

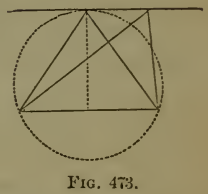
FIG. 473.

SUG.—By what is the vertical angle measured?

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922. Theo.—*The base and area of a triangle being constant, its perimeter is a minimum when the triangle is isosceles.*

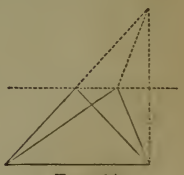
FIG. 474.

SUG'S.—The area and base being constant, the vertex remains in a line parallel to the base, for all values of the other sides. The figure will suggest the demonstration, which is based on the fact that any side of a triangle is less than the sum of the other two.

---

923. Theo.—*The difference between the distances from two points on opposite sides of a fixed line to a point in that line, is a maximum, when the lines measuring these distances make equal angles with the fixed line.*

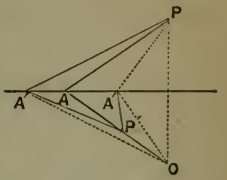

FIG. 475.

SUG'S. $P'O = AP - AP'$; but $P'O > A'O\ (= A'P) - A'P'$.

QUERY.—Having the points P, P', and the fixed line given, how is the point A found by geometrical construction?

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924. Theo.—*The lengths of two sides of a triangle being constant, the area is a maximum when the included angle is right.*

FIG. 476.

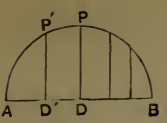

Fig. 477.

925. Theo.—*The sum of two adjacent sides of a rectangle being constant* (AB), *the area is a maximum when the sides are equal.*

ISOPERIMETRY.

926. Isoperimetric Figures are such as have equal perimeters, *i. e.*, bounding lines of equal length.

Problems in isoperimetry are a species of problems in Maxima and Minima. Thus, of all figures whose perimeters are m (say 10 inches), to find that which has the greatest area, is a problem in isoperimetry. Again, what must be the form of a pentagon whose perimeter is m, in order that its area may be a maximum?

---

927. Theo.—*Of isoperimetric triangles with a constant base, the isosceles is a maximum.*

Sug's.—By means of the figure to Theorem (***921***), we can readily show that any triangle having the same base as the isosceles triangle, and its vertex either in or beyond the line through the vertex of the isosceles triangle and parallel to its base, has a greater perimeter than the isosceles triangle. Hence, the isoperimetric triangle on the given base has its vertex below this parallel, except when isosceles; and consequently the isosceles is the maximum.

928. Cor.—*Of isoperimetric triangles, the equilateral has the maximum area* (?)

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929. Prob.—*Given any triangle with a constant base, to construct the maximum isoperimetric triangle.*

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930. Prob.—*Given any triangle, to construct the maximum isoperimetric triangle.*

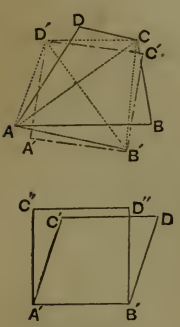

Fig. 478.

931. Theo.—*Of isoperimetric quadrilaterals, the square has the maximum area.*

Dem.—Let ABCD be any quadrilateral. If AD is not equal to DC, ADC can be replaced by the isosceles isoperimetric triangle AD′C, and the area of the quadrilateral increased. So ABC can be replaced by AB′C. Therefore AB′C′D > ABCD. In like manner if AD′ is not equal to AB′, D′AB′ can be replaced by the maximum isoperimetric triangle D′A′B′. So also D′CB′ can be replaced by D′C′B′. Therefore A′B′C′D′ > AB′CD′ > ABCD. Now, A′B′C′D′ is a rhombus (?), and the student can show that the square on A′B′ is greater than *any* rhombus with the same side.

***932. Prob.**—Having given a quadrilateral, to construct the maximum isoperimetric quadrilateral.*

***933. Theo.**—Of isoperimetric quadrilaterals with a constant base, the maximum has its three remaining sides equal each to each, and the angles which they include equal.*

DEM.—Let ABCD be the maximum isoperimetric quadrilateral on the base AD, then AB = BC = CD, and angle ABC = BCD. For, if AB is not equal to BC, draw AC, and replacing the triangle ABC with its isoperimetric isosceles triangle, we shall have a quadrilateral isoperimetric with ABCD, and greater than ABCD, *i. e.*, greater than the maximum, which is absurd.

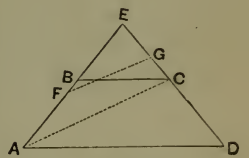

FIG. 479.

Again, if angle ABC is not equal to BCD, let ABC < BCD, whence BCE < EBC, and BE < EC. Take EF = EC, and EG = EB, whence the triangles FEG and BEC are equal, and FG = BC. Also, since AB + BC + CD = AE + ED − (EB + EC) + BC, and AF + FG + GD = AE + ED − (FE + EG) + FG, it follows that AFGD and ABCD are isoperimetrical, and, since ABCD = AED − BEC, and AFGD = AED − FEG, that AFGD and ABCD are equal. Therefore, AFGD is a maximum, and by the preceding part of the demonstration AF = FG = BC = AB, which is absurd; and there can be no inequality between angles ABC and BCD.

***934. Theo.**—Of isoperimetric polygons of a given number of sides, the regular polygon has the maximum area.*

DEM.—First, the polygon must be equilateral; for, if any two adjacent sides, as AB, BC, are unequal, the triangle ABC can be replaced by its isoperimetric isosceles triangle, and thus the area of the polygon be increased.

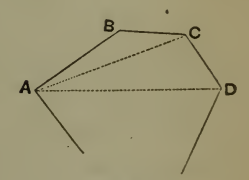

FIG. 480.

Second, the polygon must be equiangular; for, if any two adjacent angles, as B and C, are unequal, the quadrilateral ABCD can be replaced by its isoperimetric quadrilateral with B = C, and thus the area of the polygon be increased.

***935. Theo.**—Of isoperimetric regular polygons, the one of the greater number of sides is the greater.*

DEM.—Let ABC be an equilateral (regular) triangle. Join any vertex, as A, with any point, as D, in the opposite side. Replace the triangle ACD with the isosceles isoperimetric triangle AED. Then is the quadrilateral ABDE > the triangle ABC.

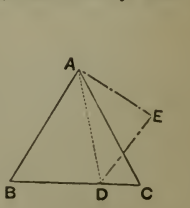

FIG. 481.

But, of isoperimetric quadrilaterals, the regular (the square) is the greater. Hence, the regular quadrilateral (the square) isoperimetric with the triangle ABC, is greater than

the triangle. In the same manner the regular pentagon isoperimetric with the square can be shown greater than the square; and thus on, *ad libitum*.

936. COR.—*Of plane isoperimetric figures, the circle has the maximum area,* since it is the limiting form of the regular polygon, as the number of its sides is indefinitely increased.

SECTION IV.

OF TRANSVERSALS.

937. DEF.—***A Transversal*** is a line cutting a system of lines. A transversal of a triangle is a line cutting its sides;* it either cuts two sides and the third side produced, or the three sides produced. In speaking of the transversal of a triangle (or polygon), the distances on any side (or side produced) from the intersection of the transversal with that side to the angles, are *Segments*. Of these there are six. *Adjacent segments* are such as have an extremity of each at the same point. *Non-adjacent segments* are such as have no extremity common.

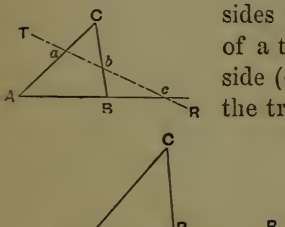

FIG. 482.

ILL'S. TR is a transversal of the triangle ABC; aA, aC, bC, bB, cA, cB are adjacent segments two and two; aC, bB, cA, and aA, bC, cB are the two groups of non-adjacent segments.

938. THE TWO FUNDAMENTAL PROPOSITIONS OF THE THEORY OF TRANSVERSALS.

939. Theo.—*The product of three non-adjacent segments of the sides of a triangle cut by a transversal, is equal to the product of the other three.*

DEM.—ABC being cut by the transversal TR, $aA \times bC \times cB = aC \times bB \times cA$. Draw BD parallel to AC, and from the similar triangles we have

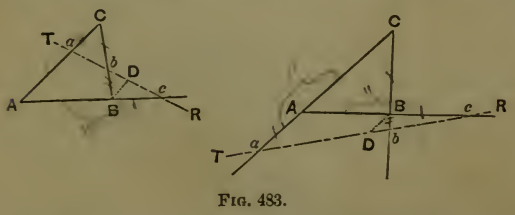

FIG. 483.

$$\frac{aA}{BD} = \frac{cA}{cB}, \text{ and } \frac{DB}{bB} = \frac{aC}{bC};$$

whence, multiplying,

$$\frac{aA}{bB} = \frac{aC}{bC} \times \frac{cA}{cB},$$

or $aA \times bC \times cB = aC \times bB \times cA$.

* Or, sides produced—this expression being usually omitted in higher Geometry as all lines are to be considered indefinite unless limited in the problem.

940. Cor.—Conversely, *If three points be taken in the sides of a triangle* (as a, b, c) *such that the product of three non-adjacent segments equals the product of the other three, the points are in the same straight line.*

For, passing a line through a and b, let it cut the third side in c'. Then, by the proposition, $aA \times bC \times c'B = aC \times bB \times c'A$. But, by hypothesis $aA \times bC \times cB = aC \times bB \times cA$. Whence $\frac{cB}{c'B} = \frac{cA}{c'A}$, and c and c' must coincide.

Sch.—This theorem is known among mathematicians as *The Ptolemaic Theorem*, and is usually attributed to Claudius Ptolemy, an Egyptian mathematician and philosopher who flourished in Alexandria during the first half of the second century. But it is thought to be more properly due to *Menelaus*, who lived a century before Ptolemy.

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941. Theo.—*The three angle-transversals* of a triangle, passing through a common point, divide the sides into segments such that the product of three non-adjacent segments equals the product of the other three.*

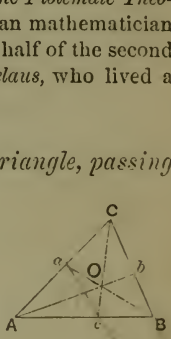

Fig. 484.

Dem.—From the triangle ACc cut by the transversal aB, we have $aA \times CO \times cB = AB \times aC \times Oc$; and from CBc cut by bA, $Oc \times bC \times AB = CO \times bB \times cA$. Multiplying, we obtain $aA \times bC \times cB = aC \times bB \times cA$.

942. Cor. 1.—Conversely, *If the three angle-transversals of a triangle divide the sides into segments such that the product of three non-adjacent segments equals the product of the other three, the transversals pass through a common point.*

For, the sides being divided at a, b, and c, so that $aA \times bC \times cB = aC \times bB \times cA$, draw Cc, and A$b$, and let O be their intersection. Now, let a' be the point in which BO cuts AC. Then, by the proposition, $a'A \times bC \times cB = a'C \times bB \times cA$.

$$\text{Whence } \frac{aA}{a'A} = \frac{aC}{a'C}, \text{ and } a \text{ and } a' \text{ coincide.}$$

943. Cor. 2.—*If any one of the sides is bisected, the line joining the other points of division is parallel to this side.*

For, let $bC = bB$. Then $aA \times bC \times cB = aC \times bB \times cA$, becomes
$aA \times cB = aC \times cA$; or $Aa : aC :: Ac : cB$.

Query.—How does this apply to the second figure?

944. Cor. 3.—*If the line joining two points of division is parallel to the third side, the latter side is bisected.*

---

* The transversals passing through the angles.

For, if ab is parallel to AB, $aC : bC :: aA : bB$, whence $aC \times bB = bC \times aA$. And, since $aA \times bC \times cB = aC \times bB \times cA$, $cA = cB$.

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945. We will now give a few problems to illustrate the use of the theory of transversals.

946. ***Prob.***—*To show that the medial lines of a triangle pass through a common point.*

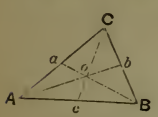

FIG. 485.

SOLUTION.—Since $aA = aC$, $bC = bB$, and $cB = cA$, by multiplying, we have $aA \times bC \times cB = aC \times bB \times cA$; whence by the last corollary these transversals pass through a common point.

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947. ***Prob.***—*To show that the bisectors of the angles of a triangle pass through a common point.*

SOLUTION.—In the last figure let aB, bA, cC be the bisectors.

Then $\frac{aA}{aC} = \frac{AB}{CB}$, $\frac{bC}{bB} = \frac{AC}{AB}$, $\frac{cB}{cA} = \frac{CB}{AC}$; multiplying, $\frac{aA \times bC \times cB}{aC \times bB \times cA} = 1$, or $aA \times bC \times cB = aC \times bB \times cA$. Therefore these transversals pass through a common point.

---

948. ***Prob.***—*To show that the altitudes of a triangle pass through a common point.*

SUG'S.—In the last figure, if aB, bA, cC, were the perpendiculars, there would be three pairs of similar triangles giving $\frac{aA}{bB} = \frac{AO}{BO}$, $\frac{bC}{cA} = \frac{CO}{AO}$, $\frac{cB}{aC} = \frac{OB}{CO}$; whence, as in the last.

---

949. ***Prob.***—*To show that the angle-transversals terminating in the points of tangency of the sides of the triangle with its inscribed circle, pass through a common point.*

SUG'S.—In the last figure, if a, b, c were the points of tangency we should have $aA = cA$, $bC = aC$, $cB = bB$; whence $aA \times bC \times cB = aC \times bB \times cA$. Which shows that the transversals pass through a common point.

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950. ***Theo.***—*If two sides of a triangle are divided proportionally, starting from the vertex, the angle-transversals from the extremities of the other side to the corresponding points of division, intersect in the medial line to this third side.*

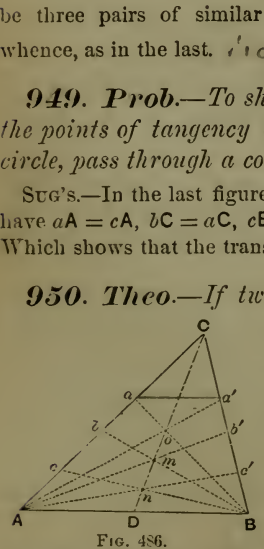

FIG. 486.

DEM.—Since AC and CB are divided proportionally at a and a', $aA \times a'C = aC \times a'B$; and as $DB = DA$, $aA \times a'C \times DB = aC \times a'B \times DA$, the angle-transversals Aa', Ba intersect in CD. The same may be shown of any other angle-transversals from A and B, dividing CB and CA proportionally.

951. Cor.—*In any trapezoid the transversal passing through the intersection of the diagonals, and the intersection of the non-parallel sides, bisects the parallel sides.*

Sug.—Joining aa' in the last figure, CD is such a transversal. The student will readily see the connection with the proposition.

---

952. Prob.—*Through a given point to draw a line which shall meet two given lines at their intersection in an invisible, inaccessible point.*

Solution.—Let Mm, Nn be the two given lines which meet in the invisible, inaccessible point S, and P the given point through which a line is to be located which will meet Mm, Nn in S. Through P draw any convenient transversal, as BF, and any other meeting this, as AF. Now, considering MS as a transversal of the triangle CDF, we have $AF \times BC \times SD = AD \times BF \times SC$; whence $\frac{SD}{SC} = \frac{AD \times BF}{AF \times BC}$. But, HD being drawn parallel to BF, we have $\frac{HD}{PC} = \frac{SD}{SC} = \frac{AD \times BF}{AF \times BC}$, or $HD = \frac{AD \times BF \times PC}{AF \times BC}$; whence HD is known, as AD, BF, PC, AF, BC can be measured. The points P and H determine the required line.

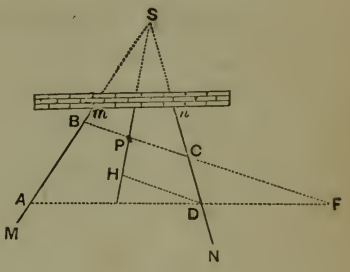

Fig. 487.

953. Def.—The ***Complete Quadrilateral*** is the figure formed by four lines meeting in six points. The complete quadrilateral has three diagonals.

Ill.—ABCDEF is a complete quadrilateral, and its diagonals are CF, BD, and AE, the latter being spoken of as the third or exterior diagonal.

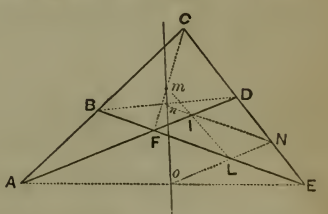

Fig. 488.

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954. Theo.—*The middle points of the three diagonals of a complete quadrilateral are in the same straight line.*

Dem.—m, n, o being the centres of the diagonals of the complete quadrilateral, in the preceding figure, are in the same straight line. Bisect the sides of the triangle FDE, as at I, N, L, and draw IN, IL, LN. Since IN is parallel to BE, and bisects DF and DE, it also bisects DB (?) and hence passes through n. For like reasons IL passes through m, and LN through o. Now, AC being a transversal of the triangle FDE gives $CD \times BE \times AF = CE \times BF \times AD$. Therefore,

noticing that $\frac{1}{2}CD = mI$, $\frac{1}{2}BE = nN$, $\frac{1}{2}AF = oL$, $\frac{1}{2}CE = mL$, $\frac{1}{2}BF = nI$, and $\frac{1}{2}AD = oN$, we have $mI \times nN \times oL = mL \times nI \times oN$. Hence these three points m, n, o lie in a transversal to the triangle ILN.

SECTION V.

HARMONIC PROPORTION AND HARMONIC PENCILS.

955. DEF.—Three quantities are in *Harmonic Proportion* when the difference between the first and second is to the difference between the second and third, as the first is to the third.

ILL.—6, 4, 3 are in harmonic proportion, since $6 - 4 : 4 - 3 :: 6 : 3$. In general, a, b, c are in harmonic proportion, if $a - b : b - c :: a : c$.

956. ***Theo.***—*If a given line be divided internally and externally in the same geometric ratio, the distance between the points of division is a harmonic mean between the distances of the extremities of the given line from the point not included between them.*

DEM. — Let AB be the given line; and let O and O′ be so taken that $AO : BO :: AO' : BO'$; then is OO′ a harmonic mean between AO′ and BO′. For $AO = AO' - OO'$, and $BO = OO' - BO'$; whence AO′, OO′, and BO′ are such that $AO' - OO' : CO' - BO' :: AO' : BO'$, that is, they are in harmonic proportion.

FIG. 490.

957. COR. 1.—AO, AB, *and* AO′ *are in harmonic proportion*, i.e., AB *is a harmonic mean between* AO *and* AO′.

For $AB - AO (= BO) : AO' - AB (= BO') :: AO : AO'$.

958. COR. 2.—*When* AO, OO′, BO′ *are in harmonic proportion*, $AO \times BO' = BO \times AO'$.

959. COR. 3.—Conversely, *When a line is divided into three parts such that the rectangle of the extreme parts equals the rectangle of the mean part into the whole line, the line is divided harmonically.*

Thus, let AO′ be the line, and $AO \times BO' = BO \times AO'$; then $AO : BO :: AO' : BO'$, whence, by the proposition, OO′ is a harmonic mean between AO′ and BO′.

DEF.—The points O and O′ are called *Harmonic Conjugates.*

960. Theo.—*If two lines be drawn, one bisecting the interior and the other the adjacent exterior angle of a triangle, and meeting the opposite side,* they divide this line harmonically.*

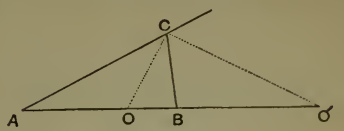

FIG. 491.

SUG.—By means of (***358, 359,*** PART II.) the student will be enabled to establish the relation AO : BO : : AO′ : BO′, whence, by the last proposition, AO′, OO′, BO′ are in harmonic proportion.

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961. Theo.—*In the complete quadrilateral, any diagonal is divided harmonically by the other two.*

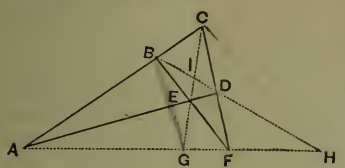

FIG. 489.

DEM.—Thus, AFH is divided harmonically at G and H. For, considering BH as a transversal of the triangle ACF, we have HF × DC × BA = HA × DF × BC. And CG, AD, FB being angle-transversals of the same triangle, we have GF × BA × DC = GA × BC × DF. Whence, dividing, $\frac{HF}{GF} = \frac{HA}{GA}$, *i.e.*, AH is divided harmonically. Again, if CH is drawn, CA, CG, CF, CH constitute a harmonic pencil, and BH, a transversal of it, is cut harmonically at B, I, D, H. Finally, if F and I be joined, FH (or FA), FB, FI, FD constitute a harmonic pencil, and hence CG is cut harmonically at C, I, E, G.

962. COR.—An angle-transversal of a triangle, and a line passing through the feet of the other angle-transversals, divide the third side harmonically.

---

963. Prob.—*Given a right line to locate two harmonic conjugate points.*

SOLUTION.—Let AB be the line. O may be taken at pleasure between A and B. We are then to find O′, so that AO : BO : : AO′ : BO′. Taking this by division, we have AO − BO : BO : : AO′ − BO′ (= AB) : BO′. The first three terms being known, the other can be constructed. Or, we may first locate O′ at pleasure, and then find O.

---

964. Theo.—*If from the given point* C *in a line the distances* CO, CB, CO′ *be taken in the same direction, so that* CO × CO′ = $\overline{CB}^2$; *and if* CA = CB *be taken in the opposite direction,* AO′ *will be divided harmonically at* O *and* B.

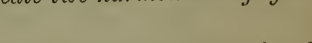

FIG. 492.

DEM. — From CO × CO′ = $\overline{CB}^2$, we readily write CO : CB : : CB : CO′, CB + CO (= AO) : CB − CO (= BO) : : CO′ + CB (= AO′) : CO′ − CB (= BO′).

* The bisector of the exterior angle meets the side *produced*; but in higher geometry, as it is always understood that lines are indefinite unless limited by hypothesis, such specifications are deemed unnecessary.

AO : OB : : AO′ : BO′

965. Cor. 1.—Conversely, *If a line* AO′ *be cut harmonically at* O *and* B, *and either of the harmonic means be bisected, as* AB *at* C, *the three segments* CO, CB, CO′ *will be in geometric proportion.*

For, since AO′ : BO′ :: AO : BO, AO′ + BO′ : AO′ − BO′ :: AO + BO : AO − BO, or 2CO′ : 2CB :: 2CB : 2CO, and CO′ : CB :: CB : CO.

966. Cor. 2.—In a given line, as AB, as O approaches the centre C, O′ recedes, and when O is at C, O′ is at infinity, since $CO' = \frac{\overline{CB}^2}{CO}$.

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967. Theo.—*The geometric mean between two lines is also the geometric mean between their arithmetic and harmonic means.*

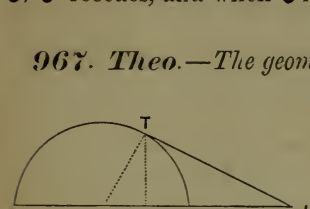

Fig. 493.

Dem.—Let AO′ and BO′ be the two lines. On their difference, AB, draw a semicircle, draw the tangent O′T and let fall the perpendicular TO. Then O and O′ are harmonic conjugates, since $CO \times CO' = \overline{CB}^2$ (?), CO′ is the arithmetic mean (that is, ½ the sum) of AO′ and BO′ (?) and TO′ is the geometric mean (?). ∴ CO′ : TO′ :: TO′ : OO′ (?).

Queries.—Which is the greatest, in general, the arithmetic, geometric, or harmonic mean between two quantities? Are they ever equal?

968. Sch.—*This proposition* affords a ready method of finding either of the harmonic conjugates O or O′, when the other is given. The student will show how.

969. Cor. 1.—*The rectangle of the harmonic means and the sum of the extremes, is equivalent to twice the rectangle of the extremes.*

For, $CO' \times OO' = \overline{TO}^2 = AO' \times BO'$, whence $2CO' \times OO' = 2AO' \times BO'$; and, since $2CO' = AO' + BO'$, $(AO' + BO') \times OO' = 2AO' \times BO'$.

970. Cor. 2.—*The rectangle of the harmonic mean and the difference of the differences of the* 1*st and* 2*nd, and the* 2*nd and* 3*rd, is equivalent to twice the rectangle of these differences.*

That is, $OO' \times [(AO' - OO') - (OO' - BO')] = 2(AO' - OO')(OO' - BO')$, or $OO' \times (AO - BO) = 2AO \times BO$. Let the student give the proof.

971. Cor. 3.—*If three quantities are in harmonic proportion their reciprocals are in arithmetic proportion* (*i.e.*, the difference between the 1st and 2nd equals the difference between the 2nd and 3rd).

For, from AO′, OO′, BO′, we have the reciprocals $\frac{1}{AO'}, \frac{1}{OO'}, \frac{1}{BO'}$. Now $\frac{1}{OO'} - \frac{1}{AO'} = \frac{AO' - OO'}{OO' \times AO'} = \frac{AO}{OO' \times AO'}$; and $\frac{1}{BO'} - \frac{1}{OO'} = \frac{OO' - BO'}{OO' \times BO'}$

$= \frac{BO}{OO' \times BO'}$. But $\frac{AO}{OO' \times AO'} = \frac{BO}{OO' \times BO'}$ since $\frac{AO}{AO'} = \frac{BO}{BO'}$ (?). $\therefore \frac{1}{OO'} - \frac{1}{AO'} = \frac{1}{BO'} - \frac{1}{OO'}$.

---

972. Prob.—*Given the harmonic mean and the difference between the extremes, to find the extremes.*

Sug's.—We have OO′ and AB, (*Fig.* 493, Art. ***967***) given. Then $CO \times CO' = \overline{CT}^2 = \frac{1}{4}\overline{AB}^2$, and $CO' - CO = OO'$, whence $\overline{CO'}^2 - OO' \times CO' = \frac{1}{4}\overline{AB}^2$. From this equation CO′ can be constructed (***832***), and the problem solved.

---

973. Theo.—*When two circles cut each other orthogonally* (*i. e.*, so that the tangents at the common point are at right angles), *any line passing through the centre of one, and cutting the other, is divided harmonically by the circumferences.*

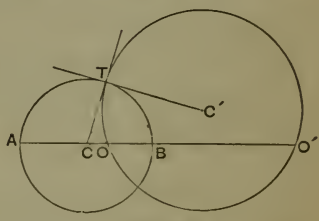

Fig. 494.

Dem.—The tangents being perpendicular to each other pass through the centres, hence $CO \times CO' = \overline{CT}^2$. But CB = CT. Therefore AO′ is cut harmonically.

---

974. Prob.—*To find the altitude of a triangle in terms of the radii of the escribed circles touching the adjacent sides.*

Solution.—Let r and r' be the radii of the escribed circles, and p the altitude. Now RT, AT, and QT are in harmonic proportion; since, considering the triangle ACT, CQ bisects its interior and RC its exterior angle (?), we have QT : QA :: RT : RA. But r, p, r', sustain the same relation to each other as RT, AT, QT; hence r, p, r' are in harmonical proportion. Therefore, by (***969***) $p(r+r') = 2rr'$; or $p = \frac{2rr'}{r+r'}$.

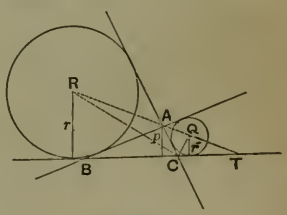

Fig. 495.

HARMONIC PENCILS.

975. Def.—***A Pencil*** of lines is a series of lines diverging from a common point.

Def.—***A Harmonic Pencil*** is a pencil of four lines cutting another line harmonically.

Ill.—In the following figure OA, OB, OC, OD constitute a *Harmonic Pencil*, if they divide the line *mn* harmonically at A, B, C, D.

976. Theo.—*A harmonic pencil divides harmonically every line which cuts it.*

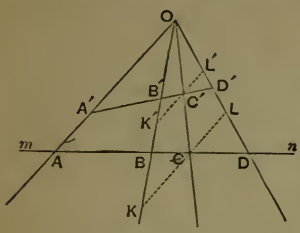

Fig. 496.

Dem.—OA, OB, OC, OD being a harmonic pencil, that is, AD, BD, CD, being in harmonic proportion, A′D′, any other line cutting the pencil, is divided harmonically, so that A′D′, B′D′, C′D′, are in harmonic proportion. Through C and C′ draw parallels to OA, as LK and L′K′. Now, from similar triangles, AB : BC : : AO : CK, and AO : CL : : AD : CD. But AD : CD : : AB : BC, since AD is harmonically divided. Hence AO : CK : : AO : CL, and CK = CL. Hence from similar triangles C′K′ = C′L′. Again A′B′ : B′C′ : : A′O : C′K′ (?), and A′D′ : C′D′ : : A′O : C′L′ (= C′K′) (?), whence A′B′ : B′C′ : : A′D′ : C′D′, or A′D′, B′D′, C′D′ are in harmonic proportion.

Sch.—If the line through C′ cut the prolongation of AO beyond O, it is still harmonically divided; and, in fact, it is scarcely necessary to make this statement, since in all general discussions lines are to be considered indefinite, unless limited by hypothesis.

977. Def.—The alternate legs of a harmonic pencil are called *conjugate*, as OA and OC, OB and OD.

---

978. Theo.—*If two conjugate legs of a harmonic pencil be at right angles, one of them bisects the angle included by the other pair, and the other the supplement of this angle.*

Sug.—This is the converse of (***962***), remembering that the bisectors of two adjacent supplemental angles are at right angles.

SECTION VI.

ANHARMONIC RATIO.

979. Def.—***The Anharmonic Ratio*** of four points in a right line is the ratio of the rectangle of the distance between the first and fourth into the distance between the second and third to the rectangle of the distance between the first and second into the distance between the second and fourth.

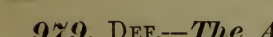

Fig. 497.

Ill.—The anharmonic ratio of the four points A, B, C, D is AD × BC : AB × CD.

980. The relation $AD \times BC : AB \times CD$ is expressed for brevity [ABCD].

ILL.—Thus [ABCD] means $AD \times BC : AB \times CD$; [ADBC] means $AC \times DB : AD \times BC$; [BACD] means $BD \times AC : BA \times CD$; etc. The ratio [ABCD], or $AD \times BC : AB \times CD$ is evidently the same as $\frac{AB}{BC} : \frac{AD}{CD}$.

981. SCH.—The appropriateness of the term *anharmonic* (*not*-harmonic) will be seen when we observe that, if AD is *harmonically* divided, $\frac{AB}{BC}$ *equals* $\frac{AD}{CD}$. If, therefore, $\frac{AB}{BC}$ is *not* equal to $\frac{AD}{CD}$, which is the *general* case of division, irrespective of the position of the points B and C, we may consider the *ratio* of $\frac{AB}{BC}$ to $\frac{AD}{CD}$. This general ratio, or, what is the same thing, $AD \times BC : AB \times CD$, is called the anharmonic ratio.

---

982. Theo.—*The anharmonic ratio of four points is not changed by interchanging two of the letters, provided the other two be interchanged at the same time.*

DEM. [ABCD] = [DCBA] = [BADC] = [CDAB], *i. e.*, $AD \times BC : AB \times CD = DA \times CB : DC \times BA = BC \times AD : BA \times DC = CB \times DA : CD \times AB$, which are evidently identical. [The student should notice the different segments of the line indicated by the different forms.]

983. SCH.—But [ACBD] is a *different* anharmonic ratio from [ABCD]; since $AD \times CB : AC \times BD$ is not necessarily equal to $AD \times CB : AB \times CD$. Now, as there can be twenty-four permutations of four letters, there may be formed six different anharmonic ratios from four given points in a line.

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984. Theo.—*If a pencil of four lines is cut by any transversal, the anharmonic ratio of the four points of intersection is constant.*

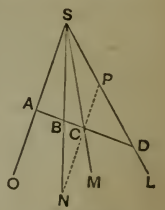

FIG. 498.

DEM. SL, SM, SN, SO, or, as we may read it, S-L,M,N,O, being such a pencil, and AD any transversal, draw through C NP parallel to SO. Then,

$$AD \times BC : AB \times CD :: \left\{ \begin{matrix} BC : AB \\ AD : CD \end{matrix} \right\} :: \left\{ \begin{matrix} CN : AS \\ AS : CP \end{matrix} \right\} :: CN : CP.$$

But CN : CP is constant for all positions of C on SM. Therefore $AD \times BC : AB \times CD$ is constant for any transversal.

985. SCH.—Other constant ratios may be written from the preceding proposition and scholium. The anharmonic ratio [ABCD] is called the anharmonic ratio of the pencil. The angles of the pencil are the six angles included by the rays.

986.—Cor. 1.—*If two pencils are mutually equiangular their anharmonic ratios are equal.*

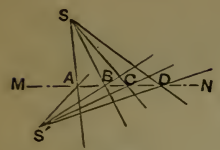

Fig. 499.

Query.—Is the converse of this corollary true? No!

987. Cor. 2.—*If two pencils have their intersections in the same right line, their anharmonic ratios are equal.*

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988. Def.—The anharmonic ratio of four points on the circumference of a circle is the anharmonic ratio of the pencil formed by joining these points with any point in the circumference.

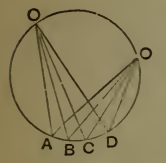

Fig. 500.

Ill.—Thus, the anharmonic ratio of the points A, B, C, D is the anharmonic ratio of the pencil O-A,B,C,D, it being immaterial where in the circumference the point O is taken, since by Cor. 1, preceding, the ratio is the same for any position of O (?).

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989. Theo.—*If four fixed tangents to a circle are cut by a fifth, the anharmonic ratio of the four points of intersection, called the anharmonic ratio of the tangents, is constant.*

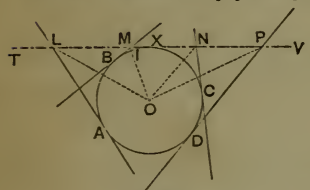

Fig. 501.

Dem. A, B, C, D being the fixed points of tangency, any transversal, as TV, cutting the tangents, has the anharmonic ratio [LMNP] constant. For the pencil O-L,M,N,P has its angles constant. Thus LOM is measured by $\frac{1}{2}$ arc (AX—BX) = $\frac{1}{2}$AB, which is constant. And in like manner MON is measured by $\frac{1}{2}$ arc BC, and NOP is measured by $\frac{1}{2}$ arc CD. Hence, by the first of the preceding corollaries, the anharmonic ratio [LMNP] is constant.

---

990. The theory of anharmonic ratio is applied with great facility to the demonstration of theorems showing that several points are in a right line, and that several lines intersect in a common point. We give three specimens of each class.

991. Theo.—*If two pencils have the same anharmonic ratio and a homologous ray common, the intersection of the other homologous rays are in the same right line.*

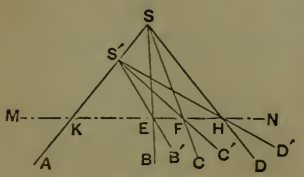

Fig. 502.

Dem.—Let S-A,B,C,D and S'-A,B',C',D' be two pencils having the same anharmonic

ratio, and the rays SA, S'A coincident; then the intersections E, F, H are in the same right line. Let the line passing through E and F intersect SA in K, and suppose it intersect SD in H', and S'D' in H''. Then, since the anharmonic ratios of the two pencils are equal [KEFH'] = [KEFH'']; whence H' and H'' are the same point, and must be the intersection of the two lines SD, S'D', that is, H.

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992. Theo.—*If in two right lines four points in the one have the same anharmonic ratio as four points in the other, and one homologous point in common, the three lines passing through the other pairs of homologous points meet in a common point.*

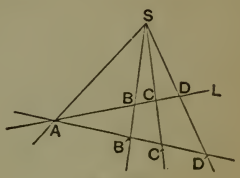

FIG. 503.

DEM.—Let A be common, and [ABCD] = [A B'C'D']. Draw SA and SD'. Call the point in which SD' cuts AL D'' (for the time being). Then [AB'C'D'] = [ABCD'']. But by hypothesis [AB'C'D'] = [ABCD]. Therefore [ABCD] = [ABCD''], and D and D'' are one and the same point. Hence the three lines which pass through B and B', C and C', D and D' meet in a common point S.

---

993. Theo.—*If the lines passing through the corresponding vertices of two triangles meet in a common point, the intersections of their homologous sides lie in the same right line.*

DEM.—Let ABC and A'B'C' be two triangles so situated that the lines AA', BB', CC' meet in the common point S; then L, M, N, the intersections of the homologous sides, are in a right line. For the pencil S-L, B, A, C being cut by the two transversals LD, LD', gives [LBAD] = [LB'A'D'] (***984***). But C-L,B,A,D, and C'-L,B',A',D', have these anharmonic ratios, hence C-L, O,M,N, and C'-L,O,M,N, their equivalents, and having a common ray CC', have equal anharmonic ratios, and consequently L, M, N are in the same right line (***991***).

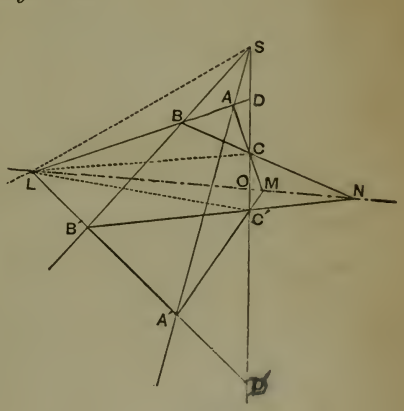

FIG. 504.

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(26) ***994. Theo.***—*If the intersection of the corresponding sides of two triangles are in the same right line, the lines passing through their corresponding angles meet in a common point.*

DEM.—In the last figure, if AB and A'B', AC and A'C', BC and B'C' have their

intersections in the same right line, as LN, the lines passing through B and B′, A and A′, C and C′ meet in a common point, as S. By (*987*) C-L,O,M,N has the same anharmonic ratio as C′-L,O,M,N, whence [LBAD] = [LB′A′D′], and the truth of the theorem follows from (992).

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995. Theo.—*The opposite sides of an inscribed hexagon have their intersections in the same straight line.*

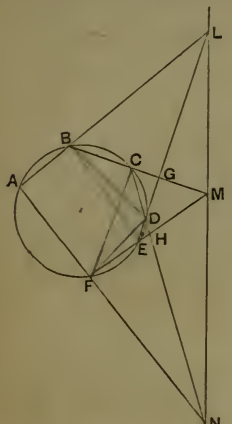

FIG. 505.

DEM.—The anharmonic ratios of the pencils B-A,E,D,C,* and F-A,E,D,C being equal (*988*), LGDE, which intersects the first, is divided in the same anharmonic ratio as NHDC, which cuts the second, or [LGDE] = [NCDH]. But these lines have a common homologous point D, hence the lines joining the other pairs of homologous points, as LN, CG, EH, meet in a common point, as M. Therefore L, M, N are in the same right line.

996. SCH.—This theorem is due to PASCAL, whose wonderful achievements in his brief life of thirty-nine years (1623–1662) have been the admiration of all succeeding generations.

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997. Theo.—*The diagonals joining the opposite vertices of a circumscribed hexagon intersect in a common point.*

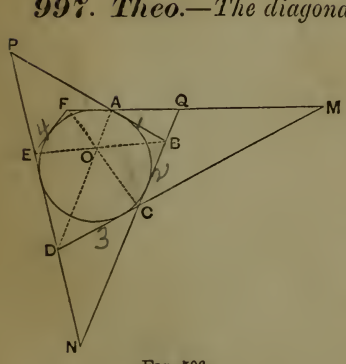

FIG. 506.

DEM.—Consider AB, BC, CD, EF four fixed tangents cut by ED and FA. Then [PNDE] = [AQMF] (*989*). Hence the anharmonic ratios of B-P,N,D,E,* and C-A,Q,M,F are equal (*985*); and since they have a common ray (CQ, BN) the intersections A, O, D, of their homologous rays, are in the same right line. Therefore the diagonals pass through a common point.

---

* The student can *conceive* the rays BE, BD, etc,, as drawn, without encumbering the figure with them.

SECTION VII.

POLE AND POLAR IN RESPECT TO A CIRCLE.

998. DEF.—If a secant to a circle be revolved about a fixed point in the plane of the circle, the locus of the harmonic conjugate of the fixed point, in reference to the intersections, is the ***Polar*** of the fixed point. The fixed point is the ***Pole*** of the *Polar Line.* The terms pole and polar as here used are correlative, and neither has any significance without the other.

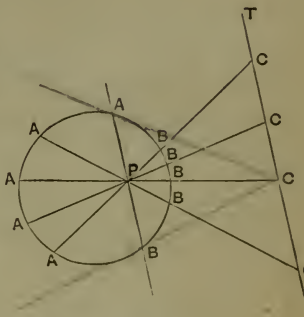

FIG. 507.

ILL.—Let AP be a secant revolving about the fixed point P, and let C be so taken that (in every position) AC : CB :: AP · BP, then is the locus of C the *Polar* of P, and P is the *Pole* of the locus of C.

---

999. ***Theo.***—*The Polar of a given point in respect to a circle is a right line.*

DEM.—Let P be the pole, AP any secant passing through P, and a point in the polar. The locus of C is required. Draw PL through the centre, and let fall the perpendicular CC′. Draw AL, AH, AC′F, and C′B. Since AC : CB :: AP : BP, C′P bisects the angle BC′F, the exterior angle of the triangle AC′B (?); hence, as LAH is a right angle, AL bisects NAC′, the exterior angle of the triangle C′AP (?). Therefore, PL is harmonically divided at C′, and H; and, C′ being a fixed point, and C any point in the locus, the locus is the perpendicular TCC′V.

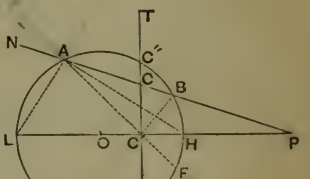

FIG. 508.

1000. COR. 1.—*Since, as the secant revolves, the points* A *and* B *will vanish in* C″, C″ *is the point at which a tangent from the pole* P *touches the circle.*

1001. COR. 2.—*Drawing* OC″, C″P, *we see that* $\overline{OC''}^2$ (*or* $\overline{OH}^2$) $= OC' \times OP$.

1002. Cor. 3.—*The polar of a point in the circumference is a tangent at that point.*

For, as $OC' \times OP$ is constant and equal to $\overline{OH}^2$, OP diminishes as OC' increases, and when $OP = OH$, $OC' = OH$ also.

1003. Prob.—*To draw the polar to a given pole in respect to a given circle.*

Cor. 1 effects the solution.

1004. Prob.—*To find the pole of a polar to a given circle.*

Through the centre draw a perpendicular to the polar. [The student should be able to complete the solution.]

1005. Def.—The point C' where the polar cuts the line passing through the pole and the centre of the circle is called the *Polar Point.*

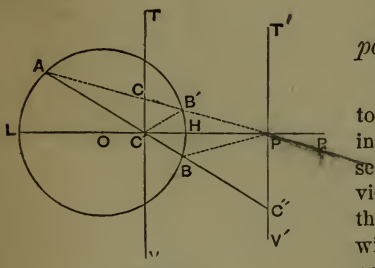

Fig. 509.

1006. Theo.—*The pole and polar point are interchangeable.*

Dem.—TV being the polar to P, we are to show that $T'V'$, parallel to TV and passing through P, is polar to C'; *i.e.*, that any secant, as $AC'C''$, passing through C', is divided harmonically in the intersections with the circumference, C', and the intersection with $T'V'$. Drawing AP, since P is the pole of TV, we have, as in the last demonstration, angle APB bisected by PC'; and consequently RPB bisected by PC''. Therefore $AC' : C'B :: AC'' : BC''$. Q. E. D.

1007. Theo.—*The polars of all the points in a right line pass through the pole of that line; and, conversely, The poles of all straight lines which pass through a given point are in the polar of that point.*

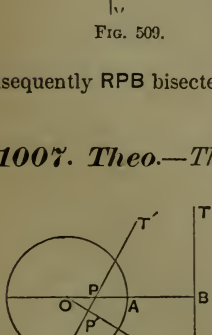

Fig. 510.

Dem.—1st. TV being a given line and P its pole, we are to show that the polar to any point, as N, passes through P. Draw through P a perpendicular to ON; then P' is the polar point to N. For, $OP : ON :: OP' : OB$ (?): whence $ON \times OP' = OP \times OB = \overline{OA}^2$. Therefore, $T'V'$ is the polar of N (?). 2nd. P being any point and TV

its polar, the pole of any line, as T′V′ passing through P, is in TV, as at N. Draw ON perpendicular to T′V′. Then, as before, $ON \times OP' = OP \times OB = \overline{OA}^2$, and N is the pole of T′V′.

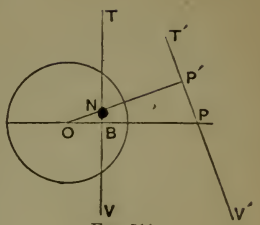

Fig. 511.

1008. Cor.—*The pole of a straight line is the intersection of the polars of any two of its points;* and, conversely, *The polar of any point is the straight line joining the poles of any two straight lines passing through that point.*

RECIPROCAL POLARS.

1009. Def.—If two polygons be constructed, one within, or inscribed in, a circle, and the other without, or circumscribed about the same circle, such that the vertices of the one are the poles of the sides of the other, the two polygons are called ***Reciprocal Polars***; and the circle is called the *Auxiliary Circle.*

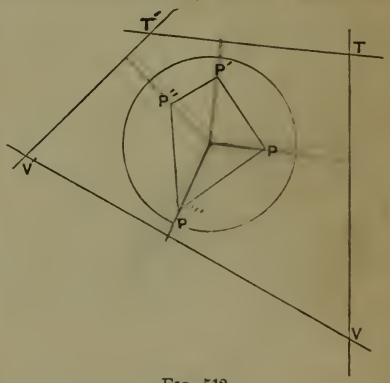

Fig. 512.

The possibility of constructing such polygons is apparent from the last theorem. When the points P, P′, P″, P‴ are in the circumference, TV, TT′, T′V′, V′V become tangents, as appears from (1002).

1010. Prob.—*Having given one of two reciprocal polars, to construct the other.*

The student should be able to make the construction.

1011. By means of the relation between reciprocal polars a large class of propositions relating to the relative positions of lines and points, become, as it were, double; *i.e.*, one proposition being proved, another can be inferred. The process by which the inference is made is called *reciprocation.* We will give an example.

1012. Prob.—*To deduce the reciprocal of Pascal's theorem* (995).

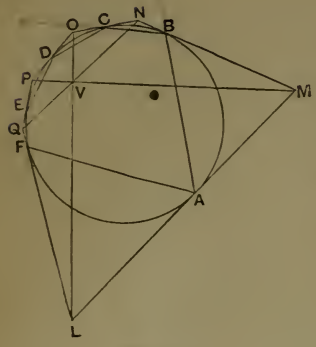

Fig. 513.

Solution.—Draw tangents at the six vertices of the inscribed hexagon. Thus, a circumscribed hexagon is formed whose sides are the polars of the vertices of the inscribed hexagon, through the vertices of which they respectively pass (1002). Now, drawing the diagonals PM, NQ, OL, they are the polars of the intersections of the opposite sides of the inscribed hexagon, as PM, polar to the intersection of DE and CB (?); and hence they pass through a common point, as V. Thus we have Brianchon's theorem, viz.: *The lines joining the opposite angles of a circumscribed hexagon pass through a common point.*

1013.—The three following theorems are of frequent use in applying the theory of reciprocation.

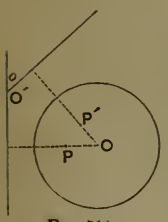

Fig. 514.

1014. Theo. — *The angle included by two straight lines is equal to the angle included by the lines joining their poles to the centre of the auxiliary circle.*

Dem.—The pole of a line being in the perpendicular from the centre of the auxiliary circle upon the line (1000), O′ is the supplement of O; hence $o = O$.

1015. Theo.—*The distances of any two points from the centre of the auxiliary circle are to each other as the distances of each point from the polar of the other.*

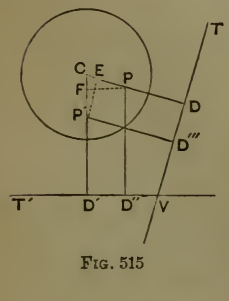

Fig. 515

Dem. P, P′ being the points, and TV, T′V their polars respectively, we are to show that

$$CP : CP' :: PD'' : P'D'''.$$

By (***1001***) $R^2 = CP \times CD = CP' \times CD'$, R being the radius of the auxiliary circle. Whence $CP : CP' :: CD' : CD$. But $CP : CP' :: CF : CE$ (?), and there follows $CF : CE :: CD' : CD$, $CD' - CF\ (= PD'') : CD - CE\ (= P'D''') :: CF : CE :: CP : CP'$.

Sch.—This is known as *Salmon's Theorem.*

1016. Theo.—*The anharmonic ratio of four points in a straight line is equal to that of the pencil formed by the four polars of these points.*

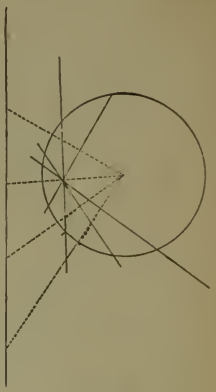

Fig. 516.

Dem.—1st. The polars pass through a common point and thus form a pencil (?). 2d. The angles included by the lines joining the four points with the centre, and those included by the polars are equal (?), hence the two pencils have the same anharmonic ratio.

SECTION VIII.

RADICAL AXES AND CENTRES OF SIMILITUDE OF CIRCLES.

1017. Def.—***The Power of a Point*** in the plane of a circle is the rectangle of the distances from the point to the intersections of the circumference by a line passing through the point.

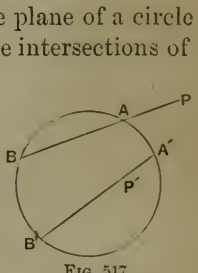

Fig. 517.

Ill.—Thus, the *power of a point* P, without the circle, is PA × PB;—the power of a point within, as P′, is P′A′ × P′B′; the power of a point in the circumference is *zero*, since one of the distances is then 0;—the power of the centre is the square of the radius.

1018. Cor.—*The power of a given point with respect to a given circle is a constant quantity.*

Thus PA × PB = the square of the tangent from P to the circle, in whatever position PB lies, so long as it passes through P. So also P′A′ × P′B′ = the rectangle of the segments of any other chord passing through P′.

1019. Def.—***The Radical Axis*** of *Two Circles* is the locus of the point whose powers with respect to the two circles are equal.

1020. Prop.—*The Radical Axis of two circles is a right line.*

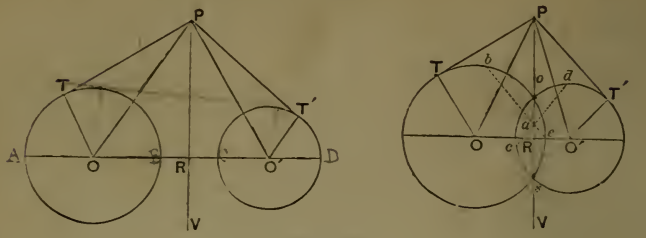

FIG 518.

DEM.—Join the centres of the two circles O, O′, and take a point R on this line such that $\overline{OR}^2 - \overline{O'R}^2 = \overline{OT}^2 - \overline{O'T'}^2$, or $\overline{OR}^2 - \overline{OT}^2 = \overline{O'R}^2 - \overline{O'T'}^2$, and erect PR perpendicular to OO′. Then P being *any* point in this perpendicular, $\overline{OP}^2 - \overline{OR}^2 = \overline{O'P}^2 - \overline{O'R}^2$. Adding this to the preceding equation, we have $\overline{OP}^2 - \overline{OT}^2 = \overline{O'P}^2 - \overline{O'T'}^2$, or $\overline{PT}^2 = \overline{PT'}^2$. ∴ PT = PT′, PT and PT′ being tangents to the circles from any point in PR. Hence PV is the radical axis of the two circles.

1021. COR.—*When the circles are exterior to each other, the Radical Axis lies between them, touching neither; when they are tangent, either externally or internally, the radical axis is the common tangent; when they cut each other, the axis is the common chord produced.*

1022. SCH.—When the circles intersect it might seem that the above demonstration fails for points within, as in the common chord. But, the powers of any point in this chord are still equal. Thus, at the intersections the powers are zero; and at any other point in the chord, as *a*, $ab \times ae = ad \times ac$, since each is equal to $ao \times as$.

1023. COR. *There are an infinite number of circles having their centres in the same right line, which have the same radical axis as any two given circles.*

Thus, in the first figure, PV being the radical axis of the circles O, O′, letting circle O remain fixed, O′ may vary indefinitely so that $\overline{O'R}^2 - \overline{O'T'}^2$ remains constant, and equal to $\overline{OR}^2 - \overline{OT}^2$.

1024. Prob.—*Given two circles, to draw their radical axis.*

SOLUTION.—Draw a common tangent, bisect it, and through the point of bisection draw a line perpendicular to the line joining the centres. When the circles are tangent to each other, the distance between the points of tangency is 0; hence the perpendicular is erected at this point. When they intersect, produce the common chord, or use the first method.

1025. Prop.—*When two circles cut each other orthogonally, that is, at right angles, the square of the radius of either is equal to the power of its centre with respect to the other.*

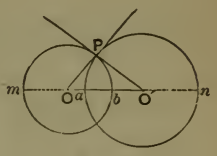

Fig. 519.

Dem.—The power of O with respect to circle O′ is $Oa \times On = \overline{OP}^2$, and of O′ with respect to circle O, $O'b \times O'm = \overline{O'P}^2$; since, as the circles cut each other orthogonally, their tangents are at right angles, and the tangent to either passes through the centre of the other.

1026. Prop.—*The radical axes of a system of three circles whose centres are not in the same straight line, intersect at a common point.*

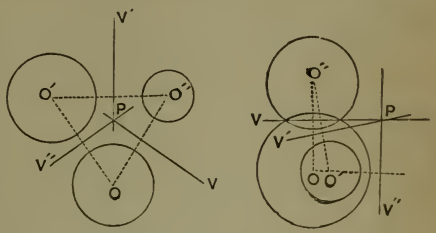

Fig. 520.

Dem.—Since O, O′, O″ are not in a straight line, the radical axes of O, O′, and O, O″, as PV″ and PV intersect. Let P be their common point. Now the power of P with respect to O′ is equal to its power with respect to O″, since each is equal to its power with respect to O. Hence P is a point in the radical axis of O′, O″.

1027. Cor.—*If the centres are in the same straight line, their radical axes are parallel, and the common point is at infinity.*

1028. Def.—The intersection of the radical axes of three circles is called their ***Radical Centre.***

CENTRES OF SIMILITUDE.

1029. Def.—If the line joining the centres of two circles be divided externally, as at C, and internally, as at C′, in the ratio of the radii, these points are respectively the *External* and the *Internal* ***Centres of Similitude*** of the two circles.

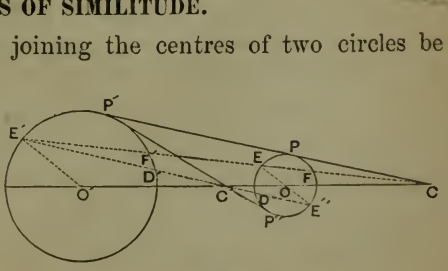

Fig. 521.

Ill.—If $CO : CO' :: EO : E'O'$, C is the external centre of similitude; and, if $C'O : C'O' :: E'O : E'O'$, C′ is the internal centre of similitude.

The student should construct the figure when the circles are tangent externally,—when they are tangent internally,—and when one is wholly within the other.

QUERY.—How are the centres of similitude situated in the three different relative positions of the circles?

1030. Prop.—*In two circles the line passing through the extremities of two parallel radii on the same side of the line passing through the centres, intersects this line in the external centre of similitude, and if the radii are on opposite sides of this line the intersection is the internal centre of similitude.*

The proof consists in showing that the line passing through the centres is divided as above. Let the student show it for the three different positions of the circles.

1031. COR. 1.—Conversely, *If any transversal be drawn from either centre of similitude, the radii drawn to the intersections are parallel.*

Thus in the last figure, since CO : CO′ : : EO : E′O′, and the triangles have the angle C common, EO and E′O′ are parallel.

1032. COR. 2.—*Tangents drawn at the alternate intersections of a transversal through the external centre of similitude are parallel; also, those at the mean intersections, and those at the extreme intersections, if the transversal be drawn through the internal centre of similitude.*

This follows as a consequence of the parallelism of the corresponding radii, to which the tangents are perpendicular. Thus, tangents at E and E′ are parallel, as are those at F and F′. So, also, tangents D and D′, and at E′ and E″ are parallel.

1033. DEF.—The extremities of two parallel radii on the same side of the line joining the centres are called *Homologous Points*, and those of non-parallel radii where the transversal cuts the circumferences, as E, F′, are called *Anti-Homologous Points.*

1034. COR. 2.—*The distances of a centre of similitude from two homologous points are to each other as the radii.*

1035. COR. 3.—*The centres of similitude and the centres of the circles are four harmonic points.*

1036. Prop.—*If a circle touch two others, the line joining their points of contact passes through the external centre of similitude of*

the latter if the contacts are both external or both internal; and through the internal centre of similitude if the contacts are the one external and the other internal.

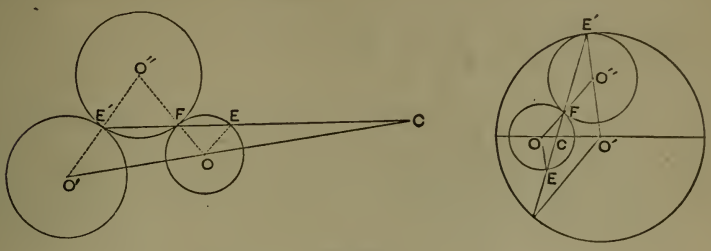

Fig. 522.

Dem.—In either case let O'' be the circle tangent to O, and O'; and through the points of tangency draw E'C. The angle O''E'F = O''FE' = EFO = FEO; whence OE and O'E' are parallel, and the similar triangles CEO, CE'O' give CO : CO' : : OE : O'E'. [The student should make the other constructions.]

1037. Prob.—*To draw a line parallel to a given line so that the distance between the extreme intersections with two given circles shall be a maximum.*

Solution.—Draw a line through the internal centre of similitude and parallel to the given line. Now, at the extreme intersections draw tangents, and it will become evident that the line first drawn is a maximum. [The student should make the figure and fill out the proof.]

If the circles are wholly exterior to each other, the distance between the mean intersections is a minimum.

1038. Concluding Note.—Our limits preclude our pursuing these topics farther. We have given enough to make the language of the Modern Geometry intelligible, and to afford some insight into its character. One of the best elementary resources for the English student who wishes to pursue the subject at greater length, is Mulcahy's *Principles of Modern Geometry*, Dublin, 1862. It is, however, much to be regretted, that there is no English treatise which presents the elements of this subject with the philosophic elegance of the French. The best of the latter is Rouché and Comberousse's Treatise on Elementary Geometry. For a more extended view of the subject, Salmon or Whitworth will furnish the English student; but he who would be proficient must read the works of Chasles and Poncelet, who are the great authorities.

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SECTION I.

DEFINITIONS AND FUNDAMENTAL RELATIONS BETWEEN THE TRIGONOMETRICAL FUNCTIONS OF AN ANGLE (OR ARC).

1. Trigonometry is a part of Geometry which has for its subject-matter, *Angles*. It is chiefly occupied in presenting a scheme for measuring and comparing angles, by means of certain auxiliary lines called *Trigonometrical Functions*, in investigating the relations between these functions, and in the solution of triangles by means of the relations between their sides and the trigonometrical functions of their angles.

2. Plane Trigonometry treats of plane angles and triangles, in distinction from *Spherical Trigonometry*, which treats of spherical angles and triangles.

3. A Function is a quantity, or a mathematical expression, conceived as depending upon some other quantity or quantities for its value.

ILL'S.—A man's wages *for a given time* is a function of the amount received per day; or, in general, his wages is a function of *both* the *time* of service and the *amount* received per day. Again, in the expressions $y = 2ax^2$, $y = x^3 - 2bx + 5$, $y = 2\log ax$, $y = a^x$, y is a function of x; since, the numbers 2, 5, a and b being considered fixed or constant, the value of y depends upon the value we assign to x. For a like reason such expressions as $\sqrt{a^2 - x^2}$, and $3ax^2 - 2\sqrt{x}$, may be spoken of as functions of x. Once more, the area of a triangle is a function of its base and altitude.

4. Angles as Functions of Arcs.—We have learned in Geometry (PART II., SEC. VI.), that angles and arcs may be treated as functions of each other; and that, if the angles be taken at the

centre of the same or equal circles, the arcs intercepted have the same ratio as the angles themselves, and hence may be taken as their measures or representatives. For trigonometrical purposes, an angle is considered as measured by an arc struck with a radius 1, from the angular point as a centre.

5. A Degree being the $\frac{1}{360}$ part of the circumference of a circle, becomes the measure of $\frac{1}{90}$ of a right angle; and, for convenience, it is customary to speak of such an angle as *an angle of one degree*, of four times as large an angle as an angle of four degrees, etc., applying the term directly to the angle. A small circle written at the right and a little above a number indicates degrees (°).

6. A Minute is $\frac{1}{60}$ part of a degree. Minutes are designated by an accent (′). *A Second* is $\frac{1}{60}$ part of a minute. Seconds are indicated by a double accent (″). Smaller divisions of angles (or arcs) are most conveniently represented as decimals of a second, though the designations *thirds, fourths*, etc., are sometimes met with, and signify further subdivisions into 60ths. 5° 12′ 16″ 13‴ is read, "5 degrees, 12 minutes, 16 seconds, and 13 thirds."

Ill's.—In *Fig.* 1 AOB is an angle of 35°, because the measuring arc *ab* contains 35 of the 360 equal parts into which the circumference whose radius is *Oa*, could be divided. In like manner BOC is an angle of 7°. BOC = $\frac{1}{5}$AOB = $\frac{1}{6}$AOC. Hence, it becomes evident that we may use the numbers 35, 7, and 42, to represent the respective angles AOB, BOC, and AOC, or the corresponding arcs *ab*, *bc*, and *ac*.

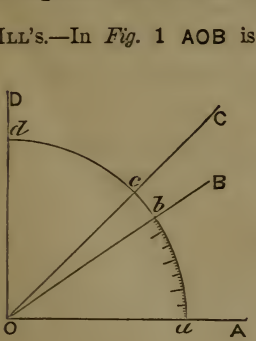

Fig. 1.

7. A Quadrant is an arc of 90°, and is the measure of a right angle; hence, a right angle is called an angle of 90°. Thus arc *ad*, *Fig.* 1, = 90°, or angle AOD = 90°.

8. The Complement of an angle or arc is what remains after subtracting the angle or arc from 90°. The *Supplement* of an angle or arc is what remains after subtracting the angle or arc from 180°.

Ill's.—In *Fig.* 1, the angle BOD is the complement of AOB, and the arc *bd* is the complement of arc *ab*. The complement of 35° is 90° − 35° = 55°. The supplement of 35° is 180° − 35° = 145°.

9. A Quadrant is often represented by $\frac{1}{2}\pi$, since π is the semi-circumference when the radius is unity. When this notation is used, the ***Unit Arc*** becomes $\frac{180°}{\pi} = 57°.29578$ nearly, or 57° 17′ 44″.8 +, which is an arc equal in length to the radius.

10. For trigonometrical purposes, an angle is conceived as generated by the revolution of a line about the angular point, and hence *may have any value whatever,* not only from 0° to 180°, but from 0° to 360°, and even to any number of degrees greater than 360°, as 1280°, etc. An angle of 45° is generated by $\frac{1}{8}$ of a revolution, 90° by $\frac{1}{4}$ of a revolution, 180° by $\frac{1}{2}$ a revolution, 270° by $\frac{3}{4}$ of a revolution, 360° by one revolution, 450° by $1\frac{1}{4}$ revolutions, 1280° by $3\frac{5}{9}$ revolutions, etc., etc.

11. In accordance with the conception of an angle as generated by a revolving line, the measuring arc is considered as *originating* at the first position of the revolving line (*i. e.,* with one side of the angle), and terminating in the line after it has generated the angle under consideration (*i. e.,* with the other side of the angle). The first extremity is called the ***Origin*** of the arc, and the other the ***Termination.***

Ill's.—In *Fig.* 1, let the angle AOB be considered as generated by a line starting from the position OA, and revolving around the point O, from right to left,* till it reaches the position OB. O*a* being taken as unity, the arc *ab* is the measuring arc of the angle AOB; *a* is its *origin,* and *b* its *termination.*

12. In the generation of angles by means of a revolving line, the normal motion is considered to be from right to left, and the quadrants are numbered 1st, 2d, 3d, and 4th, in the order in which they are generated.

13. ***The Trigonometrical Functions*** are eight in number; viz., *sine, cosine, tangent, cotangent, secant, cosecant, versed-sine,* and *coversed-sine.* These lines are functions of angles, or, what amounts to the same thing, of arcs considered as measures of angles, and are the characteristic quantities of trigonometry.

14. ***The Sine*** of an angle (or arc) is a perpendicular let fall from the termination of the measuring arc upon the diameter passing through the origin of the arc. Thus in *Fig.* 2, *bd* is in each case the sine of the angle AOB, or of the arc *axb.*

* The pupil will understand that, if he imagines himself standing at the centre of motion, as the moving body or point passes before him, the distinctions "from right to left," and "from left to right," are easily made.

15. The Trigonometrical Tangent of an angle (or arc) is a tangent drawn to the measuring arc at its origin, and limited

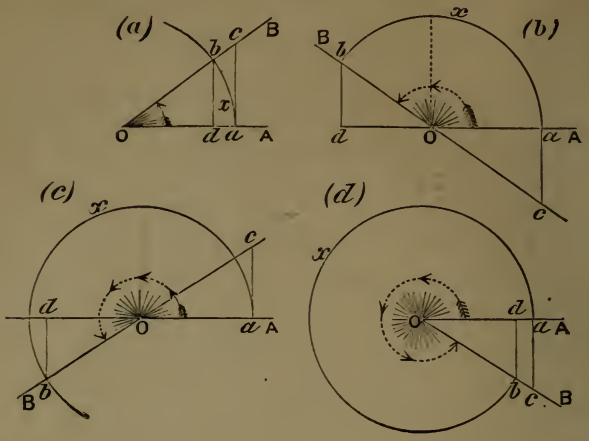

Fig. 2.

by the produced diameter passing through the termination of the arc. Thus in *Fig. 2*, *ac* is in each case the tangent of the angle AOB, or of the arc *axb*.

16. The Secant of an angle (or arc) is the distance from the angular point, or centre of the measuring circle, to the extremity of the tangent of the same angle (or arc). Thus in *Fig.* 2, O*c* is in each case the secant of the angle AOB, or of the arc *axb*.

17. The Versed-Sine of an angle (or arc) is the distance from the foot of the sine of the same angle (or arc) to the origin of the measuring arc. Thus, in *Fig.* 2, *da* is in each case the versed-sine of the angle AOB, or of the arc *axb*.

18. The prefix *co*, in the names of the four trigonometrical functions in which it occurs, is an abbreviation for the word *complement*. Thus cosine means complement-sine, *i. e.*, the sine of the complement; cotangent means tangent of the complement; etc. The cosine of 40° is the sine of 90° — 40°, or 50°; the cosine of 110° is the sine of 90° — 110°, or — 20°; the cotangent of 30° is the tangent of 60°; the cosecant of 200° is the secant of — 110°.

19. Construction of the Complementary Functions. —Let us now see how the complementary functions are constructed with reference to their primitives, premising that all arcs in *Fig.* 3, reckoned from A, are

to be reckoned around from right to left in this discussion. 1st. Let AP be any arc less than 90° ; then 90° − AP = aP is its complement. Now considering a as the origin and P the termination of this complementary arc, Pd is its sine, at its tangent, Ot its secant, and ad its versed-sine. Hence, Pd, at, Ot, and ad are respectively the cosine, cotangent, cosecant, and coversed-sine of the arc AP, or the angle AOP. 2d. Letting APP′ be any arc between 90° and 180°, its complement is 90° − APP′ or − aP′, the − sign signifying that the arc is reckoned backward from P′ to a. But as the *values* of the functions will be the same whether the origin be taken at P′ or at a, we may take a as the origin of this complementary arc, and P′ as its termination, whence P′d' becomes its sine, at' its tangent, Ot' its secant, and ad' its versed-sine. Therefore P′d', at', Ot', and ad', are respectively the cosine, cotangent, cosecant, and coversed-sine of the arc APP′, or the angle AOP′. 3d. In like manner, aP″ is shown to be the complement of arc APP′P″ ; and as P″d'', at'', Ot, and ad'' are respectively the sine, tangent, secant, and versed-sine of this complement, they are the corresponding cofunctions of the arc APP′P″, or the salient angle AOP″. 4th. In the same way, it appears that P‴d''', at''', Ot', and ad''' are the cosine, cotangent, cosecant, and coversed-sine of the arc APP′P″P‴, or the salient angle AOP‴. *Observe that* a, *a point on the measuring arc* 90° *from the primitive origin, is the origin of all the complementary functions.*

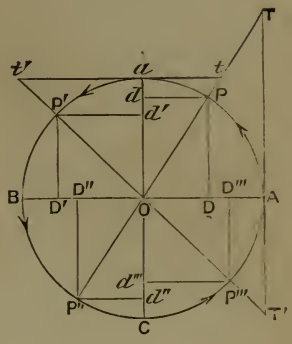

FIG. 3.

SCH.—It will readily appear from the figure that the cosine of an angle (or arc) is always equal to *the distance from the foot of the sine to the vertex of the angle* (or the centre of the measuring arc). This is the more convenient practical definition. Thus the cosine of AP is Pd = DO ; the cosine of APP′ is P′d' = D′O, etc.

20. Notation.—Letting x represent any angle (or arc), the several trigonometrical functions of it are written sin x, cos x, tan x, cot x, sec x, cosec x, vers x, and covers x. They are read "sine x," "cosine x," "tangent x," "cotangent x," etc.

FUNDAMENTAL RELATIONS BETWEEN THE TRIGONOMETRICAL FUNCTIONS OF AN ANGLE (OR ARC).

[Note.—These fundamental relations must be made perfectly familiar. They must be memorized, and be as familiar as the Multiplication Table. The student can do nothing in trigonometry without them.]

☞ The discussions in this treatise all proceed upon one general plan; viz.,—***First obtain the particular property of the***

sine and cosine, and from this deduce all the others according to the dependencies shown in the following proposition.

21. Prop.—*The Fundamental Relations which the Trigonometrical Functions sustain to each other are:*

(1) $\sin^2 x + \cos^2 x = 1$;

(2) $\tan x = \dfrac{\sin x}{\cos x}$;

(3) $\cot x = \dfrac{\cos x}{\sin x}$;

(4) $\cot x = \dfrac{1}{\tan x}$;

(5) $\sec x = \dfrac{1}{\cos x}$;

(6) $\operatorname{cosec} x = \dfrac{1}{\sin x}$;

(7) $\sec^2 x = 1 + \tan^2 x$;

(8) $\operatorname{cosec}^2 x = 1 + \cot^2 x$;

(9) $\operatorname{vers} x = 1 - \cos x$;

(10) $\operatorname{covers} x = 1 - \sin x$.

(The forms $\sin^2 x$, $\sec^2 x$, etc., signify the square of the sine, the square of the secant of x, etc., and are read "sine square x," "secant square x," etc. The student should distinguish between $\sin^2 x$, and $\sin x^2$.)

DEM.—In *Fig.* 4, let x represent any arc as AP, less than 90°. Then PD = $\sin x$, OD or Pd = $\cos x$, AT = $\tan x$, OT = $\sec x$, at = $\cot x$, Ot = $\operatorname{cosec} x$, AD = $\operatorname{versin} x$, and ad = $\operatorname{coversin} x$.

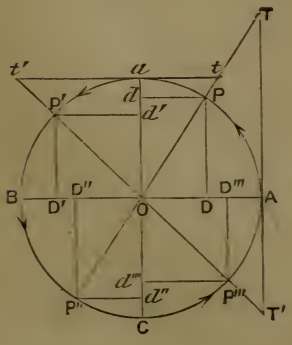

FIG. 4.

(1). In the right-angled triangle POD, $\overline{PD}^2 + \overline{OD}^2 = \overline{OP}^2$, or $\sin^2 x + \cos^2 x = 1$, since OP = radius = 1.

(2). From the similar triangles POD and TOA,

$$\frac{AT}{OA} = \frac{PD}{OD}, \text{ or } \tan x = \frac{\sin x}{\cos x}.$$

(3). From the similar triangles POd and tOa,

$$\frac{at}{Oa} = \frac{Pd}{Od}, \text{ or } \cot x = \frac{\cos x}{\sin x}.$$

(4). Multiplying (2) and (3) together,

$$\tan x \cot x = \frac{\sin x \cos x}{\cos x \sin x} = 1, \text{ or } \tan x = \frac{1}{\cot x}$$

(5). From the similar triangles OTA and OPD,

$$\frac{OT}{OA} = \frac{OP}{OD}; \text{ but OP and OA each} = 1, \therefore \sec x = \frac{1}{\cos x}.$$

(6). From the similar triangles Ota and OPd,

$$\frac{Ot}{Oa} = \frac{OP}{Od}, \text{ or } \operatorname{cosec} x = \frac{1}{\sin x}.$$

(7). From the right-angled triangle OAT,

$$\overline{OT}^2 = \overline{OA}^2 + \overline{AT}^2, \text{ or } \sec^2 x = 1 + \tan^2 x.$$

(8). From the right-angled triangle Oat,

$$\overline{Ot}^2 = \overline{Oa}^2 + \overline{at}^2, \text{ or } \operatorname{cosec}^2 x = 1 + \cot^2 x.$$

(9). $AD = AO - OD$, or $\operatorname{vers} x = 1 - \cos x$.

(10). $ad = aO - Od$, or $\operatorname{covers} x = 1 - \sin x$.

Thus the fundamental relations of the functions are established for an arc less than 90°. But it will readily appear that the relations are the same for any other arc. For example, let $x = AP'$ be any arc between 90° and 180°. Then the triangle $P'D'O$ gives $\sin^2 x + \cos^2 x = 1$, since $P'D' = \sin x$, and $OD' = \cos x$. The similar triangles $P'D'O$ and OAT' give $\frac{AT'}{OA} = \frac{P'D'}{D'O}$, or $\tan x = \frac{\sin x}{\cos x}$; and the similar triangles $P'd'O$ and $t'aO$ give $\cot x = \frac{\cos x}{\sin x}$. In like manner let the student observe the relations when $x = APP'P''$, or an arc between 180° and 270°. So also when $x = APP'P''P'''$, or an arc between 270° and 360°.

22. Cor. 1.—*The tangent and cotangent of the same angle are reciprocals of each other; so also are the secant and cosine, and the cosecant and the sine.* Thus, if $\tan x = 3$, $\cot x = \frac{1}{3}$; since $\cot x = \frac{1}{\tan x}$. If $\sec x = 2$, $\cos x = \frac{1}{2}$; since $\sec x = \frac{1}{\cos x}$, or $\cos x = \frac{1}{\sec x}$.

23. Cor. 2.—*Sines and cosines cannot exceed* 1. *Tangents and cotangents can have any values from* 0 *to* ∞. *Secants and cosecants can have any values between* 1 *and* ∞. *Versed-sines and coversed-sines can have any values between* 0 *and* 2. These conclusions will readily appear from the definitions, and an inspection of *Fig.* 4.

SIGNS OF THE TRIGONOMETRICAL FUNCTIONS.

24. *Prop.*—*Angles* (*or arcs*) *considered as generated from right to left being called positive* and marked* +, *those considered as generated from left to right are to be called negative and marked* −.

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* This is purely an arbitrary convention. We might equally well reverse it

DEM.—This is a direct application of the significance of the + and − signs. (See Complete School Algebra, pp. 20–23.) Thus, in *Fig.* 5, if the angle AOP, considered as generated by the revolution of a line from the position OA in the direction of the arrow-head (from right to left), is called positive and marked +, an angle generated by the motion of a line from the position OA in the *opposite* direction (from left to right), as the angle AOP″ thus generated, is to be considered negative and marked −. Let it be carefully observed that it is the assumed *direction* of the motion of the generatrix that determines the sign of the angle (or arc). Two lines meeting at a common point may be considered as designating either a + or a − angle, according to the direction of motion assumed. Thus the lines OA and OP′, *Fig.* 5, may form the positive angle measured by the arc APP′, or the negative, salient angle measured by the negative arc AP‴P″P′. Q. E. D.

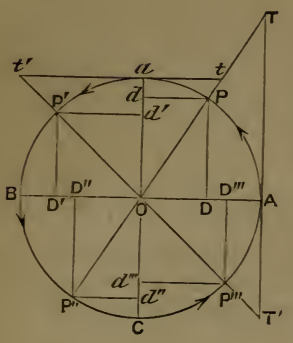

FIG. 5.

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25. Prop.—*Radius being considered as always extending in the same direction, viz., from the centre toward the circumference, is always positive.*

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26. Prop.—*The sign of the sine of an angle between* 0° *and* 180° *being* +, *that of an angle between* 180° *and* 360° *is* −.

DEM.—In *Fig.* 5, we observe that the sines of all angles terminating in the 1st and 2d quadrants, *i. e.*, between 0° and 180°, may be considered as measured from the primary diameter AB, *upward*, while those of angles terminating in the 3d and 4th quadrants, *i. e.*, between 180° and 360°, are reckoned downward from the same line; hence, the former being called +, the latter should be −, *as the two species are estimated in opposite directions.* Q. E. D.

A more elegant conception is to consider the sine as projected upon the diameter vertical to that passing through the origin, as aC; whence Od is the sine of AOP (or arc AP). Now this line evidently is 0 when the angle is 0; and as the angle increases, the sine increases, being generated from O *upward*, and hence is called +. This is the same conception as we use in the case of the cosine. Adopting it, we see that sines reckoned from O upward are +, and downward −. Cosines reckoned from O to the *right*, are +, and to the left, −.

27. COR.—*The cosecant of an arc has the same sign as its sine,* since $\operatorname{cosec} x = \frac{1}{\sin x}$; and as 1, being the radius, is +, the sign of $\frac{1}{\sin x}$ is the same as the sign of $\sin x$.

28. Prop.—*The sign of the cosine of an angle between* 0° *and* 90°, *and between* 270° *and* 360°, *is* +, *while that of an angle between* 90° *and* 270° *is* −.

DEM.—In *Fig.* 5, we observe that the cosines of all angles terminating in the 1st and 4th quadrants, may be considered as estimated from the centre toward the right, as OD, OD‴; while correspondingly, the cosines of angles terminating in the 2d and 3d quadrants will be estimated from the centre toward the left, as OD′, OD″. Hence, by reason of this opposition of direction, the former are called +, and the latter −. Q. E. D.

29. COR.—*The secant of an angle has the same sign as its cosine,* since these functions are reciprocals of each other. (See ***27.***)

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30. Prop.—*The sign of the tangent of an angle between* 0° *and* 90°, *and also between* 180° *and* 270°, *is* +; *while that of an angle between* 90° *and* 180°, *and between* 270° *and* 360°, *is* −.

DEM.—Since $\tan x = \frac{\sin x}{\cos x}$, when $\sin x$ and $\cos x$ have like signs, $\tan x$ is +, by the rules of division; and when $\sin x$ and $\cos x$ have different signs, $\tan x$ is −. Now, in the 1st and 3d quadrants* the signs of $\sin x$ and $\cos x$ are alike, hence in these quadrants $\tan x$ is plus; but in 2d and 4th quadrants $\sin x$ and $\cos x$ have unlike signs, and consequently in these $\tan x$ is −. Q. E. D.

31. COR.—*The sign of the cotangent is the same as the sign of the tangent of the same angle,* since $\cot x = \frac{1}{\tan x}$.

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32. Prop.—*Versed-sine and coversed-sine are always* +.

DEM.—$\text{Vers}\, x = 1 - \cos x$; and as $\cos x$ cannot exceed 1, $1 - \cos x$ is always +. In like manner, $\text{covers}\, x = 1 - \sin x$; and as $\sin x$ cannot exceed 1, $1 - \sin x$ is always +. Q. E. D.

SCH. 1.—It is essential that the law of the signs, as explained above, be well understood, and the facts fixed in memory. *Fig.* 6 will aid the student in fixing the law in the memory. Having this constantly before the mind, and remembering that tan and cot are + when sin and cos have like signs, and − when they have unlike, and that cos and sec have like signs, as also sin and cosec, or, more simply, that $\tan = \frac{\sin}{\cos}$, $\cot = \frac{1}{\tan}$, $\sec = \frac{1}{\cos}$, and $\text{cosec} = \frac{1}{\sin}$, the student cannot fail to know the sign of a function at a glance.

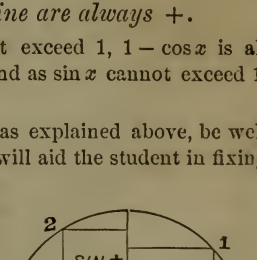

FIG. 6.

It will be of service to remember that versed-sine and coversed-sine, and *all* the functions of angles of the 1st quadrant, are +; but that of the other functions than the versed-sine and coversed-sine, of angles terminating in the other quadrants, but *two* are + in each quadrant.

* This is a convenient elliptical form for "an angle whose measuring arc terminates in the 1st quadrant," etc.

Sch. 2.—The signs of functions of angles greater than 360° are readily determined by observing in what quadrant the measuring arc terminates. Thus, sin 570° is −, since an arc of 570° terminates in the 3d quadrant. In any given case, the sign of the function is the same as the sign of the same function of the remainder left after dividing the arc by 360°, or 2π. Thus tan 1180° is the same as tan 100°; *i. e.*, it is −. The same may also be said of the *value* of the function.

LIMITING VALUES OF THE TRIGONOMETRICAL FUNCTIONS.

33. Prop.—*Sin* $0° = \mp 0$, *sin* $90° = 1$, *sin* $180° = \pm 0$, *sin* $270° = -1$, *sin* $360° = \mp 0$, *and the limits within which the sines of all angles are comprised, are* $+1$ *and* -1.

Dem.—Let the point P be supposed to start from A and move around the circumference from right to left, and let x represent the angle (or arc) generated.

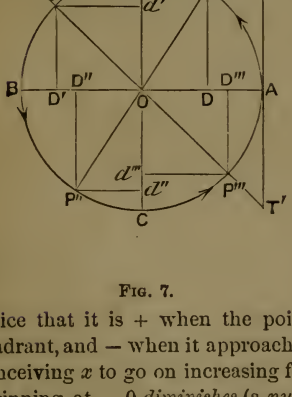

Fig. 7.

Now, when P is at A, $x = 0$, and PD = 0. Moreover, if we consider the sine P''' D''' as reaching its limit, by the moving of P''' from some point in the fourth quadrant to the origin, the sine of 0° becomes − 0, since *what is true of a varying quantity all the way as it approaches its limit, is assumed true at the limit.* But, if the sine reaches its limit by the passage of the point P from some point in the first quadrant to the origin, the sine of 0° is to be considered as +, since the function is + as it approaches its limit. ∴ sin $0° = \mp 0$.* As x increases from 0° to 90° (*i. e.*, as P passes from A to *a*), PD is + and increases from 0 to 1 (+ 0 to + 1). ∴ sine $90° = 1$. As x continues to increase from 90°, sin x diminishes and becomes 0 at 180°. To determine the sign of sin 180°, we notice that it is + when the point P approaches this limit from the second quadrant, and − when it approaches it from the third quadrant, ∴ sin $180° = \pm 0$. Conceiving x to go on increasing from 180°, sin x appears *below* AB and is −, and beginning at − 0 *diminishes* (a *numerical increase* of a negative quantity being considered an absolute decrease) till at 270° it becomes − 1. ∴ sin $270° = -1$. As x passes from 270° to 360°, sin x *increases* (see above) from − 1 to 0. The *sign* of this limit is ambiguous, as appears by regarding the limit as reached by the angle passing from something less than 360° to 360°, whence we have −, and also as reached by the angle passing *back* from something *greater* than 360° to 360°, whence the sign is +. ∴ sin $360° = \mp 0$. Finally, as it is evident that these values would recur in the same order as the point P passed around again, the above comprise all real values of sines of angles. Q. E. D.

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* It has been customary to disregard the sign of 0, in such a series as $-m \ldots -3, -2, -1, \mp 0, +1, +2, +3, \ldots + m$, whereas it should be regarded as ambiguous, as appears above.

34. Prop.—*Cos* $0° = 1$, *cos* $90° = \pm 0$, *cos* $180° = -1$, *cos* $270° = \mp 0$, *cos* $360° = 1$, *and the cosines of all angles are comprised between* $+1$ *and* -1.

DEM.—Using the same figure and the same conception as in the last demonstration, it is evident that as P approaches A, *from either direction*, that is as x approaches 0°, the cosine OD approaches to equality with the radius and reaches it at $x = 0$, being + in either case. $\therefore \cos 0 = 1$. As x approaches 90° from a less value, *i. e.*, from the first quadrant, $\cos x$ is +, and approaching + 0; but as x approaches 90° from some greater value, *i. e.*, from the second quadrant, $\cos x$ is − and approaching − 0. $\therefore \cos 90° = \pm 0$. In like manner we observe that as x increases from 90° to 180°, $\cos x$ *decreases* (see DEM. of 33) from 0 to − 1, which it reaches at 180°, and this, whether the point is reached in one way or the other. $\therefore \cos 180° = -1$. Again, $\cos 270° = \mp 0$; since it is − 0 if x passes to 270° from some value *less* than 270°, and + 0, if it passes from some value *greater* than 270. While x passes through the fourth quadrant, $\cos x$ passes from 0 to + 1, reaching the latter value when $x = 360°$. $\therefore \cos 360° = 1$. Finally, as it is evident that the above values would recur in the same order as the generating point passed around again, this discussion comprises all real values of a cosine.

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35. Prop.—*Tan* $0° = \mp 0$, *tan* $90° = \pm \infty$, *tan* $180° = \mp 0$, *tan* $270° = \pm \infty$, *tan* $360° = \mp 0$, *and the tangents of all angles are comprised between* $+\infty$ *and* $-\infty$.

DEM.—$\text{Tan } 0° = \dfrac{\sin 0°}{\cos 0°} = \dfrac{\mp 0}{1} = \mp 0$. $\text{Tan } 90° = \dfrac{\sin 90°}{\cos 90°} = \dfrac{1}{\pm 0} = \pm \infty$ $\text{Tan } 180° = \dfrac{\sin 180°}{\cos 180°} = \dfrac{\pm 0}{-1} = \mp 0$. $\text{Tan } 270° = \dfrac{\sin 270°}{\cos 270°} = \dfrac{-1}{\mp 0} = \pm \infty$. $\text{Tan } 360° = \dfrac{\sin 360°}{\cos 360°} = \dfrac{\mp 0}{1} = \mp 0$. From these results it appears that the tangent may have any value whatever from $+\infty$ to $-\infty$; and as subsequent revolutions of the generatrix would evidently only repeat these values, these are all the real values of a tangent. (Moreover, all real values are comprised between $+\infty$ and $-\infty$). Q. E. D.

It is easy to observe the direction of the change in the tangent (whether it is increasing or decreasing) by observing the fraction $\dfrac{\sin}{\cos}$. As the arc increases in the first quadrant, the sine increases and the cosine decreases, for both of which reasons the tangent increases, and hence changes more rapidly than either sine or cosine. The student should observe the change in each of the four quadrants in the same manner.

SCH.—These values of the tangent are illustrated by *Fig.* 7. Thus AT, which is +, becomes + 0 when P returns to A, or $x = 0$. Also AT′, which is −, and may be considered as the tangent of AP‴, a negative arc, becomes − 0, when $\text{AP}''' = 0$. Again, if AP passes to Aa, AT passes to $+\infty$. But, if P′ passes *back* to a, so that AaP′ passes to Aa, or 90°, AT′ passes to $-\infty$. $\therefore$ We see that tan 0 may be considered ∓ 0, and $\tan 90° = \pm \infty$. In like manner the other limits are illustrated.

36. Prop.—*Cot* $0° = \mp \infty$, *cot* $90° = \pm 0$, *cot* $180° = \mp \infty$, *cot* $270° = \pm 0$, *cot* $360° = \mp \infty$, *and the limits of the cotangent are* $+ \infty$, *and* $- \infty$.

Dem.—These values are the reciprocals of the corresponding values of the tangent; or they may be deduced from the relation $\cot = \frac{\cos}{\sin}$, in a manner altogether analogous to those of the tangent. *Fig.* 7 also affords geometrical illustrations of them. The student should not fail to deduce and illustrate them, and also to observe the law of change.

---

37. Prop.—*Sec* $0° = 1$, *sec* $90° = \pm \infty$, *sec* $180° = -1$, *sec* $270° = \mp \infty$, *sec* $360° = 1$, *and all real values of this function are comprised between* 1 *and* $\pm \infty$, *and* -1 *and* $\mp \infty$.

Dem.—These values are the reciprocals of the corresponding values of the cosine. The student should obtain them from the cosine, and illustrate them from *Fig.* 7, observing the law of change. Thus beginning at OA = 1, which is the secant of 0°, the secant increases till $x = 90°$, when the secant OT becomes $+ \infty$, if we consider this limit as reached thus; but $- \infty$, if we consider such an arc as A*a*P′, whose secant is $-$ OT′, which becomes $- \infty$, when the point P′ passes back to *a*.

Sch.—The series which represent the values of secants are, for the first and second quadrants, $+ 1, + 2, + 3, + \ldots + m, \pm \infty, - m, - \ldots - 3, - 2, - 1$; for the third and fourth quadrants, $- 1, - 2, - 3, - \ldots - m, \mp \infty, + m, + \ldots + 3, + 2, + 1$, understanding the numbers in these series as representing a few terms of series which have an infinite number of terms of all values extending, in the first case, from + 1 to $\pm \infty$, and thence to $- 1$. It will be a good exercise for the pupil to write the series representing the values of each of the trigonometrical functions. Thus, for the sine, we have $\mp 0, + \frac{1}{4}, + \frac{1}{2}, + \frac{3}{4}, + 1, + \frac{3}{4}, + \frac{1}{2}, + \frac{1}{4}, \pm 0$, for the first and second quadrants, understanding that all values intermediate between those represented are included. For the second and third quadrants, we have, $+ 1, + \frac{3}{4}, + \frac{1}{2}, + \frac{1}{4}, \pm 0, - \frac{1}{4}, - \frac{1}{2}, - \frac{3}{4}, - 1$.

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38. Prop.—*Cosec* $0° = \mp \infty$, *cosec* $90° = 1$, *cosec* $180° = \pm \infty$, *cosec* $270° = -1$, *cosec* $360° = \mp \infty$, *and all the real values of this function are comprised between* 1 *and* $\pm \infty$, *and* -1 *and* $\mp \infty$.

Dem.—Let the student demonstrate and illustrate as in the preceding article. Do not neglect to go through the whole in detail; it is an important and excellent exercise.

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39. Prop.—*Versin* $0° = 0$, *versin* $90° = 1$, *versin* $180° = 2$, *versin* $270° = 1$, *versin* $360° = 0$, *and the real limits of the function are* 0 *and* 2.

DEM.—The student will readily deduce these results from the relation vers $x = 1 - \cos x$. Thus when $x = 0$, vers $0 = 1 - \cos 0 = 1 - 1 = 0$, etc.

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40. Prop.—*Covers* $0° = 1$, *covers* $90° = 0$, *covers* $180° = 1$, *covers* $270° = 2$, *covers* $360° = 1$, *and the limits of the real values of this function are* 0 *and* 2.

DEM.—The student should be able to give it.

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41. GENERAL SCHOLIUM.—It is important to observe that in the case of each of the above functions *it changes its sign by passing through* 0 *or* ∞. In fact, it is assumed, in mathematics, that a varying quantity which passes from + to −, does so by passing through 0 or ∞. The converse, however, is by no means true; viz., that whenever a varying quantity passes through 0 or ∞, its sign necessarily changes.*

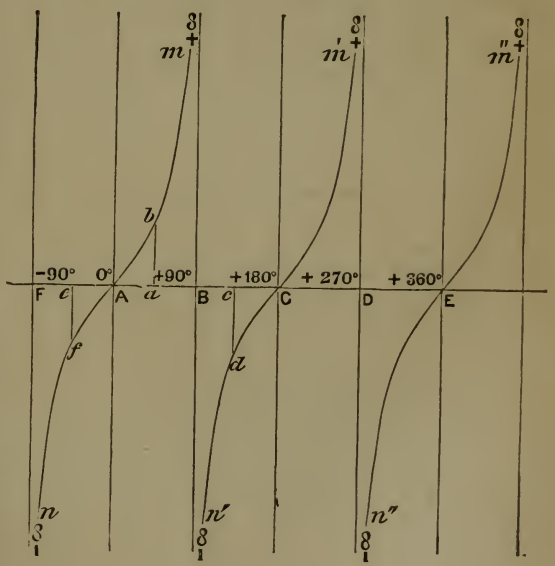

FIG. 8.

* The Co-ordinate Geometry affords elegant illustrations of the theory of the change in value and sign of these functions. (See Gen. Geom., **23**, etc.) The annexed Figure represents a curve, (or as the student may be disposed to consider it, a *series of curves*), constructed as follows: On the indefinite line AE, a circumference is developed (as it were straightened out), the origin being at A, and AE being the length of the circumference. The curves mn, $m'n'$, $m''n''$ are drawn by erecting at every few degrees from A, a line equal to the tangent of the same number of degrees, *above* the line AE when the tangent is +, and *below* AE when the tangent is −. Thus $Aa = 45°$ in length, and $ab = \tan 45°$; $Ac = 135°$, and $cd = \tan 135°$. Such lines as ab, cd, etc., are called ordinates of the curve. The law of change in these ordinates is manifestly the same as the law of change in tangents. We see that as we pass from 0° to 90°, the ordinate (tangent) passes from 0 to $+\infty$. *At* 90° the ordinate (tangent) is *both* + and −, *i. e.*, $\pm\infty$. So also at 270°, and at other similar points. A similar device illustrates the changes in the other trigonometrical functions. Some may see the propriety of distinguishing ∞ as + and −, who, nevertheless, do not see why it is necessary to make the same distinction in the case of 0. But a moment's reflection will show that one distinction involves the other, since ∞ and 0 are mutually reciprocals of each other.

TABLE.

x	SIN x	COS x	TAN x	COT x	SEC x	COSEC x	VERSIN x	COVER-SIN x	
$0°$ to $\frac{\pi}{2}$	∓ 0 to $+1$ $+$	$+1$ to ± 0 $+$	∓ 0 to $\pm \infty$ $+$	$\mp \infty$ to ± 0 $+$	$+1$ to $\pm \infty$ $+$	$\mp \infty$ to $+1$ $+$	$+0$ to $+1$ $+$	$+1$ to $+0$ $+$	*At* the Limits. *Between* "
$\frac{\pi}{2}$ to π	$+1$ to ± 0 $+$	± 0 to -1 $-$	$\pm \infty$ to ∓ 0 $-$	± 0 to $\mp \infty$ $-$	$\pm \infty$ to -1 $-$	$+1$ to $\pm \infty$ $+$	$+1$ to $+2$ $+$	$+0$ to $+1$ $+$	*At* the Limits. *Between* "
π to $\frac{3\pi}{2}$	± 0 to -1 $-$	-1 to ∓ 0 $-$	∓ 0 to $\pm \infty$ $+$	$\mp \infty$ to ± 0 $+$	-1 to $\mp \infty$ $-$	$\pm \infty$ to -1 $-$	$+2$ to $+1$ $+$	$+1$ to $+2$ $+$	*At* the Limits. *Between* "
$\frac{3\pi}{2}$ to 2π	-1 to ∓ 0 $-$	∓ 0 to $+1$ $+$	$\pm \infty$ to ∓ 0 $-$	± 0 to $\mp \infty$ $-$	$\mp \infty$ to $+1$ $+$	-1 to $\mp \infty$ $-$	$+1$ to $+0$ $+$	$+2$ to $+1$ $+$	*At* the Limits. *Between* "

See Art. 32, Sch's 1 and 2.

42. Sch. The results of the preceding discussion of the signs and limits of the trigonometrical functions, **21** to **41**, are exhibited in the annexed

TRIGONOMETRICAL FUNCTIONS OF NEGATIVE ARCS (OR ANGLES).

43. Prop.—*Changing the sign of an arc (or angle) changes the sign of its sine, and consequently of its cosecant; i. e.,* $\sin(-x) = -\sin x$,* and $\operatorname{cosec}(-x) = -\operatorname{cosec} x$.

DEM.—1st, If the angle (or arc) is numerically *less* than 90°, and +, it ends in the first quadrant, and hence its sine is + ; but, if the angle (or arc) is numerically *less* than 90° and −, it ends in the fourth quadrant, and hence its sine is −. That is, x being numerically less than 90°, $\sin(-x) = -\sin x$. 2d, If x is between 90° and 180° in numerical value, the arc ends in the second quadrant when x is +, and hence its sine is + ; but when x is −, the arc ends in the third quadrant, whence its sine is −. ∴ In this case, also, $\sin(-x) = -\sin x$. 3d, If x is between 180° and 270° in numerical value and +, the arc ends in the third quadrant, whence $\sin x$ is −, $(-\sin x)$; but if the arc is −, it ends in the second quadrant, whence its sign is +, $[+\sin(-x)]$. ∴ In this case $-\sin x = \sin(-x)$, or $\sin x = -\sin(-x)$. 4th, In like manner the student will observe that $-\sin x = \sin(-x)$, or $\sin x = -\sin(-x)$, when x is between 270° and 360°. Moreover, since this order will recur as we pass around again, we learn that in any case the sign of the sine of a negative angle is the opposite of that of an equal positive angle. Finally, $\operatorname{cosec}(-x) = \dfrac{1}{\sin(-x)} = \dfrac{1}{-\sin x} = -\dfrac{1}{\sin x} = -\operatorname{cosec} x$. Q. E. D.

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44. Prop.—*Changing the sign of an angle (or arc) does not change the sign of its cosine or secant; i. e.,* $\cos(-x) = \cos x$, and $\sec(-x) = \sec x$.

DEM.—1st, When $x < 90°$ and +, the arc ends in the first quadrant, and hence its cosine is + ; so, also, if $x < 90°$ and −, though the arc ends in the fourth quadrant, its cosine is still +. ∴ $\cos(-x) = \cos x$. (The student can supply the other three cases.) Finally, $\sec(-x) = \dfrac{1}{\cos(-x)} = \dfrac{1}{\cos x} = \sec x$. Q. E. D.

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45. Prop.—*Changing the sign of an angle (or arc) changes the sign of its tangent, and consequently of its cotangent; i. e.,* $\tan(-x) = -\tan x$, and $\cot(-x) = -\cot x$.

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* The student should be careful to note the exact meaning of this expression. It is read "sine minus x = minus sine x," and means that the sine of a negative angle (or arc) is equa (numerically) to the sine of the same positive angle (or arc), but has the opposite sign.

DEM.—This is an immediate consequence of the fact that changing the sign of the angle (or arc) changes the sign of its sine, but *not* of its cosine. Thus, $\tan(-x) = \frac{\sin(-x)}{\cos(-x)} = \frac{-\sin x}{\cos x} = -\frac{\sin x}{\cos x} = -\tan x$. Also, $\cot(-x) = \frac{1}{\tan(-x)} = \frac{1}{-\tan x} = -\frac{1}{\tan x} = -\cot x$; or $\cot(-x) = \frac{\cos(-x)}{\sin(-x)} = \frac{\cos x}{-\sin x} = -\frac{\cos x}{\sin x} = -\cot x$.

SCH.—The proper sign of a function of a negative angle can always be ascertained by observing in what quadrant the measuring arc ends, in a manner altogether similar to that in which the signs of the functions of positive angles are determined.

CIRCULAR FUNCTIONS.

46. Circular Functions are angles (or arcs) expressed as functions of sines, cosines, tangents, or other trigonometrical lines.

ILL'S.—In the expression $\sin x$, we designate a *sine*, *i. e.*, a right line, merely using the x to tell *what* sine, as the sine of 20°, of 135°, etc. But we often wish to speak of an angle (or arc) which has a particular sine, tangent, or other trigonometrical line. Thus, we say, "the angle (or arc) whose sine is $\frac{1}{2}$," "the angle (or arc) whose tangent is 3," etc. In this form of expression, it is evidently the angle (or arc) which is the thing mainly thought of; and it is conceived as dependent upon its trigonometrical line.

47. Notation.—The circular functions are written $\sin^{-1}y$, $\cos^{-1}x$, $\tan^{-1}z$, etc.; and are read "the angle (or arc) whose sine is y," "the angle whose tangent is z," etc.

ILL'S.—The expressions $x = \sin^{-1}y$, and $y = \sin x$, are ultimately equivalent, since the first is "x = angle whose sine is y," and the second, "y = the sine of x." The only difference is, that in the first form the *angle* (x) is the thing thought of, and the sine (y) is used merely to tell *what* angle; but in the second form, the *sine* (y) is the prominent thing, and the *angle* (x) is used simply to tell *what* sine. This mutual relation has caused the circular functions to be called also *Inverse Functions*.

SCH.—This notation is rather an unfortunate one, inasmuch as it is the same as has been already adopted in the theory of exponents. The student will however observe that the signification in this instance is altogether different from the former. Thus, since we write "the square of sine x," $\sin^2 x$; according to the theory of exponents, $\sin^{-2}x$ would be $\frac{1}{\sin^2 x}$. So also $\sin^{-1}x$ should mean $\frac{1}{\sin x}$. Now, the former of these expressions would actually signify as indicated (though it were better to write it $(\sin x)^{-2}$), while the latter does not mean at all what the theory of exponents would make it. Unfortunate as the notation is, it is probably best to retain it. It, doubtless, was suggested thus: If we have $y = a^2x$, we may write $x = a^{-2}y$, so also $y = ax$, may be written $x = a^{-1}y$. This affords a parallelism in form, but not in signification.

EXERCISES.

1. What is the complement of 150° 21′ 13″.5? What the supplement? Give the complements and also the supplements of 125° 15′, 283° 21′ 11″, 36° 05′ 02″, and 89° 00′ 12″.

2. How many degrees in the angle (or arc) $\frac{\pi}{3}$? In $\frac{3}{2}\pi$? In the arc 2π? In $\frac{1}{4}\pi$? In $1\frac{1}{4}\pi$?

3. How many times is the radius contained in 108°? How many times in 2π? How many in 460°? How many in 210°?

4. Radius being taken as the measure of the arc, by what are 45° represented? By what 90°? By what 180°? By what 225°? How many degrees does 1 represent, radius being the measure?

5. Find the length of a degree of the meridian upon a globe of 18 inches diameter.

6. Express 12° 22′ 13″ 11‴ 05^{iv} in °, ′, ″, and decimals of a second. So also express 53°.51 in °, ′, ″, etc.

7. How many degrees, minutes, and seconds in an arc equal to twice radius? three times radius? Show that 27° is equivalent to $\frac{3}{20}\pi$. That 10° to radius 10 ft. = 1.75 ft. What radius gives 1° = 1 inch?

8. Draw any angle, and construct its sine, cosine, tangent, or any other trigonometrical function, and then determine as nearly as practicable the numerical values of the function by actual measurement.

Solution.—Given the angle MON, to find the numerical value of the tangent as near as practicable by measurement. Taking any convenient unit, as OA, for a radius, and striking the arc Aa, draw AT tangent to aA at A. Now apply OA to AT and find their ratio (Part I., *36*). In this case AT = $1\frac{1}{3}$ approximately, ∴ tan MON = $1\frac{1}{3}$.

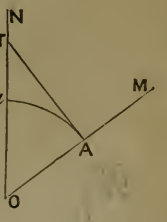

Fig. 9.

[Note.—The student should practice upon such examples, finding the values of all the functions until the process, and the meaning of the numerical value of any function of an angle, are clearly seen.]

9. Construct an angle whose sine is $\frac{2}{3}$, *i. e.*, $\sin^{-1}\frac{2}{3}$.

Solution.—Let O be the required angular point, and OA one side of the angle. Lay off from O on OA 3 measures of any convenient length, making Oa. Using Oa as a radius, describe the indefinite arc aM. Erect OC perpendicular to OA and take OC = $\frac{2}{3}$ of Oa. Through C draw CP parallel to OA. Finally, draw OB through P. AOB is the angle required, since Oa being 1, the sine of AOB, PD, is $\frac{2}{3}$. AOB = $\sin^{-1}\frac{2}{3}$.

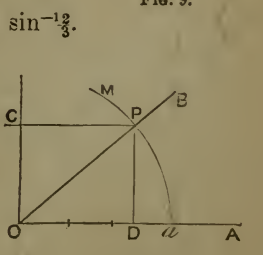

Fig. 10.

10. Construct an angle whose cosine is $\frac{2}{3}$. That is, construct $\cos^{-1}\frac{2}{3}$.

SUG.—The construction is the same as in the last example, except that instead of OC being drawn to limit the arc, a perpendicular is erected to Oa at $\frac{2}{3}$ (the second point of division) from O, and the point P located where this perpendicular intersects the arc.

11. Construct an angle whose tangent is 2. That is, construct $\tan^{-1}2$.

SUG.—To construct this at O on the line OA, *Fig.* 10, take any convenient radius, Oa, and strike the indefinite arc. Then erect at a a tangent, and make it equal to twice the radius used. Through the extremity of this tangent and O draw a line, and the angle between this and OA is $\tan^{-1}2$.

12. Construct $\sec^{-1}2$; $\cot^{-1}3$; $\operatorname{cosec}^{-1}1\frac{1}{2}$; an obtuse angle $\sin^{-1}\frac{1}{2}$; $\tan^{-1}(-3)$.

SUG.—To construct $\sin^{-1}\frac{1}{2}$, see *Fig.* 7. Let OA be one side of the angle, and O the vertex. With any convenient radius draw the semicircumference AaB, and draw the perpendicular Oa. Bisect this perpendicular, and through the point of bisection draw a parallel to AB, intersecting the arc in the 2d quadrant. Through this intersection draw a line, as OP′. Then AOP′ (assuming the construction as specified, and not as in the figure) $= \sin^{-1}\frac{1}{2}$.

13. Construct the following: $\tan^{-1}1$; $\tan^{-1}(-1)$; $\tan^{-1}\frac{3}{4}$; $\tan^{-1}(-2)$; $\tan^{-1}(-\frac{1}{2})$; $\cos^{-1}(-\frac{1}{3})$; $\sec^{-1}(-2)$; $\operatorname{cosec}^{-1}(-3)$; $\operatorname{versin}^{-1}\frac{1}{4}$; $\operatorname{versin}^{-1}1\frac{1}{2}$.

14. From the fundamental relations (***21***) deduce the following:

$\sin x = \sqrt{1-\cos^2 x}$; $\cos x = \sqrt{1-\sin^2 x}$; $\tan x \cot x = 1$; $\tan x \cos x = \sin x$;

$\cos x = \dfrac{\sin x}{\tan x}$; $\sin x = \dfrac{\cos x}{\cot x}$; $\sin x = \dfrac{1}{\sqrt{1+\cot^2 x}}$; $\cos x = \dfrac{1}{\sqrt{1+\tan^2 x}}$;

$\tan x = \sin x \sec x$; $\cot x = \cos x \operatorname{cosec} x$; $\tan x = \dfrac{\sec x}{\operatorname{cosec} x}$; $\sin x = \dfrac{\cos x}{\sqrt{\operatorname{cosec}^2 x - 1}}$;

$\operatorname{vers} x = \dfrac{\sec x - 1}{\sec x} = \dfrac{\operatorname{cosec} x - \cot x}{\operatorname{cosec} x}$; $\cot x = \dfrac{\sqrt{1-\sin^2 x}}{\sin x}$.

15. Given $\tan x = \frac{3}{4}$, to find the other trigonometrical functions of x.

Results: Sec $x = \frac{5}{4}$; $\cos x = \frac{4}{5}$; $\sin x = \frac{3}{5}$; $\operatorname{cosec} x = \frac{5}{3}$; $\cot x = \frac{4}{3}$; $\operatorname{vers} x = \frac{1}{5}$; $\operatorname{covers} x = \frac{2}{5}$.

16. Given $\sin x = \frac{2}{3}$, to find the other trigonometrical functions of x.

17. Given $\sec x = 2$, to find the other trigonometrical functions of x.

18. Given $\tan x = -1$, to find the other trigonometrical functions of x.

Results.—Cot $x = -1$; $\sec x = \mp\sqrt{2}$; $\operatorname{cosec} x = \pm\sqrt{2}$; $\sin x = \pm\frac{1}{2}\sqrt{2}$; $\cos x = \mp\frac{1}{2}\sqrt{2}$; versin $x = 1 \pm \frac{1}{2}\sqrt{2}$; coversin $x = 1 \mp \frac{1}{2}\sqrt{2}$.

[NOTE.—Observe closely the signs of the functions in Ex. 18.]

19. In the preceding examples the constructions required have been limited to angles less than 180°, but it is evident that an infinite number of angles (or arcs) according to the more comprehensive trigonometrical view, correspond to the same function. What angles or arcs have their tangents each 1? What, each −1? Construct an angle between 180° and 270°, whose sine is $-\frac{1}{2}$. What other angle less than 360° has the same sine?

20. Having given a sine, how many angles less than 180° correspond to it? Construct the angle or angles less than 180° whose sine is $\frac{2}{3}$. How many angles less than 180° have the same cosine? tangent? cotangent? (In each of the last three cases *only one.*) Construct $y = \cos^{-1}\frac{1}{2}$; $y = \cos^{-1}(-\frac{1}{2})$. How are these angles related to each other? Construct $y = \tan^{-1}3$; $y = \tan^{-1}(-3)$. How are these angles related to each other? (Restrict the constructions and questions in this example to angles less than 180°.)

21. Given $y = \sin^{-1}x$, show that $\cos y = \sqrt{1-x^2}$; $\tan y = \frac{x}{\sqrt{1-x^2}}$; $\sec y = \frac{1}{\sqrt{1-x^2}}$; $y = \operatorname{cosec}^{-1}\frac{1}{x}$; $\operatorname{vers} y = 1 - \sqrt{1-x^2}$; $y = \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$.

22. What are trigonometrical functions of 450°? Of 1350°? Of 900°?

23. Show that the following are true for all integral values of n, including 0: $\sin 4n\frac{\pi}{2} = 0$; $\sin(4n+1)\frac{\pi}{2} = 1$; $\sin(4n+2)\frac{\pi}{2} = 0$; $\sin(4n+3)\frac{\pi}{2} = -1$; $\cos 4n\frac{\pi}{2} = 1$; $\cos(4n+1)\frac{\pi}{2} = 0$; $\cos(4n+2)\frac{\pi}{2} = -1$; $\cos(4n+3)\frac{\pi}{2} = 0$; $\tan 2n\frac{\pi}{2} = 0$; $\tan(2n+1)\frac{\pi}{2} = \infty$.

24. What are the signs of the several trigonometrical functions of −110°? Of −35°? Of −500°? Of −2000?

25. Prove that $\sin 30° = \frac{1}{2}$, $\cos 30° = \frac{1}{2}\sqrt{3}$, $\tan 30° = \frac{1}{3}\sqrt{3}$, and $\cot 30° = \sqrt{3}$, $\sec 30° = \frac{2}{3}\sqrt{3}$, and $\operatorname{cosec} 30° = 2$.

SUG.—Observe that the chord of 60° = 1, and that the sine of 30° = ½ the chord of 60°. Make the figure.

26. Prove that $\sin 45° = \frac{1}{2}\sqrt{2}$, $\cos 45° = \frac{1}{2}\sqrt{2}$, $\tan 45° = 1$, $\cot 45° = 1$, $\sec 45° = \sqrt{2}$, and $\operatorname{cosec} 45° = \sqrt{2}$.

SUG.—Observe that $\sin 45° = \cos 45°$; hence $\sin^2 45° + \cos^2 45° = 1$, becomes $2\sin^2 45° = 1$.

SCH.—The values obtained in the last two examples should be retained in the memory, as they are of frequent use. The functions of 30° and of 45° are always assumed to be known in any trigonometrical operation.

SECTION II.

RELATIONS BETWEEN THE TRIGONOMETRICAL FUNCTIONS OF DIFFERENT ANGLES (OR ARCS).

(*a*) FUNCTIONS OF THE SUM OR DIFFERENCE OF ANGLES (OR ARCS).

48. Prop.—*The sine of the sum of two angles (or arcs) is equal to the sine of the first into the cosine of the second, plus the cosine of the first into the sine of the second.* Thus letting x and y represent any two angles (or arcs),

$$\sin(x + y) = \sin x \cos y + \cos x \sin y.$$

DEM.—Let AOB and BOC be the two angles represented respectively by x and y. Draw the measuring arc aP′, and the sines PD, and P′E of the angles. AOC = AOB + BOC, is the *sum* of the two angles. Draw P′D′, *the sine of the sum* of the two angles. Then PD = $\sin x$, P′E = $\sin y$, OD = $\cos x$, OE = $\cos y$, P′D′ = $\sin(x + y)$, and OD′ = $\cos(x + y)$.

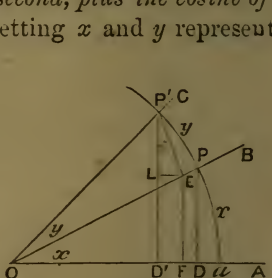

FIG. 11.

Now $\sin(x + y)$ = P′D′ = EF + P′L. But from the similar triangles EOF, and POD, we have $\frac{EF}{OE} = \frac{PD}{OP}$, or $\frac{EF}{\cos y} = \frac{\sin x}{1}$. ∴ EF = $\sin x \cos y$. Also, from the similar triangles P′EL and POD, we have $\frac{P'L}{P'E} = \frac{OD}{OP}$, or $\frac{P'L}{\sin y} = \frac{\cos x}{1}$. ∴ P′L = $\cos x \sin y$. Substituting these values of EF and P′L, we have $\sin(x + y) = \sin x \cos y + \cos x \sin y$. Q. E. D.*

* This demonstration may seem defective, since the sum of the angles x and y, as represented in the diagram, is less than 90°; nevertheless, in the General (Analytical) Geometry, we constantly proceed in a manner entirely analogous; viz., first produce the equation of a locus from some particular figure, and then make it general in application. The demonstrations in cases in which the sum of the angles is greater than 90°, etc., are similar, and some of them will be given in the EXERCISES at the close of the section. It is not thought best to cumber the purely theoretical part of the subject with such matter.

COR. $Sin(90°+x)=\cos x$. $Sin(180°+x)=-\sin x$. $Sin(270°+x)=-\cos x$. $Sin(360°+x)=\sin x$.

DEM.—From the proposition we have,

$$\sin(90°+x)=\sin 90°\cos x+\cos 90°\sin x=\cos x;$$

since $\sin 90°=1$, and $\cos 90°=0$.

In like manner,

$$\sin(180°+x)=\sin 180°\cos x+\cos 180°\sin x=-\sin x;$$

since $\sin 180°=0$, and $\cos 180°=-1$.

[The student will readily produce the other forms.]

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49. Prop.—*The sine of the difference between two angles (or arcs) is equal to the sine of the first into the cosine of the second, minus the cosine of the first into the sine of the second.* Thus, letting x and y represent the angles,

$$\sin(x-y)=\sin x\cos y-\cos x\sin y.$$

DEM.—In $\sin(x+y)=\sin x\cos y+\cos x\sin y$, substituting $-y$ for y, we have, $\sin(x-y)=\sin x\cos(-y)+\cos x\sin(-y)=\sin x\cos y-\cos x\sin y$; since $\cos(-y)=\cos y$, and $\sin(-y)=-\sin y$. (*43*, *44*.)

COR. $Sin(90°-x)=\cos x$. $Sin(180°-x)=\sin x$. $Sin(270°-x)=-\cos x$. $Sin(360°-x)=-\sin x$.

DEM.—This is simply an application of the proposition, as the preceding corollary is of its proposition. (The student will make it.) Or we may deduce the results from the corollary under the preceding proposition by merely substituting $-x$ for x. Thus, $\sin(90°-x)=\cos(-x)=\cos x$; $\sin(180°-x)=-\sin(-x)=-(-\sin x)=\sin x$; etc.

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50. Prop.—*The cosine of the sum of two angles (or arcs) is equal to the rectangles of their cosines, minus the rectangle of their sines.* Thus, letting x and y represent the angles,

$$\cos(x+y)=\cos x\cos y-\sin x\sin y.$$

DEM.—Taking the formula $\sin(x-y)=\sin x\cos y-\cos x\sin y$, and substituting $90°-x$ for x, we have, $\sin(90°-x-y)=\sin(90°-x)\cos y-\cos(90°-x)\sin y$. Now $90°-x-y=90°-(x+y)$; and $\sin[90°-(x+y)]=$ the sine of the complement of $(x+y)$, or $\cos(x+y)$. Also $\sin(90°-x)=\cos x$ and $\cos(90°-x)=\sin x$. ∴ Substituting, $\cos(x+y)=\cos x\cos y-\sin x\sin y$.

COR. $Cos(90°+x)=-\sin x$. $Cos(180°+x)=-\cos x$. $Cos(270°+x)=\sin x$. $Cos(360°+x)=\cos x$.

DEM.—Apply the proposition.

51. Prop.—*The cosine of the difference of two angles (or arcs) is equal to the rectangle of their cosines, plus the rectangle of their sines.* Thus, letting x and y represent the angles,

$$\cos(x-y) = \cos x \cos y + \sin x \sin y.$$

Dem.—In $\cos(x+y) = \cos x \cos y - \sin x \sin y$, substituting $-y$ for y, we have, $\cos(x-y) = \cos x \cos(-y) - \sin x \sin(-y)$, or $\cos(x-y) = \cos x \cos y + \sin x \sin y$; since $\cos(-y) = \cos y$, and $\sin(-y) = -\sin y$. [Notice that the last term becomes $-\sin x(-\sin y)$, which equals $+\sin x \sin y$.]

Cor. $Cos(90°-x) = sin\, x$. $Cos(180°-x) = -cos\, x$. $Cos(270°-x) = -sin\, x$. $Cos(360°-x) = cos\, x$.

Dem.—Apply the proposition.

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52. Prop.—*The tangent of the sum of two angles (or arcs) is equal to the sum of their tangents, divided by* 1 *minus the rectangle of their tangents.* Thus, x and y being the angles, we have,

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}.$$

Dem. $\operatorname{Tan}(x+y) = \dfrac{\sin(x+y)}{\cos(x+y)} = \dfrac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}$. Dividing numerator and denominator of the last fraction by $\cos x \cos y$,* we have, $\tan(x+y) = \dfrac{\frac{\sin x}{\cos x} + \frac{\sin y}{\cos y}}{1 - \frac{\sin x \sin y}{\cos x \cos y}} = \dfrac{\frac{\sin x}{\cos x} + \frac{\sin y}{\cos y}}{1 - \frac{\sin x}{\cos x} \times \frac{\sin y}{\cos y}} = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$; since $\dfrac{\sin x}{\cos x} = \tan x$, etc., and $\dfrac{\sin x \sin y}{\cos x \cos y} = \dfrac{\sin x}{\cos x} \times \dfrac{\sin y}{\cos y}$. Q. E. D.

Cor. $Tan(90°+x) = -cot\, x$. $Tan(180°+x) = tan\, x$. $Tan(270°+x) = -cot\, x$. $Tan(360°+x) = tan\, x$.

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* The three following forms may readily be obtained by dividing respectively by $\sin x \sin y$, $\sin x \cos y$, and $\cos x \sin y$: viz., $\tan(x+y) = \dfrac{\cot y + \cot x}{\cot x \cot y - 1}$, $\tan(x+y) = \dfrac{1 + \cot x \tan y}{\cot x - \tan y}$, and $\tan(x+y) = \dfrac{\tan x \cot y + 1}{\cot y - \tan x}$. Any one of these may be reduced to the one given in the proposition, by substituting in it $\dfrac{1}{tan}$ for *cot*, and reducing. Notice that the form in the *Prop.* is in terms of the tangents. Also observe *why* dividing by a certain term gives a particular form, and by which to divide to get a required form.

DEM. $\text{Tan}\,(90° + x) = \dfrac{\sin(90° + x)}{\cos(90° + x)}* = \dfrac{\cos x}{-\sin x} = -\cot x$. $\text{Tan}\,(180° + x) = \dfrac{\sin(180° + x)}{\cos(180° + x)} = \dfrac{-\sin x}{-\cos x} = \tan x$, etc., etc.

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53. *Prop.*—*The tangent of the difference of two angles (or arcs) is equal to the difference of their tangents, divided by* 1 *plus the rectangle of their tangents. Thus, x and y being the angles,

$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}.$$

DEMONSTRATION. $\text{Tan}\,(x - y) = \dfrac{\sin(x - y)}{\cos(x - y)} = \dfrac{\sin x \cos y - \cos x \sin y}{\cos x \cos y + \sin x \sin y} = \dfrac{\dfrac{\sin x \cos y}{\cos x \cos y} - \dfrac{\cos x \sin y}{\cos x \cos y}}{\dfrac{\cos x \cos y}{\cos x \cos y} + \dfrac{\sin x \sin y}{\cos x \cos y}} = \dfrac{\dfrac{\sin x}{\cos x} - \dfrac{\sin y}{\cos y}}{1 + \dfrac{\sin x}{\cos x} \times \dfrac{\sin y}{\cos y}} = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$. Q. E. D. (See foot-notes to preceding proposition.) This proposition is also readily deduced from the preceding by substituting in the formula $\tan(x + y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$, $-y$ for y, and remembering that $\tan(-y) = -\tan y$.

COR. *Tan*$(90° - x) = \cot x$. *Tan*$(180° - x) = -\tan x$. *Tan*$(270° - x) = \cot x$. *Tan* $(360° - x) = -\tan x$.

DEM. $\text{Tan}\,(90° - x) = \dfrac{\sin(90° - x)}{\cos(90° - x)} = \dfrac{\cos x}{\sin x} = \cot x$. $\text{Tan}\,(180° - x) = \dfrac{\sin(180° - x)}{\cos(180° - x)} = \dfrac{\sin x}{-\cos x} = -\tan x$. $\text{Tan}\,(270° - x) = \dfrac{\sin(270° - x)}{\cos(270° - x)} = \dfrac{-\cos x}{-\sin x} = \cot x$. $\text{Tan}\,(360° - x) = \dfrac{\sin(360° - x)}{\cos(360° - x)} = \dfrac{-\sin x}{\cos x} = -\tan x$.

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* These and the kindred *formulæ* may be produced by a direct application of the proposition. Thus, $\tan(90° + x) = \dfrac{\tan 90° + \tan x}{1 - \tan 90° \tan x} = \dfrac{\tan 90°}{-\tan 90° \tan x} = \dfrac{1}{-\tan x} = -\cot x$. (The reason for dropping $\tan x$ and 1 is that they are finite terms connected with infinities, as $\tan 90° = \infty$. Or, otherwise, $\tan(90° + x) = \dfrac{\tan 90° + \tan x}{1 - \tan 90° \tan x} = \dfrac{1 + \dfrac{\tan x}{\tan 90°}}{\dfrac{1}{\tan 90°} - \tan x} = \dfrac{1 + \dfrac{\tan x}{\infty}}{\dfrac{1}{\infty} - \tan x} = \dfrac{1}{-\tan x}$, since a finite divided by an infinite equals 0. Again, $\tan(180° + x) = \dfrac{\tan 180° + \tan x}{1 - \tan 180° \tan x} = \dfrac{\tan x}{1} = \tan x$, since $\tan 180° = 0$, etc., etc.

54. Prop.—*The cotangent of the sum of two angles (or arcs) is equal to the rectangle of their cotangents minus* 1, *divided by their sum.* Thus, x and y being the angles,

$$\cot(x+y) = \frac{\cot x \cot y - 1}{\cot y + \cot x}.$$

DEM. $\operatorname{Cot}(x+y) = \dfrac{\cos(x+y)}{\sin(x+y)} = \dfrac{\cos x\cos y - \sin x \sin y}{\sin x \cos y + \cos x \sin y} = \dfrac{\dfrac{\cos x \cos y}{\sin x \sin y} - \dfrac{\sin x \sin y}{\sin x \sin y}}{\dfrac{\sin x \cos y}{\sin x \sin y} + \dfrac{\cos x \sin y}{\sin x \sin y}}$

$$= \frac{\dfrac{\cos x}{\sin x} \times \dfrac{\cos y}{\sin y} - 1}{\dfrac{\cos y}{\sin y} + \dfrac{\cos x}{\sin x}} = \frac{\cot x \cot y - 1}{\cot y + \cot x}.$$ Q. E. D. Or we may deduce it thus,

$$\cot(x+y) = \frac{1}{\tan(x+y)} = \frac{1 - \tan x \tan y}{\tan x + \tan y} = \frac{1 - \dfrac{1}{\cot x \cot y}}{\dfrac{1}{\cot x} + \dfrac{1}{\cot y}} = \frac{\cot x \cot y - 1}{\cot y + \cot x}$$

Q. E. D.

COR. $Cot(90° + x) = -\tan x$. $Cot(180° + x) = \cot x$. $Cot(270° + x) = -\tan x$. $Cot(360° + x) = \cot x$.

DEM. Divide cosine by sine, or take the reciprocals of the corresponding tangents Thus, $\cot(90° + x) = \dfrac{1}{\tan(90° + x)} = \dfrac{1}{-\cot x} = -\tan x$, etc.

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55. Prop.—*The cotangent of the difference of two angles (or arcs) is equal to the rectangle of their cotangents plus* 1, *divided by their difference.* Thus, x and y being the angles,

$$\cot(x-y) = \frac{\cot x \cot y + 1}{\cot y - \cot x}.$$

DEM. Substitute $-y$ for y, in the preceding formula; or, divide $\cos(x-y)$ by $\sin(x-y)$ and reduce; or, take the reciprocal of $\tan(x-y)$, and substitute $\dfrac{1}{cot}$ for *tan.*

COR. $Cot(90° - x) = \tan x$. $Cot(180° - x) = -\cot x$. $Cot(270° - x) = \tan x$. $Cot(360° - x) = -\cot x$.

DEM. Same as above.

SCH. 1. The *formulæ* for the secant and cosecant of the sum and of the difference of two angles (or arcs) are not of sufficient importance to warrant their introduction here; some of them will be given in the exercises, as also the extension of those already given to the case of the sum of three or more angles, or arcs.

Sch. 2. The results reached in this discussion are so important that we will collect them into a

TABLE.

(A) $\sin(x + y) = \sin x \cos y + \cos x \sin y.$
(B) $\sin(x - y) = \sin x \cos y - \cos x \sin y.$
(C) $\cos(x + y) = \cos x \cos y - \sin x \sin y.$
(D) $\cos(x - y) = \cos x \cos y + \sin x \sin y.$
(E) $\tan(x + y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}.$
(F) $\tan(x - y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}.$
(G) $\cot(x + y) = \dfrac{\cot x \cot y - 1}{\cot y + \cot x}.$
(H) $\cot(x - y) = \dfrac{\cot x \cot y + 1}{\cot y - \cot x}.$

(I)

x	$90° - x$	$90° + x$	$180° - x$	$180° + x$	$270° - x$	$270° + x$	$360° - x$	$360° + x$
sine	$\cos x$	$\cos x$	$\sin x$	$-\sin x$	$-\cos x$	$-\cos x$	$-\sin x$	$\sin x$
cosine	$\sin x$	$-\sin x$	$-\cos x$	$-\cos x$	$-\sin x$	$\sin x$	$\cos x$	$\cos x$
tangent	$\cot x$	$-\cot x$	$-\tan x$	$\tan x$	$\cot x$	$-\cot x$	$-\tan x$	$\tan x$
cotangent	$\tan x$	$-\tan x$	$-\cot x$	$\cot x$	$\tan x$	$-\tan x$	$-\cot x$	$\cot x$

It will not be found difficult to memorize and extend set (I), if the student observes, that, when the number of whole quadrants is *odd* (as 90°, 270°, etc.), the function *changes name* (as from sin to cos, from cos to sin, etc.); but, when the number of whole quadrants is *even*, the function retains the same name. The sign of the sine and cosine is readily determined according to fundamental principles by observing where the arc ends, assuming $x < 90°$. Thus $180° + x$ ends in the third quadrant; hence its sine (which in numerical value is $\sin x$) is $-$, and its cosine is also $-$. As the signs of the tangent and cotangent of the same arc are alike, we have only to observe whether the sine and cosine, in any given case, have like or unlike signs, in order to determine the sign of the tangent and cotangent. For example, what is $\cot(630° + x)$ equal to? The number of quadrants being odd (7), the function changes *name*, and since the arc ends in the fourth quadrant, its sine is $-$, and its cosine $+$; therefore $\cot(630° + x) = -\tan x$. If in any given case $x > 90°$, determine the character of the function as above, on the hypothesis $x < 90°$, and then modify the result for the particular value of x. Thus in the last case, if x was between 90° and 180°, its tangent would be $-$, and for such a value $\cot(630° + x) = \tan x$. Or, we may consider at first where the arc ends, taking into consideration the given value of x.

(*b*) FUNCTIONS OF DOUBLE AND HALF ANGLES.

56. Prop.—*Letting x represent any angle (or arc),*

(K) $\sin 2x = 2\sin x \cos x$;

(L) $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1$, or $1 - 2\sin^2 x$;

(M) $\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$;

(N) $\cot 2x = \dfrac{\cot^2 x - 1}{2\cot x}$.

DEM. These results are readily deduced from (A), (C), (E), and (G) Thus, in $\sin(x + y) = \sin x \cos y + \cos x \sin y$, if we make $y = x$, we have $\sin 2x = \sin x \cos x + \cos x \sin x = 2\sin x \cos x$. (In like manner produce the others.)

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57. Prop.—*Letting x represent any angle (or arc), we have,*

(O) $\sin \frac{1}{2}x = \pm\sqrt{\frac{1}{2}(1 - \cos x)}$;

(P) $\cos \frac{1}{2}x = \pm\sqrt{\frac{1}{2}(1 + \cos x)}$;

(Q) $\tan \frac{1}{2}x = \pm\sqrt{\dfrac{1 - \cos x}{1 + \cos x}}$;

(R) $\cot \frac{1}{2}x = \pm\sqrt{\dfrac{1 + \cos x}{1 - \cos x}}$.

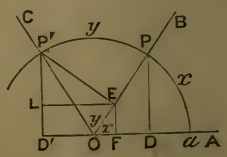
DEM. From (L), $2\sin^2 x = 1 - \cos 2x$, or $\sin x = \pm\sqrt{\frac{1}{2}(1 - \cos 2x)}$. Putting $\frac{1}{2}x$ for x, this becomes $\sin \frac{1}{2}x = \pm\sqrt{\frac{1}{2}(1 - \cos x)}$. In like manner, from the same formula (L), $2\cos^2 x = 1 + \cos 2x$; whence, $\cos \frac{1}{2}x = \pm\sqrt{\frac{1}{2}(1 + \cos x)}$. Again, $\tan \frac{1}{2}x = \dfrac{\sin \frac{1}{2}x}{\cos \frac{1}{2}x} = \pm\sqrt{\dfrac{1 - \cos x}{1 + \cos x}}$; and $\cot \frac{1}{2}x = \dfrac{1}{\tan \frac{1}{2}x} = \pm\sqrt{\dfrac{1 + \cos x}{1 - \cos x}}$. Q. E. D.

SCH. The sign of the function in the case of each of these is + if $x < 180°$; but can only be determined by the value of x in any given case.

EXERCISES.

1. Prove from *Fig.* (*a*) that $\sin(x + y) = \sin x \cos y + \cos x \sin y$, when x and y are each $< 90°$, but $x + y > 90°$.

2. Same as in *Ex.* 1, from (*b*), when $x < 90°$, $x + y > 90°$, and $< 180°$, and $y > 90°$ and $< 180°$.

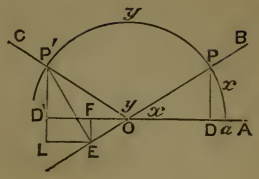


(*a*).

SUG. In this case, $\sin(x + y) = P'D' = P'L - EF$. In other respects the demonstration is identical with the preceding. This gives $\sin(x + y) = \cos x \sin y - \sin x \cos y$. But the

— sign is accounted for in the general formula, $\sin(x+y) = \sin x \cos y + \cos x \sin y$, by noticing that $\cos y$ is —, when $y > 90°$ and $< 180°$.

3. Same as in the preceding, when $x > 90°$, $y < 90°$, and $(x + y) < 180°$.

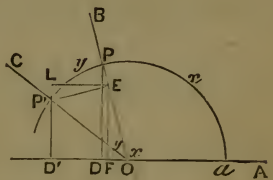


(b).

Sug. Here $\sin(x + y) = P'D' = EF - P'L$. In all other respects the demonstration is identical with the other cases. The — sign in this case arises from x being between 90° and 180°, whence $\cos x$ is —.

[Note. A number of other cases may be devised, but the above illustrate the varieties.]

4. From *Fig.* 11, Art. 48, demonstrate geometrically the formula $\cos(x + y) = \cos x \cos y - \sin x \sin y$. The same for each of the cases in Ex's 1, 2, and 3, above.

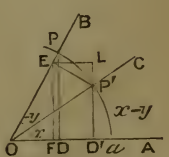


(c).

Sug. In *Fig.* 11, $\cos(x + y) = OD' = OF - LE$. $\frac{OF}{OE} = \frac{OD}{OP}$, or $OF = \cos x \cos y$. $\frac{LE}{P'E} = \frac{PD}{OP}$, or $LE = \sin x \sin y$.

5. Prove geometrically the relation $\sin(x - y) = \sin x \cos y - \cos x \sin y$.

Sug. Let $aP = x$, and since y is to be subtracted we measure it *back* from P, and $y = PP'$. Now $\sin(x - y) = P'D' = EF - P'L$.



6. Prove from *Fig.* (*d*) that $\cos(x - y) = \cos x \cos y + \sin x \sin y$.

(d).

7. Given $\sin 45° = \sqrt{\frac{1}{2}}$, and $\sin 30° = \frac{1}{2}$ to find $\sin 75°$, and $\sin 15°$ Also tan, and cot. *Result,* $\sin 75° = .97$, $\sin 15° = .26$, nearly.

Sug. Use *formulæ* (A . . . . H).

8. Given $\sin 30° = \frac{1}{2}$, to find sine, cosine, tangent, and cotangent, of 15°, 7° 30′, 3° 45′, and 1° 52′ 30″. *Results,* $\sin 15° = .2588$, $\cos 15° = .97$, nearly.

Sug. Use the *formulæ* in (57). Compare results with those found in the Table of Natural Sines, etc.

9. Of what angles may the trigonometrical functions be found from $\sin 45° = \frac{1}{2}\sqrt{2}$, by means of the *formulæ* in (57)? How?

10. Prove that $\sin(x+y+z) = \sin x\cos y\cos z + \cos x\sin y\cos z + \cos x\cos y\sin z - \sin x\sin y\sin z$. Also, $\cos(x+y+z) = \cos x\cos y\cos z - \sin x\sin y\cos z - \sin x\cos y\sin z - \cos x\sin y\sin z$. Also that $\tan(x+y+z) = \dfrac{\tan x + \tan y + \tan z - \tan x\tan y\tan z}{1 - \tan x\tan y - \tan x\tan z - \tan y\tan z}$.

Sug. $\sin(x+y+z) = \sin[(x+y)+z] = \sin(x+y)\cos z + \cos(x+y)\sin z$.

Sch. Since if $(x+y+z) = \pi$, $\tan(x+y+z) = 0$, we have from the last form, $\tan x + \tan y + \tan z = \tan x\tan y\tan z$; *i. e.*, if a semicircumference be divided into any three parts, the sum of the tangents of the three parts equals the products of the tangents of the same.

11. Prove that $\sec(x+y) = \dfrac{\sec x\sec y\operatorname{cosec} x\operatorname{cosec} y}{\operatorname{cosec} x\operatorname{cosec} y - \sec x\sec y}$.

12. Prove that $\sin 3x = 3\sin x - 4\sin^3 x$. Also that $\cos 3x = 4\cos^3 x - 3\cos x$. Also, $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$. Also, $\cot 3x = \dfrac{\cot^3 x - 3\cot x}{3\cot^2 x - 1}$.

Sugs. $\sin 3x = \sin(2x+x) = \sin 2x\cos x + \cos 2x\sin x = 2\sin x\cos x\cos x + (1 - 2\sin^2 x)\sin x = 2\sin x\cos^2 x + \sin x - 2\sin^3 x = 2\sin x(1 - \sin^2 x) + \sin x - 2\sin^3 x = 3\sin x - 4\sin^3 x$.

$\tan 3x = \tan(2x+x) = \dfrac{\tan 2x + \tan x}{1 - \tan 2x\tan x}$. In the latter substitute for $\tan 2x$ its value in terms of $\tan x$.

13. Prove that $\sin 4x = 4(\sin x - 2\sin^3 x)\cos x$.

14. Prove that $\sin x = \dfrac{2\tan \frac{1}{2}x}{1 + \tan^2 \frac{1}{2}x} = \dfrac{2}{\cot \frac{1}{2}x + \tan \frac{1}{2}x}$.

Sug. $\sin x = 2\sin \frac{1}{2}x\cos \frac{1}{2}x = \dfrac{2\sin \frac{1}{2}x}{\sec \frac{1}{2}x} = \dfrac{2\dfrac{\sin \frac{1}{2}x}{\cos \frac{1}{2}x}}{\dfrac{\sec \frac{1}{2}x}{\cos \frac{1}{2}x}} = \dfrac{2\tan \frac{1}{2}x}{\sec^2 \frac{1}{2}x} = \dfrac{2\tan \frac{1}{2}x}{1+\tan^2 \frac{1}{2}x}$. To produce the last form divide numerator and denominator of $\dfrac{2\tan \frac{1}{2}x}{1+\tan^2 \frac{1}{2}x}$ by $\tan \frac{1}{2}x$.

15. Prove that $\tan \frac{1}{2}x = \dfrac{1 - \cos x}{\sin x}$. Also $\cot \frac{1}{2}x = \dfrac{1 + \cos x}{\sin x}$.

Sug. From (L), (*56*), $2\sin^2 \frac{1}{2}x = 1 - \cos x$, and from (K), $2\sin \frac{1}{2}x\cos \frac{1}{2}x = \sin x$. Divide the former by the latter.

16. Find the trigonometrical functions of 18°.

SOLUTION.—Letting $x = 18°, 2x = 36°$, and $3x = 54°$, hence $\sin 2x = \cos 3x$ But $\sin 2x = 2\sin x \cos x$; and $\cos 3x = 4\cos^3 x - 3\cos x$; hence $2\sin x \cos x = 4\cos^3 x - 3\cos x$, or $2\sin x = 4\cos^2 x - 3 = 4 - 4\sin^2 x - 3$. From which $4\sin^2 x + 2\sin x = 1$. Solving this quadratic, we have $\sin x$, or $\sin 18° = \frac{\sqrt{5}-1}{4}$ neglecting the $-$ root, since $\sin 18°$ is $+$. From this, $\cos 18° = \frac{1}{4}\sqrt{10 + 2\sqrt{5}}$. These may be put in approximate decimal fractions.

17. Having given the functions of 18°, and 15° (Ex. 8), find those of 3°; then of 6°, 12°, 24°, etc.

Compare the results obtained with the values as given in the Table of Natural Sines, etc., obtaining all the values in decimal fractions.

SECTION III.

FORMULÆ FOR RENDERING CALCULABLE BY LOGARITHMS THE ALGEBRAIC SUM OF TRIGONOMETRICAL FUNCTIONS.

58. Since multiplication, division, involution, and evolution are the only elementary combinations of number which we can effect by means of logarithms, if we wish to add or subtract trigonometrical (or other) quantities, we have first to discover what products, quotients, powers, or roots, are equivalent to the proposed sums or differences.

59. Prop.—*To render* $\sin x \pm \sin y$, *and* $\cos x \pm \cos y$ *calculable by logarithms.*

SOLUTION. From (***55***, SCH. 2) we have $\sin(x + y) = \sin x \cos y + \cos x \sin y$, and $\sin(x - y) = \sin x \cos y - \cos x \sin y$. Adding these formulas $\sin(x + y) + \sin(x - y) = 2\sin x \cos y$. Now putting $x + y = x'$, and $x - y = y'$, whence $x = \frac{1}{2}(x' + y')$, and $y = \frac{1}{2}(x' - y')$; we have $\sin x' + \sin y' = 2\sin \frac{1}{2}(x' + y') \cos \frac{1}{2}(x' - y')$; or, dropping the accents, as the results are general,

(A′) $$\sin x + \sin y = 2\sin \tfrac{1}{2}(x + y) \cos \tfrac{1}{2}(x - y).$$

Again, by subtracting formula B (***55***, SCH. 2) from formula A, and making the same substitutions, we have,

(B′) $$\sin x - \sin y = 2\cos \tfrac{1}{2}(x + y) \sin \tfrac{1}{2}(x - y).$$

In like manner adding $\cos(x+y) = \cos x \cos y - \sin x \sin y$, and $\cos(x-y) = \cos x \cos y + \sin x \sin y$, and making the same substitutions, we have,

(C′) $\qquad \cos x + \cos y = 2\cos \tfrac{1}{2}(x+y) \cos \tfrac{1}{2}(x-y).$

Finally subtracting formula C (**55**, Sch. 2) from formula D, and making the same substitutions, we have,

(D′) $\qquad \cos y - \cos x = 2\sin \tfrac{1}{2}(x+y) \sin \tfrac{1}{2}(x-y)$, or

$\qquad \cos x - \cos y = -2\sin \tfrac{1}{2}(x+y) \sin \tfrac{1}{2}(x-y).$

60. Cor. 1.—*The sum of the sines of two angles is to their difference, as the tangent of one-half the sum of the angles is to the tangent of one-half their difference.*

Dem.—Dividing A′ by B′, we have,

$$\frac{\sin x + \sin y}{\sin x - \sin y} = \frac{\sin \tfrac{1}{2}(x+y) \cos \tfrac{1}{2}(x-y)}{\cos \tfrac{1}{2}(x+y) \sin \tfrac{1}{2}(x-y)} = \frac{\sin \tfrac{1}{2}(x+y)}{\cos \tfrac{1}{2}(x+y)} \times \frac{\cos \tfrac{1}{2}(x-y)}{\sin \tfrac{1}{2}(x-y)} =$$

$$\tan \tfrac{1}{2}(x+y) \cot \tfrac{1}{2}(x-y) = \tan \tfrac{1}{2}(x+y) \times \frac{1}{\tan \tfrac{1}{2}(x-y)} = \frac{\tan \tfrac{1}{2}(x+y)}{\tan \tfrac{1}{2}(x-y)}.$$ Q. E. D.

61. Cor. 2.—*The difference of the cosines of two angles divided by their sum is numerically equal to the product of the tangent of one-half the sum of the two angles into the tangent of one-half their difference.*

Dem.—Dividing D′ by C′, we have,

$$\frac{\cos x - \cos y}{\cos x + \cos y} = \frac{-\sin \tfrac{1}{2}(x+y) \sin \tfrac{1}{2}(x-y)}{\cos \tfrac{1}{2}(x+y) \cos \tfrac{1}{2}(x-y)} = -\frac{\sin \tfrac{1}{2}(x+y)}{\cos \tfrac{1}{2}(x+y)} \times \frac{\sin \tfrac{1}{2}(x-y)}{\cos \tfrac{1}{2}(x-y)} = -$$

$\tan \tfrac{1}{2}(x+y) \tan \tfrac{1}{2}(x-y)$. Q. E. D. (Observe the opposition in signs.)

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62. Prob.—*To render* $\tan x \pm \tan y$ *calculable by logarithms.*

Dem.—$$\text{Tan } x \pm \tan y = \frac{\sin x}{\cos x} \pm \frac{\sin y}{\cos y} = \frac{\sin x \cos y \pm \cos x \sin y}{\cos x \cos y} = \frac{\sin (x \pm y)}{\cos x \cos y}$$

Q. E. D.

EXERCISES.

Let the student deduce the following relations:

1. $\text{Cot } x + \cot y = \dfrac{\sin (x+y)}{\sin x \sin y}.$

2. $\text{Sec } x + \sec y = \dfrac{2 \cos \tfrac{1}{2}(x+y) \cos \tfrac{1}{2}(x-y)}{\cos x \cos y}.$

3. $\text{Sec } x - \sec y = \dfrac{2 \sin \tfrac{1}{2}(x+y) \sin \tfrac{1}{2}(x-y)}{\cos x \cos y}.$

4. $1 + \cos x = 2\cos^2 \frac{1}{2}x$. (See **57**.)

5. $1 - \cos x = 2\sin^2 \frac{1}{2}x$.

6. $\frac{\sin x + \sin y}{\cos x + \cos y} = \tan \frac{1}{2}(x + y)$. (Divide A′ by C′, **59**.)

7. $\frac{\sin x - \sin y}{\cos x + \cos y} = \tan \frac{1}{2}(x - y)$.

8. $\frac{\sin x + \sin y}{\cos x - \cos y} = -\cot \frac{1}{2}(x - y)$.

9. $\frac{\sin x - \sin y}{\cos x - \cos y} = -\cot \frac{1}{2}(x + y)$.

SECTION IV.

CONSTRUCTION AND USE OF TRIGONOMETRICAL TABLES.

[NOTE.—In order to read this and the subsequent sections, the student needs a knowledge of the nature of logarithms, and the method of using common logarithmic tables. If he is familiar with the last chapter in THE COMPLETE SCHOOL ALGEBRA of this series, he is prepared to go on. If he has not this knowledge, he should read the introduction preceding the table of Logarithms before reading this section.]

63. A Table of Trigonometrical Functions is a table containing the values of these functions corresponding to angles of all different values. In consequence of the incommensurability of an arc and its functions, these results can be given only approximately; yet it is possible to attain any degree of accuracy which practical science requires.

64. There are two tables of trigonometrical functions in common use, the *Table of Natural Functions*, and the *Table of Logarithmic Functions*.

65. A Table of Natural Trigonometrical Functions is a table in which are written the values of these functions for angles of various values, the radius of the circle being taken as the measuring unit, and the function being expressed in natural numbers extended to as many decimal places as the proposed degree of accuracy requires.

66. A Table of Logarithmic Trigonometrical Functions is the same as a table of natural functions, except that the *logarithms* of the values of the functions are written instead of the functions themselves, and to avoid the frequent occurrence of nega-

tive characteristics, the characteristic of each logarithm is increased by 10. For example, sines and cosines being always less than unity, except at the limit (*33*), and tangents of angles less than 45° and cotangents of angles greater than 45° being also less than unity, the logarithms of all such functions have negative characteristics. To obviate the necessity of writing these with their sign, the characteristic of each logarithm is increased by 10.

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67. Prob.—*To compute a table of natural trigonometrical functions for every degree and minute of the quadrant.*

Solution.—It is evident that an arc is longer than its sine, but that this disparity diminishes as the arc grows less. Thus, in a circle whose radius is 1 inch, the length of the sine of an arc of 1° would not differ appreciably from the arc. Much less should we be able to distinguish between the sine of 1′ and the arc. Now, since when the radius is 1, a semicircumference $= \pi = 3.1415926$, and also $= 180°$, or $180 \times 60 = 10800'$, we have the length of an arc of $1' = \frac{3.141592}{10800} = 0.0002908882$ approximately. Assuming this as the sine of 1′, we obtain the cosine thus,

$$\cos 1' = \sqrt{1 - \sin^2 1'} = \sqrt{(1 + \sin 1') \times (1 - \sin 1')} = \sqrt{1.0002908882 \times .9997091118} = 0.9999999577.$$

Having thus obtained sufficiently accurate values of sin 1′ and cos 1′, we can continue the operation as follows: from the *formulæ* $\sin(x + y) + \sin(x - y) = 2 \sin x \cos y$, and $\cos(x + y) + \cos(x - y) = 2 \cos x \cos y$, we have

$$\sin(x + y) = 2 \sin x \cos y - \sin(x - y),$$
$$\cos(x + y) = 2 \cos x \cos y - \cos(x - y).$$

Now letting y remain constantly equal to 1′, and letting x take successively the values 1′, 2′, 3′, etc., we have

$$\text{For } x = 1', \begin{cases} \sin 2' = 2 \cos 1' \sin 1' - \sin 0' = 0.0005817764 \\ \cos 2' = 2 \cos 1' \cos 1' - \cos 0' = 0.9999998308 \end{cases}$$

$$\text{For } x = 2', \begin{cases} \sin 3' = 2 \cos 1' \sin 2' - \sin 1' = 0.0008726646 \\ \cos 3' = 2 \cos 1' \cos 2' - \cos 1' = 0.9999996193 \end{cases}$$

$$\text{For } x = 3', \begin{cases} \sin 4' = 2 \cos 1' \sin 3' - \sin 2' = 0.0011635526 \\ \cos 4' = 2 \cos 1' \cos 3' - \cos 2' = 0.9999993232 \end{cases}$$

$$\text{For } x = 4', \begin{cases} \sin 5' = 2 \cos 1' \sin 4' - \sin 3' = 0.0014544407 \\ \cos 5' = 2 \cos 1' \cos 4' - \cos 3' = 0.9999989425 \end{cases}$$

etc., etc.

These operations present no difficulties except the labor of performing the numerical operations.

Of course 60 operations are required for every degree, and for 30°, 1800. But having computed the sines and cosines for every degree and minute up to 30°

we can complete the work by simple subtraction of values already found. For example, letting $x = 30°$, the first formula used above becomes

$$\sin (30° + y) = \cos y - \sin (30° - y),$$

and from $\cos (x + y) - \cos (x - y) = -2 \sin x \sin y$, we have

$$\cos (30° + y) = \cos (30° - y) - \sin y.$$

Now making y successively $= 1', 2', 3'$, etc., these give

$$\begin{cases} \sin 30° 1' = \cos 1' - \sin 29° 59' \\ \cos 30° 1' = \cos 29° 59' - \sin 1' \end{cases}$$

$$\begin{cases} \sin 30° 2' = \cos 2' - \sin 29° 58' \\ \cos 30° 2' = \cos 29° 58' - \sin 2' \end{cases}$$

$$\begin{cases} \sin 30° 3' = \cos 3' - \sin 29° 57' \\ \cos 30° 3' = \cos 29° 57' - \sin 3' \end{cases}$$

etc., etc.

All of these values which occur in the second members having been determined in reaching sin 30° and cos 30°, those in the first members can be found by performing the requisite subtractions.

Proceeding in this way till we reach 45°, the numerical values of *all* sines and cosines become known, since the sine of any angle between 45° and 90°, being the cosine of the complementary angle, will have been computed in reaching 45°. And so also the cosines of angles between 45° and 90° will have been computed as sines of the complementary angles below 45°.

The sines and cosines being computed, the corresponding tangents, cotangents, and, if need be, the secants, cosecants, versed-sines, and coversed-sines, can be calculated from the relations $\tan x = \frac{\sin x}{\cos x}$, $\cot x = \frac{1}{\tan x}$ or $\frac{\cos x}{\sin x}$, $\sec x = \frac{1}{\cos x}$, $\operatorname{cosec} x = \frac{1}{\sin x}$, $\operatorname{vers} x = 1 - \cos x$, and $\operatorname{covers} x = 1 - \sin x$.

68. Sch.—If it is desired to obtain the natural functions of angles estimated to seconds, it is necessary that the values in the tables computed as above be extended to 7 decimals at least. From such a table we may make interpolations for seconds with sufficient accuracy for most practical ends, except for values near the limits, where the disparity between the variation of the arc and that of the function changes very rapidly. For example, let it be required to find sin 34° 24′ 12″ from the data sin 34° 24′ = .5649670, and sin 34° 25′ = .5652070. We observe that an increase of 1′ upon the angle of 34° 24′ makes an increase of .5652070 − .5649670 = .0002400 in the sine. Hence an increase of 12″, or $\frac{1}{5}$ of 1′, makes an increase of $\frac{1}{5}$ of .0002400, or .0000480, *approximately*. Adding, we have sin 34° 24′ 12″ = .5650150. The student must be careful to notice whether an increase of the angle makes a numerical increase or a decrease of the function, and add or *subtract* as the case may require.

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69. Prob.—*To construct a table of logarithmic trigonometrical functions.*

Solution.—Compute the natural sines and cosines as in the preceding problem. Take the logarithms of the values thus obtained, and add 10 to each

characteristic. The results are the ordinary tabular logarithmic sines and cosines. For example, we find from the table of natural functions that sin 34° 24′ = .5649670. The logarithm of this number is $\bar{1}.752023$. Adding 10 to the characteristic, we have log sin 34° 25′ = 9.752023, as usually given in the tables. In like manner the cosines are obtained.

To obtain the tabular logarithmic tangents, we have from $\tan x = \frac{\sin x}{\cos x}$ log $\tan x$ = log $\sin x$ − log $\cos x$. If we now take the log $\sin x$ from the table as computed by the preceding part of this solution, and from it subtract the corresponding log $\cos x$, the result is the *true* log $\tan x$, since the extra 10 in the tabular log sin and log cos is destroyed by the subtraction. Therefore, to this difference we must add 10 to get the tabular log tan, as above explained. For example, the tabular log sin 34° 24′ = 9 752023, and log cos 34° 24′ = 9.916514. Hence, the tabular log tan 34° 24′ = 9.752023 − 9.916514 + 10 = 9.835509. In like manner the tabular log cot x = log $\cos x$ − log $\sin x$ + 10.

If the logarithmic secants are required they can be obtained from the relation $\sec x = \frac{1}{\cos x}$, which gives log sec x = 0 − log $\cos x$. In applying this by means of the tabular functions, it must be observed that the log $\cos x$, as we get it from the table, is 10 too great; hence, the *true* log sec x = 0 − log $\cos x$ + 10. In tabulating log secants and cosecants, it is not necessary to add 10, since, as these functions are never less than 1, their logarithms are never negative.

70. Sch.—The interpolations for seconds are usually made in the same way when using the logarithmic functions, as explained above for the natural functions. But to facilitate the operation, the approximate change of the logarithm for a change of 1″ of the angle is commonly written in the table, in a column called *Tabular Differences*, and marked D.

EXERCISES.

1. Find from the tables at the close of the volume the natural trigonometrical functions of 25° 18′.

Solution.—*To find the sine and cosine* we look in Table II., and find 25° at the top of the page. In the extreme left-hand column we find the minutes, and passing down to 18, find opposite, in the column headed N. sin (natural sine) 42736; also in the column N. cos, we find 90408. Now, as these are the lengths of the sine and cosine as compared with radius, we know they are fractions. ∴ Sin 25° 18′ = .42736, and cos 25° 18′ = .90408.

To find the tangent we turn to Table IV., and finding 25° at the top of the page, pass down the column of minutes, on the left-hand of the page, to 18, opposite which, and under the column headed 25°, we find 2698. To this we prefix the figures 47, which stand in the same column, opposite 11′, and belong to the tangents of all the angles from 25° 10′ to 25° 19′, and are omitted in the table simply to relieve the eye and to economize space. Thus we find tan 25° 18′ = .472698, the number being known to be a fraction because the angle is less than 45°

To find the cotangent we look at the *bottom* of the page in the same table till we find 25°, and then passing *up* the minutes column at the *right* hand, find $\cot 25° 18' = 2.11552$.

If the secant were required we should be obliged to obtain it by dividing 1 by the cosine, as our tables do not include this function. Thus $\sec 25° 18' = \frac{1}{\cos 25° 18'} = \frac{1}{.90408} = 1.1061$.

[NOTE.—Tables of secants and cosecants are sometimes given, but they are not of sufficient importance to justify their introduction into an elementary text-book.]

2. Show that $\sin 37° 43' = .61176$; $\cos 37° 43' = .79105$; $\tan 37° 43' = .773353$; $\cot 37° 43' = 1.29307$; $\sec 37° 43' = 1.264142$; $\operatorname{cosec} 37° 43' = 1.634628$; $\operatorname{vers} 37° 43' = .20895$; $\operatorname{covers} 37° 43' = .38824$.

3. Find that $\sin 64° 36' = .90334$; $\cos 64° 36' = .42894$; $\tan 64° 36' = 2.10600$; $\cot 64° 36' = .474835$; $\sec 64° 36' = 2.331328$; $\operatorname{cosec} 64° 36' = 1.107003$; $\operatorname{vers} 64° 36' = .57106$; $\operatorname{covers} 64° 36' = .09666$.

SUG.—In looking for sines and cosines of angles above 45°, seek the degrees at the *bottom* of the page, and be careful to observe that the columns of sines and cosines, as named at the top, change names when read from the bottom. The foundation of this arrangement will be readily perceived. Thus, turning in Table II. to 24° 32′, we find $\sin 24° 32' = .41522$. But $\sin 24° 32' = \cos(90 - 24° 32') = \cos 65° 28' = .41522$. Thus the degrees and minutes read from the bottom of the page are the complements of those read from the top.

4. Find that $\sin 42° 27' 12'' = .67499$; $\cos 42° 27' 12'' = .73783$; $\tan 42° 27' 12'' = .914834$; $\cot 42° 27' 12'' = 1.09309$.

SUG.—$\sin 42° 27' = .67495$, and $\sin 42° 28' = .67516$. ∴ An increase of 1′ in the angle makes an increase of 21 (hundred-thousandths) in the sine, and 12″ will make $\frac{12}{60}$ or $\frac{1}{5}$ as great an increase, approximately. Observe that in the case of cosine an increase of the arc makes a *decrease* of the function.

5. Find that $\sin 143° 24' = 0.596225$; $\cos 151° 23' = .877844$; $\tan 132° 36' = 1.08749$; and $\cot 116° 7' = .490256$.

SUG.—$\sin 143° 24' = \sin(180° - 143° 24') = \sin 36° 36'$. Also the trigonometrical function of any angle is numerically equal to the same function of its supplement (**56**).

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6. Find the logarithmic trigonometrical functions of 32° 15′ 22″ from the tables at the end of the volume.

SOLUTION.—Turning to Table II. we find 32° at the top of the page, and opposite 15′, and in the column L. sin (logarithmic sine), we get 9.727228; *i. e.*, log sin 32° 15′ = 9.727228. Now from the column of differences, D. 1″, we learn that an increase of 1″ of the arc at this point makes, approximately, an increase of 3.34 (millionths) in the logarithm of its sine. Hence, we *assume* that an increase of 22″ makes 22 × 3.34 = 73 (millionths). ∴ log sin 32° 15′ 22″ = 9.727228 + .000073 = 9.727301. In a similar manner we have log cos 32° 15′ = 9.927231. An increase of 1″ in the arc makes a *decrease* of 1.33 (millionths) in the log cos. ∴ an increase of 22″ makes 29 (millionths) *decrease* in the log cos, and log cos 32° 15′ 22″ = 9.927202. Log tan 32° 15′ 22″ = 9.800100; and log cot 32° 15′ 22″ = 10.199901.

7. Find that log sin 24° 27′ 34″ = 9.617051; log cos 26° 12′ 20″ = 9.952897; log tan 26° 12′ 20″ = 9.692125; log cot 126° 23′ 50″ = 9.867579.

SUG.—Observe cot (126° 23′ 50″) = cot (180° − 126° 23′ 50″) = cot (53° 36′ 10″). Also that angles above 45° are found at the bottom of the table; and remember to subtract the correction for co-functions, if an increase of arc is assumed.

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8. Given the natural sine .45621, to find the angle from the tables.

SOLUTION.—Looking for this sine in the table of natural sines, we find the next less sine to be .45606, and the angle corresponding, 27° 8′. Now, at this point, an increase of 1′ in the arc makes an *increase* of 26 (hundred thousandths) in the natural sine. But the given sine .45621 is only 15 (hundred thousandths) greater than .45606, the sine of 27° 8′. Hence the required angle is but $\frac{15}{26}$ of 1′ or 60″ = 35″, greater than 27° 8′. ∴ $\sin^{-1}.45621 = 27° 8′ 35″$, and its supplement 152° 51′ 25″, which has the same sign, and these arcs increased by every multiple of 2π.

9. Find $\sin^{-1}.62583$; $\cos^{-1}.34268$; $\tan^{-1}.468531$; $\cot^{-1}.876434$.

Results. $\sin^{-1}.62583 = 38° 44′ 35″$, and 141° 15′ 25″; and these arcs increased by every multiple of 2π.

$\cos^{-1}.34268 = 69° 57′ 36″$, and 360° − 69° 57′ 36″ = 290° 2′ 24″, and these arcs increased by every multiple of 2π.

$\tan^{-1}.468531 = 25° 6′ 16″$, and 180° + 25° 6′ 16″ = 205° 06′ 16″, and these arcs increased by every multiple of 2π.

$\cot^{-1}.876434 = 48° 46′ 3″$, and 180° + 48° 46′ 3″ = 228° 46′ 03″, and these arcs increased by every multiple of 2π.

Sug.—Observe that an *increase* of the arc makes a *decrease* of its co-functions. In the table of tangents as given, Table IV., the proportional parts given at the bottom of each column are the approximate changes which the functions undergo for a change of 1″ in the function. Thus, in finding $\cot^{-1}.876434$, we find $\cot^{-1}.876462 = 48° 46'$; and at the bottom we find that a change of 8.64 (millionths) in the function makes a change of 1″ in the angle. Hence, as the given cotangent is 28 (millionths) *less* than the cotangent of 48° 46′, the angle required is $28 \div 8.64 = 3$ (seconds), *greater* than 48° 46′.

71. Sch.—It is usually best to take from the table that function which is nearest in value to the given function, and then increase or diminish the corresponding arc as the case may require. If we always take from the table the next less function than that given in the example for sine, tangent, and secant, and the next *greater* for the cosine, cotangent, and cosecant, corrections for seconds will require always to be *added*. If we always take from the tables the functions *next less* than the one given, the corrections for seconds must be *added* for sine, tangent, and secant, and *subtracted* for the co-functions. If we were always to take from the tables the next *greater* function than the one given, the seconds corrections would be *added* for the co-functions, and *subtracted* for the others.

[Note.—It is very important that the pupil become so familiar with the nature of these tables as to use them intelligently, and not mechanically. For this reason we refrain from giving the usual specific, mechanical directions for their use, and substitute illustrations showing how they are used in accordance with the principles upon which they are constructed.]

10. Find $\sin^{-1}(-.34256)$; $\cos^{-1}(-.62584)$; $\tan^{-1}(-3.41621)$; $\cot^{-1}(-1.21648)$.

Results. $\sin^{-1}(-.34256) = 200° 1' 58''$, and $339° 58' 02''$, and these arcs increased by every multiple of 2π.

$\cos^{-1}(-.62584) = 128° 44' 38''$, and $231° 15' 22''$, and these arcs increased by every multiple of 2π.

$\tan^{-1}(-3.41621) = 106° 18' 57''$, and $286° 18' 57''$, and these arcs increased by every multiple of 2π.

$\cot^{-1}(-1.21648) = 140° 34' 42''$, and $320° 34' 42''$, and these arcs increased by every multiple of 2π.

Sug.—To obtain these results the pupil will need to recall the principles in the corollaries to (***48—55***). Thus, to find $\cot^{-1}(-1.21648)$, we find from the table that $\cot^{-1}(1.21648) = 39° 25' 18''$; and from (***55***) Cor., we learn that $\cot(180° - x) = -\cot x$. $\therefore$ $\text{Cot}^{-1}(-1.21648) = 180° - 39° 25' 18'' = 140° 34' 42''$. Again, from the same corollary, we learn that $\cot(360° - x) = -\cot x$. $\therefore$ $\text{Cot}^{-1}(-1.21468) = 360° - 39° 25' 18'' = 320° 34' 42''$.

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11. Given the logarithmic sine 9.451234, to find the corresponding angle.

SOLUTION.—The next nearest log sin found in Table II., is 9.451204 = log sin 16° 25′. Now we learn from the table that an increase of 1″ in the angle at this point, makes an *increase* in its log sin of 7.14 (millionths). But the given log sin, 9.451234, is 30 (millionths) greater than log sin 16° 25′. ∴ The required angle is 30 ÷ 7.14 = 4 (seconds) greater than 16° 25′; and we have sin 16° 25′ 4″ = 9.451234. Again, as sin 16° 25′ 4″ = sin (180° − 16° 25′ 4″) = sin 163° 34′ 56″ the latter angle has for its log sin 9.451234. Finally, either of these angles increased by any multiple of 2π has the same logarithmic sine.

12. Show from the table that the angle whose log cos is 9.778151, is 53° 7′ 49″, and also 306° 52′ 11″, and each of these angles increased by any multiple of 2π.

13. What angles correspond to the logarithmic cosines 9.246831 and 9.889372?

14. Find from the table what angles have for their logarithmic tangents 9.895760, 10.531054, and 11.216313.

Results. The first two are the log tans of 38° 11′ 20″, and 73° 35′ 43″, and also of 180° + either of these angles, and each increased by any multiple of 2π.

15. Find the angles corresponding to the logarithmic cotangents 10.008688, 9.638336, and 9.436811.

Results. The first two are the log cots of 44° 25′ 37″, and 66° 29′ 54″, and also of 180° + either of these angles, and each increased by any multiple of 2π.

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72. SCH.—Strictly speaking, negative numbers have no logarithms; since no base can be assumed, such that all negative numbers can be represented by said base affected with exponents. It is therefore customary to say that negative numbers have no logarithms. Nevertheless, *we do apply logarithms to negative trigonometrical functions.* Thus, if we have $-\cos x$, the $-$ sign is interpreted as simply telling in what quadrants x may end; while, in other respects, the function is treated exactly like $+\cos x$.

16. Given $\log(-\cos x) = 9.346251$, to find x.

SOLUTION.—The logarithmic cosine 9.346261, considered independently of its sign, corresponds to 77° 10′ 35″. But the $-$ sign requires that the arc shall end in the 2d or 3d quadrant, for such angles, and such only, have negative cosines. ∴ The angles required are $180° \mp 77° 10′ 35″ = 102° 49′ 25″$, and 257° 10′ 35″, and these increased by entire circumferences, as all these angles have logarithmic cosines, which are numerically equal to 9.346261, and the cosines themselves are negative.

17. What angle less than 180° has a negative cosine whose tabular logarithmic value is 9.653825? *Ans.* 116° 47′ 4″.

18. What angle less than 180° has a negative tangent whose tabular logarithmic value is 9.884130? *Ans.* 142° 33′ 15″

19. What angle less than 180° has a negative sine whose tabular logarithmic value is 9.341627?

20. What angle less than 180° has a negative cotangent whose tabular logarithmic value is 9.564299? *Ans.* 110° 8′ 15″.

21. Find values of $x < 180°$ which fulfil the following conditions:

$$\log(-\cos x) = 9.562468;\ \log(-\tan x) = 10.764215;$$
$$\log(-\sin x) = 8.886432;\ \log(-\cot x) = 11.152161.$$

Results. 111° 25′; 99° 45′ 54″; none; 175° 58′ 14″.

22. Having at hand only the common logarithmic tables of trigonometrical functions, and the table of logarithms of numbers, I wish to find the number of degrees, minutes, and seconds corresponding to the natural tangent 2.16145. How is it done, and what is the result?

Answer: Find the logarithm of 2.16145, to this add 10, and find the angle corresponding to this tabular logarithmic tangent. The angle is 65° 10′ 20″.

23. From the same tables as above find the natural cosine of 35° 23′. Also what angle corresponds to natural tangent 2.

24. From the same tables as above find the angle corresponding to natural tangent − 1.82645. Also to natural cosine − .42536.

25. Why is it in the table of logarithmic functions that the sine of an angle minus its cosine + 10 gives the tangent? Why that cosine − the sine + 10 gives the cotangent? Why that the sum of the tangent and cotangent of any angle = 20? Why is but one column of tabular differences needed for tangents and cotangents, while the sines and cosines require each a separate column?

FUNCTIONS OF ANGLES NEAR THE LIMITS OF THE QUADRANT.

TABLE III.

[NOTE.—This subject may be omitted in an elementary course, the first time going over, if thought best.]

73. Failure of Table II.—The method which has been given in the preceding pages for finding the logarithmic functions of angles involving seconds, by means of the *Tabular Differences*, Table II., is sufficiently accurate in most cases for practical purposes, but is

entirely too rude for the sines, tangents, and cotangents of angles near the beginning of the quadrant (those less than 2° or 3°), and for cosines, tangents, and cotangents of angles near the close of the quadrant (those between 87° or 88° and 90°). An example will render this clear. Suppose we wish to find log sin 1′ 12″. We find from Table II., log sin 1′ = 6.463726; and also that the *average* increase of the log sin between 1′ and 2′ is 5017.17 (millionths) for every second increase of the angle. But this *average* rate of increase of the function during the minute is much *less* than its *real* rate of increase *in the first part of the minute*, as from 1′ to 1′ 12″, and much *greater* than the real rate of increase *in the latter part*, as the angle approaches 2′. In fact, we see from this table, that we should use 2934.85 as the increase of log sin 2′ for 1″ increase of the arc. Now, in our proposed example, we want the increase of the log sin while the angle is passing from 1′ to 1′ 12″. This, as shown above, is considerably *more* than 5017.17 (millionths) for every second.

The cosine being the sine of the complement is subject to the same law of change near the *close* of the quadrant, that governs the sine at the beginning.

The case of the *tangent* of a small angle is similar to that of the sine; and since the cotangent is the reciprocal of the tangent, it has the same *law* of change, only that the one increases as the other decreases. Thus, since doubling a small arc, as 1″, doubles its tangent (approximately), it divides its cotangent by 2.

Finally, while the law of change in the sine is very different near the close of the quadrant from what it is near the beginning, the sine changing very rapidly at the beginning and very slowly at the close, and the cosine is just the opposite, the tangent, and cotangent have the same law of change at both extremities of the quadrant. Thus, if near the beginning of the quadrant a certain small increase of the arc *increases* the tangent at a particular rate, it *decreases* the cotangent *at the same rate*, since these functions are reciprocals of each other. Moreover, since tan 1° = cot (90° − 1°) = cot 89°, cot 89° changes according to the same law as tan 1°; and tan 89 changes reciprocally with cot 89°.

74. Description of Table III.—The first page of the table enables us to find the sines of angles less than 2° 36′ 15″ (and consequently the cosines of angles between 87° 23′ 45″ and 90°) with very great accuracy. The columns headed *Angles* contain the degrees, minutes, and seconds of the proposed angles, and the columns at their right give the same angles in seconds. The columns headed

Diff. contain the corrections to be used according to the following problems. The second and third pages answer a similar purpose with reference to tangents and cotangents of arcs within 2° 36′ 20″ of the limits of the quadrant.

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75. Prop.—*Letting x represent any number of seconds less than* 2° 36′ 15″, *we have,*

$$\log \sin x'' = 4.685575 + \log x - \text{Diff.}$$

Dem.—The length of 1″ of an arc to radius unity is 3.14159265358979 (the length of the semicircumference) ÷ 648000 (the number of seconds in 180°), and = .00000484812. For practical purposes this fraction may also be taken as the sine of 1″. Though, actually, the sine is less than the arc, the expressions for arc 1″ and sine 1″ agree to as many places of decimals as we have here. Again, for these small arcs the sine increases at *nearly* the same rate as the arc, so that $\sin 3'' = .00000484812 \times 3$ *nearly*; $\sin 102'' = .00000484812 \times 102$ *nearly*; these results being slightly in excess of the true values. It is the correction for this excess that is furnished by Table III. in the columns marked *Diff.* But this table is adapted to logarithmic computation; hence we have $\log \sin x'' = \log \sin 1'' + \log x -$ Diff. In this expression $\log \sin x''$ is the logarithm of the natural sine of x'' (not increased by 10, as each function in Table II. is); $\log \sin 1'' + \log x$, that is, the logarithmic sine of 1″ plus the logarithm of the number of seconds, corresponds to multiplying the $\sin 1''$ by the number of seconds, and gives the logarithm of the product, or strictly, the logarithm of the length of the arc of x''. Now, the sine of x'' being less than the arc, its logarithm is less than the logarithm of the length of the arc. Just how much less the table tells. This difference, therefore, between the logarithm of the arc and the logarithm of its sine, which is given in the table, is to be subtracted. Finally, to make this result agree with Table II. we must add 10 to the result. Now, $\log \sin 1'' = \log .00000484812 = \bar{6}.685575$, and adding 10, we have 4.685575.

$$\therefore \log \sin x'' = 4.685575 + \log x - \text{Diff.},$$

a result which agrees with the logarithmic functions in Table II. Q. E. D.

76. Cor. 1.—*To obtain the log cos of an angle between* 87° 23′ 45′ *and* 90°, *from this table, take the log sin of its complement.*

77. Cor. 2.—*To obtain the log tan of an angle less than* 2° 36′ 20″, *from this table, use the formula,*

$$\log \tan x'' = 4.685575 + \log x + \text{Diff.}$$

Dem.—For as small an arc as 1″, sine, arc, and tangent are practically equal; hence, $\log \tan 1'' = \log \sin 1'' = 4.685575$ (10 being added). Moreover, for these small arcs the tangents increase (like the sines) in *nearly* the same ratio as the arcs; hence, we add $\log x$. Finally, the tangent is a little in excess of the arc, which excess is given in the table, and is to be *added*. Q. E. D.

78. Cor. 3.—*To obtain the log cot of an angle less than* 2° 36′ 15″, *from this table, use the formula,*

$$\log \cot x'' = 15.314425 - \log x - \text{Diff.}$$

Demonstration.—Since $\cot x'' = \frac{1}{\tan x''}$, $\log \cot x'' = \log 1 - \log \tan x''$ $= 20 - (4.685575 + \log x + \text{Diff.}) = 15.314425 - \log x - \text{Diff.}$ The 20 arises from adding 10 twice to $\log 1 (= 0)$. One 10 is added because 4.685575 is 10 in excess of the true log tan 1″; and the other 10 is added in order to make the log cot x'' agree with the ordinary tabulated value, as in Table II. Q. E. D.

79. Cor. 4.—*To obtain the log tan of an angle between* 87° 23′ 45″ *and* 90°, *take the log cot of its complement; and to obtain the log cot of an angle between the same values, take the log tan of its complement.*

---

80. Prob.—*Having given a log sin less than* 8.657397 (*the log sin* 2° 36′ 15″), *to find the corresponding angle.*

Solution.—From $\log \sin x'' = 4.685575 + \log x - \text{Diff.}$, we have, $\log x = \log \sin x'' - 4.685575 + \text{Diff.}$ Hence, if from the given log sin, we subtract 4.685575, and then add the proper correction as furnished by Table III., we have the logarithm of the number of seconds sought. But we cannot tell what Diff. to take till we know the number of seconds. To meet this difficulty, find the angle corresponding to the given log sin from Table II., and reduce it to seconds. This will be sufficiently accurate to furnish the required Diff.

81. Cor. 1.—*Having given a log cos less than* 8.657397 (*the log cos of* 87° 23′ 45″), *to find the corresponding angle, treat it as if it were a log sin, and having found the corresponding angle, take its complement.*

82. Cor. 2.—*For log tan and log cot, the formulæ in* (***77, 78***), *give,*

$$\log x = \log \tan x'' - 4.685575 - \text{Diff.},$$
$$\text{and, } \log x = 15.314425 - \log \cot x'' - \text{Diff.}$$

These are applied as in (***80***); that is, the Diff. to be subtracted is found by getting from Table II. the required angle in seconds, as near as may be, and then take from Table III. the corresponding Diff.

EXAMPLES.

1. Find the log sin, tan, and cot of 1° 11′ 15″.

SOLUTION.—Log sin 1° 11′ 15″ = 4.685575 + log 4275 − .000031 = 8.316480. 1° 11′ 15″ = 4275″. Since 4275″ is between 4230″ and 4300, the Diff. is 31 (millionths).

Log tan 1° 11′ 15″ = 4.685575 + log 4275 + .000062 = 8.316573.

Log cot 1° 11′ 15″ = 15.314425 − log 4275 − .000062 = 11.683427.

Or, log cot can be found by subtracting log tan from 20.

2. Verify the following by using Table III.:

log sin 56′ 26″ = 8.215242; log tan 56′ 26″ = 8.215301;

log cot 56′ 26″ = 11.784699.

3. Verify the following by using Table III.:

log cos 88° 17′ 44″ = 8.473396; log tan 88° 17′ 44″ = 11.526412;

log cot 88° 17′ 44″ = 8.473588.

---

4. Having given the logarithmic sine 7.246481 to find the angle.

SOLUTION.—From Table II. we find 6′ 5″ = 365″ as the angle. But this is subject to the inaccuracy exhibited in (*73*). To obtain the correct result from Table III., we have (*80*),

$$\log x = 7.246481 - 4.685575 + 0 = 2.560906.$$

$$\therefore x = 363.8;$$ or the angle is 6′ 3″.8.

5. Given the logarithmic tangent 7.805487, to find the corresponding angle.

SOLUTION.—Table II. gives 21′ 58″.3 = 1318″.3 as the angle. From Table III. the Diff. corresponding to this is 6 (millionths). Hence,

$$\log x = \log \tan x'' - 4.685575 - \text{Diff.} \quad (82)$$

becomes, $\log x = 7.805487 - 4.685575 - .000006 = 3.119906.$

$$\therefore x = 1318;$$ and the angle is 1318″ = 21′ 58″.

6. Given the logarithmic cotangent 12.197148, to find the corresponding arc.

Table II. gives the angle 21′ 49″.8, *but the true angle as given by Table III. is* 21′ 50″.

SECTION V.

TRIGONOMETRICAL SOLUTION OF PLANE TRIANGLES.

83. There are six parts in every plane triangle: three sides and three angles; one side and any other two of which being given, the remaining parts can be found by means of the relations which exist between the sides and tabulated trigonometrical functions. To exhibit these highly important practical operations is the object of this section. We shall treat first of right angled plane triangles, and then of oblique angled plane triangles.

OF RIGHT ANGLED TRIANGLES.

Prop. 84.—*The relations between the sides and the trigonometrical functions of the oblique angles of a right angled triangle are as follows:*

$$(1)\ \text{sine} = \frac{\text{side opposite}}{\text{hypotenuse}};\quad \therefore\ \text{cosecant} = \frac{\text{hypotenuse}}{\text{side opposite}};$$

$$(2)\ \text{cosine} = \frac{\text{side adjacent}}{\text{hypotenuse}};\quad \therefore\ \text{secant} = \frac{\text{hypotenuse}}{\text{side adjacent}};$$

$$(3)\ \text{tangent} = \frac{\text{side opposite}}{\text{side adjacent}};\quad \therefore\ \text{cotangent} = \frac{\text{side adjacent}}{\text{side opposite}}.$$

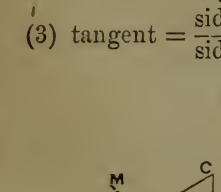


FIG. 12.

DEM.—Let CAB, *Fig.* 12, be a triangle, right angled at A. Let αM be the measuring arc of the angle B, PD = sin B, and BD = cos B. From the similar triangles PDB and CAB, we have $\frac{PD}{BP} = \frac{CA}{BC}$; *i. e.*, $\sin B = \frac{\text{side opposite}}{\text{hypotenuse}}$, since BP = 1. From the same triangles $\frac{BD}{BP} = \frac{BA}{BC}$; *i. e.*, $\cos B = \frac{\text{side adjacent}}{\text{hypotenuse}}$. Tangent being equal to sine divided by cosine, we have $\tan B = \frac{\text{side opposite}}{\text{hypotenuse}} \div \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{\text{side opposite}}{\text{side adjacent}}$. The other functions being the reciprocals of these three, are as given in the proposition. Finally, as a similar construction could be made about the other oblique angle, C, this demonstration may be considered general. Q. E. D.

SCH. 1.—These *formulæ* are so important that it is well to have them fixed in the memory, not only as written above, but also as follows:

(1). Side-opp = hy × sin, or $\frac{\text{hy}}{\text{cosec}}$, or tan × side-adj, or $\frac{\text{side-adj}}{\text{cot}}$;

(2). Side-adj = hy × cos, or $\frac{\text{hy}}{\text{sec}}$, or cot × side-opp, or $\frac{\text{side-opp}}{\text{tan}}$;

of which the relations side-opp = hy × sin, and side-adj = hy × cos are of the most frequent use.

SCH. 2.—The six ratios given in this proposition are frequently made the definitions of the trigonometrical functions. Thus, referring to a right angled triangle, a sine of an angle may be defined to be the ratio of the side opposite to the hypotenuse; the cosine as the ratio of the side adjacent to the hypotenuse, etc.

SCH. 3.—The student will be aided in remembering these important relations by observing that the side opposite the angle is analogous to the sine, and the side adjacent to the cosine. Now, the sine $= \frac{\text{its analogue}}{\text{hy}}$, and so also the cosine. Tangent equals sine divided by cosine, and in this case it is the part analogous to the sine, divided by the part analogous to the cosine. One should not make the blunder of saying that $\sin = \frac{\text{hy}}{\text{side-opp}}$, since that would make the sine always more than 1; but we have seen that it never can exceed 1. Similar checks against error may be made in the case of the other relations.

EXERCISES.

[NOTE.—The first five of these exercises are mainly designed to illustrate the proposition, and familiarize the mind with the relations.]

1. In a right angled triangle whose sides are 3, 4, and 5, what are the trigonometrical functions of the angles? What are the functions when the sides are 6, 8, and 10?

2. In a right angled triangle the hypotenuse is 12, and the angle at the base $\sin^{-1}\frac{1}{2}$. What are the sides? What is the sine of the other angle?

SUG.—Represent the angles by B, A, and C, A being the right angle; and the sides opposite by b, a, and c. Then $\sin \mathrm{B} = \frac{b}{a}$, or $\frac{1}{2} = \frac{b}{12}$; whence, $b = 6$. $\sin \mathrm{C} = \cos \mathrm{B} = \frac{1}{2}\sqrt{3}$, $c = 6\sqrt{3}$.

3. In a right angled triangle whose hypotenuse is 12, and the angle at the base $\tan^{-1}2$, what are the other parts?

Ans. $\cos^{-1}\frac{2}{5}\sqrt{5}$, $\frac{1}{5}^2\sqrt{5}$, and $\frac{2}{5}^4\sqrt{5}$.

4. The sides of a right angled triangle are 20 and 32. What are the angles and hypotenuse? Obtain the hypotenuse by means of the secant.

Ans. $\mathrm{Tan}^{-1}\frac{5}{8}$, $\tan^{-1}\frac{8}{5}$, and $4\sqrt{89}$.

5. The hypotenuse of a right angled triangle is 120, and one side 100. Show that the angle opposite the latter is $\tan^{-1}\frac{5}{11}\sqrt{11}$, the adjacent angle $\mathrm{cosec}^{-1}\frac{6}{11}\sqrt{11}$, and the remaining side $20\sqrt{11}$. Obtain these results in the order given. Use a trigonometrical function to obtain the last.

EXAMPLES.

(*a*) BY MEANS OF THE TABLE OF NATURAL FUNCTIONS.

1. In a right angled triangle ABC, the hypotenuse BC is 235, and the angle B is 43° 25′. Find the angle C, and the sides AB and AC.

C = 46° 35′; AB = 170.7; AC = 161.52.

SOLUTION.—*To find* C, we have but to remember that the angles of a right angled triangle are complements of each other; whence, C = 90° − B = 46° 35′.

To find AB, we have $\cos B = \frac{AB}{BC}$, or AB = 235 × cos 43° 25′. Now, from the table of natural functions we find cos 43° 25′ = .72637; whence AB = 235 × .72637 = 170.7. *To find* AC, we have $\sin B = \frac{AC}{BC}$; whence AC = 235 × .6873 = 161.52.

2. In a right angled triangle ABC, the hypotenuse AC is 94.6, and the angle C is 56° 30′. Find the angle A, and the sides AB and BC.

A = 33° 30′; AB = 78.88; BC = 52.21.

3. In a right angled triangle BDF, the hypotenuse BF is 127.9, and the angle B is 40° 10′ 30″. Find the angle F, and the sides BD and DF.

F = 49° 49′ 30″; BD = 97.72; DF = 82.51.

4. In the triangle CDE, right angled at E, given the side DE 75, the side CE 50.59, to find the other parts.

Hypotenuse = 90.47.

5. In the right angled triangle CDE, given the hypotenuse CD 264, the side CE 135.97, to find the other parts.

DE = 226.28.

6. Given the hypotenuse 435, and one of the acute angles 44°, to find the other parts.

7. Given the hypotenuse 64, and the base 51.778, to find the other parts.

8. Given the hypotenuse 749 feet, and the base 548.255 feet, to find the other parts.

9. Given the hypotenuse 125.7 yards, and one of the acute angles 75° 12′ 23″, to find the other parts.

10. Given one side 388.875, and the adjacent angle 27° 38′ 50″, to find the other parts of a right angled triangle.

(*b*) BY MEANS OF A TABLE OF LOGARITHMIC FUNCTIONS.

11. In a right angled triangle, given an oblique angle 54° 27′ 39″, and the side opposite 56.293, to find the other parts.

SOLUTION.—The other oblique angle is $90° - 54° 27′ 39″ = 35° 32′ 21″$.

To find the hypotenuse, we have $\sin 54° 27′ 39″ = \frac{56.293}{hy}$, or $hy = \frac{56.293}{\sin 54° 27′ 39″}$.

Applying logarithms to facilitate computation, $\log hy = \log 56.293 - \log \sin 54° 27′ 39″ + 10$. The 10 is added since the log sin 54° 27′ 39″ found in the table is 10 too great. Now, looking in Table I., we find $\log 56.293 = 1.750454$; and in Table II., $\log \sin 54° 27′ 39″ = 9.910473$. Hence,

$$\log hy = 1.750454 - 9.910474 + 10 = 1.839980, \text{ and } hy = 69.18.$$

To find the other side we have, $\tan \text{angle} = \frac{\textit{side opp}}{\textit{side adj}}$, or $\tan 54° 27′ 39″ = \frac{56.293}{\textit{side adj}}$; whence $\textit{side adj} = \frac{56.293}{\tan 54° 27′ 39″}$. Applying logarithms, $\log \textit{side adj} = \log 56.293 - \log \tan 54° 27′ 39″ + 10 = 1.750454 - 10.146104 + 10 = 1.604350$.

$\therefore$ *Side adj* $= 40.2115$.

12. In a right angled triangle ABC, the hypotenuse AC is 340, and the side AB is 200. Find the acute angles A and C, and the perpendicular BC.

A = 53° 58′ 6″; C = 36° 1′ 54″; BC = 274.95.

13. In a right angled triangle ABC, the perpendicular AB is 736.3 and BC 500. Find the acute angles A and C, and the hypotenuse AC.

A = 34° 10′ 45″; C = 55° 49′ 15″; AC = 890.02.

14. In a right angled triangle BDF, the perpendicular BD is 246.32, and DF 380.07. Find the acute angles E and F, and the hypotenuse BF.

B = 57° 3′ 11″; F = 32° 56′ 49″; BF = 452.91.

15. In a right angled triangle ABC, the side AB is 249, and the angle A is 29° 14′. Find the perpendicular BC, and the hypotenuse AC.

BC = 139.35; AC = 285.341.

16. In a right angled triangle ABC, the hypotenuse AC is 95.75, and the side BC 60. Find the acute angles A and C, and the perpendicular AB.

$A = 38° 48' 7''$; $C = 51° 11' 53''$; $AB = 74.62$.

17. In a right angled triangle ABC, the side BC is 364.3, and the angle A is 50° 45′. Find the perpendicular AB, and the hypotenuse AC.

$AB = 297.645$; $AC = 470.433$

[NOTE.—The first ten examples may be solved by logarithms if additional exercises are needed, or these by means of the natural functions. Also any one of the examples will afford several others by giving and requiring different parts. Thus, from *Ex.* 17, we can give $AB = 297.645$, $A = 50° 45'$, and require the other parts, etc.].

GENERAL APPLICATIONS.

1. Find the area of a parallelogram whose adjacent sides are 28 and 30 feet, and the included angle 75°.

SUG.—First find the altitude.

2. A railroad track 463 feet 3 inches in length has a uniform grade of 3°. Show that the vertical rise is 24 feet 3 inches, nearly.

3. A railroad track makes a vertical rise of 150 feet, by uniform grade, in 3,000 feet of track. What is the grade?

4. Find the apothem and radius of the circumscribed circle of a regular heptagon one of whose sides is 12 feet.

5. Find the area of a regular dodecagon inscribed in a circle whose radius is 12.

6. The angle of elevation to the top of a steeple is 47° 30′, as measured from a point in the same horizontal plane as its base, and at a distance of 200 feet from it. What is the height of the steeple?

Ans. 218.26. ft.

7. A tower 103 feet high throws a shadow 51.5 feet long, upon the horizontal plane of its base: what is the angle of elevation of the sun?

8. The angle at the vertex of a right cone is 52° 23′, and the slant height 126 feet: what is the diameter of the base, and what the altitude?

9. In *Fig.* 13, letting EO represent the earth and M the moon, the radius of the earth EO = 3956.2, and the angle EMO * = 57′, required to find the distance OM, E being a right angle.

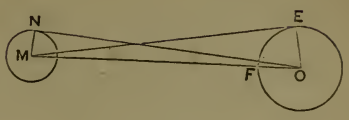


Fig. 13.

The distance of the moon from the earth, as given by this computation, is 238,613 *miles.*

10. In *Fig.* 13, letting ON represent a tangent to the moon's disc at N, the angle NOM is readily measured, being half the moon's apparent diameter. The apparent diameter of the moon being 31′ 20″, and its distance from the earth as found in the last example, what is the diameter?

Ans. 2176 miles.

OF OBLIQUE ANGLED PLANE TRIANGLES.

IMPORTANT RELATIONS EXISTING BETWEEN THE SIDES AND TRIGONOMETRICAL FUNCTIONS OF THE ANGLES OF OBLIQUE ANGLED PLANE TRIANGLES.

85. Prop.—*The sides of any plane triangle are proportional to the sines of the angles opposite.*

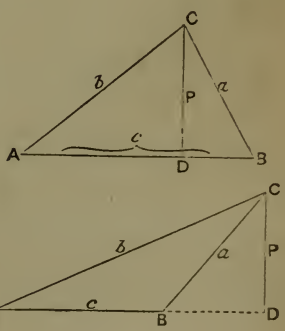


Fig. 14.

Dem.—Let ABC be any plane triangle. Let fall from either angle, as C, a perpendicular upon the opposite side, or upon that side produced. Designate the angles by A, B, and C, the sides opposite by a, b, and c, and the perpendicular by P.

Now, from the right angled triangle ADC we have $P = b \sin A$; also from CDB, $P = a \sin B$; sin ABC in the second figure being = sin CBD. Hence, equating the values of P, $b \sin A = a \sin B$, or $a : b :: \sin A : \sin B$. Q. E. D.

---

* This angle is called the moon's horizontal parallax, and is readily measured. Some rude notion of the manner in which parallax becomes apparent, may be got from considering the difference in direction from two observers to the moon, one observer standing directly under the moon, as at F, and the other at E, seeing the moon in his horizon. The angular displacement of the moon due to these different points of observation is horizontal parallax.

86. Prop.—*The sum of any two sides of a plane triangle is to their difference, as the tangent of half the sum of the angles opposite is to the tangent of half their difference.*

DEM.—Letting a and b represent any two sides of a plane triangle, and A and B the angles opposite, we have $a : b :: \sin A : \sin B$. Taking this both by composition and division, we have $a + b : a - b :: \sin A + \sin B : \sin A - \sin B$. But from (***60***), $\sin A + \sin B : \sin A - \sin B :: \tan \frac{1}{2}(A + B) : \tan \frac{1}{2}(A - B)$. $\therefore a + b : a - b :: \tan \frac{1}{2}(A + B) : \tan \frac{1}{2}(A - B)$. Q. E. D.

---

87. Prop.—*The tangent of half of any angle of a plane triangle equals k, divided by half the perimeter of the triangle minus the side opposite the angle; in which k is the radius of the inscribed circle, and equals the square root of the continued product of half the perimeter minus each side separately, divided by half the perimeter.*

DEM.—Let ABC be any plane triangle. Represent the angles by A, B, and C, the sides opposite by a, b, and c, the perimeter by p, and the segments of the sides made by the radii of the inscribed circle, by x, y, and z, as in the figure.

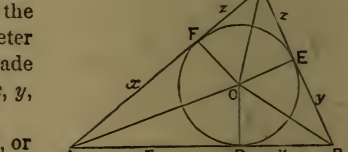


FIG. 15.

Then $a + b + c = 2x + 2y + 2z = p$, or $x + y + z = \frac{1}{2}p$.

Whence $x = \frac{1}{2}p - a$, $y = \frac{1}{2}p - b$, and $z = \frac{1}{2}p - c$; since $y + z = a$, $x + z = b$, and $x + y = c$.

Now from the triangles AOD, DOB, and COE, $\tan \frac{1}{2}A = \frac{k}{x} = \frac{k}{\frac{1}{2}p - a}$, $\tan \frac{1}{2}B = \frac{k}{y} = \frac{k}{\frac{1}{2}p - b}$, and $\tan \frac{1}{2}C = \frac{k}{z} = \frac{k}{\frac{1}{2}p - c}$.

To find k. $\frac{1}{2}A + \frac{1}{2}B + \frac{1}{2}C = 90°$, or $\frac{1}{2}A = 90° - (\frac{1}{2}B + \frac{1}{2}C)$; whence, $\tan \frac{1}{2}A = \frac{k}{x} = \tan [90° - (\frac{1}{2}B + \frac{1}{2}C)] = \cot (\frac{1}{2}B + \frac{1}{2}C) = \frac{1 - \tan \frac{1}{2}B \tan \frac{1}{2}C}{\tan \frac{1}{2}B + \tan \frac{1}{2}C}$ (***54***, DEM).

Substituting for $\tan \frac{1}{2}B$, and $\tan \frac{1}{2}C$, $\frac{k}{y}$, and $\frac{k}{z}$, we have $\frac{k}{x} = \frac{1 - \frac{k^2}{yz}}{\frac{k}{y} + \frac{k}{z}}$; whence $k^2 = \frac{xyz - xk^2}{z + y}$; $(x + y + z)k^2 = xyz$; and $k = \sqrt{\frac{xyz}{x + y + z}}$. In this value of k, substituting for x, y, and z their values, we have $k = \sqrt{\frac{(\frac{1}{2}p - a)(\frac{1}{2}p - b)(\frac{1}{2}p - c)}{\frac{1}{2}p}}$.

88. Sch. 1.—These three propositions (***85***, ***86***, ***87***), furnish the most elegant and expeditious means for finding the unknown parts of an oblique angled plane triangle, when a sufficient number of parts are given or known (***83***)

89. Sch. 2.—***Important Practical Suggestions.***

1st. Two angles of a triangle being given, the third is known by implication, it being the supplement of the sum of the other two.

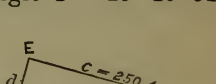
2nd. When two of the known, or given, parts are opposite each other, the first proposition (***85***) effects the solution.

3rd. When two sides and the included angle are given, the solution is effected by means of the second proposition (***86***).

4th. When the three sides are given, the angles are found by the third proposition (***87***).

EXERCISES.

1. In the plane triangle CDE, given the angle D = 15° 19′ 51″, C = 72° 44′ 05″, and the side c, opposite C, 250.4, to find the other parts.



Solution.—*First*, E = 180° − (D + C) = 91° 56′ 04″ (***89***, 1st).

Second, To find side d opposite angle D (***89***, 2nd).

$$\sin C : \sin D :: c : d, \text{ or}$$

$$\sin 72° 44′ 05″ : \sin 15° 19′ 51″ :: 250.4 : d.$$

This proposition may be solved for d, by taking the natural sine of 15° 19′ 51″, multiplying it by 250.4, and dividing the product by the natural sine of 72° 44′ 05″; or, more expeditiously, by logarithms, as follows:

log 250.4 =	2.398634	
log sin 15° 19′ 51″ =	9.422249	
log sin 72° 44′ 05″ (ar. comp.)* =	0.020024	
log d =	† 1.840907	∴ d = 69.328.

Third, To find side e opposite angle E (***89***, 2nd).

$$\sin C : \sin E :: c : e, \text{ or}$$

$$\sin 72° 44′ 05″ : \sin 91° 56′ 04″ :: 250.4 : e.$$

Making the computation by logarithms,

log 250.04 =	2.398634	
log sin 91° 56′ 04″ ‡	9.999752	
log sin 72° 44′ 05″ (ar. comp.) =	0.020024	
log e =	2.418410.	∴ e = 262.066.

---

* See Introduction to Table I. (17).

† The student must bear in mind the fact that all the log. trig. funcs. are 10 *too large*, and must see exactly what corrections to make in his results, on this account. In this case the *ar. comp.* is 10 *too small*, since the logarithm we took from the table for log sin 72° 44′ 05″ was 10 too large. But our log sin 15° 19′ 51″ is 10 *too large*, and just corrects the latter. Hence, we have to reject only 10 from the entire sum 11.840907, and this on account of the use of *ar. comp.*

‡ Take the sine of the supplement, or the *cosine* of the given angle minus 90°

2. In the plane triangle ABC, A = 35° 42′, B = 76° 27′, and AB = 142. What are the other parts?

Ans. C = 67° 51′; AC = 149.05; BC = 89.47.

3. Given two angles of a plane triangle, 23° 40′ 32″ and 69° 39′ 51″, and the included side 100, to find the other parts.

4. In a plane triangle ABC, the side AB is 254.3, the side AC 396.8, and the angle B 94° 29′. Find the angles A and C, and the side BC.

A = 45° 48′ 21″; C = 39° 42′ 39″; BC = 285.37.

SUG'S.—To find the angle C, we have

396.8 : 254.3 : : sin 94° 29′ : sin C.

From this proportion we get log sin C = 9.805443. Now, as we have seen before, there are an infinite number of angles corresponding to any given sine, how shall we know what one to take in this case? First, no angle of a triangle can exceed 180°; hence, there are but *two* angles, one an acute angle, and the other its supplement, which can come into consideration in the solution of plane triangles. But which of *these two* are we to take? Thus, in this case, both the angles 39° 42′ 39″ and 140° 17′ 21″ correspond to log sin 9.805443. In this example the ambiguity is resolved by observing that the given angle B is obtuse, and a plane triangle can have but one obtuse angle. ∴ C = 39° 42′ 39″.

5. In a plane triangle BDF, the angle B is 40°, the side BD is 400, and the side DF 350. Find the angles D and F, and the side BF.

SUG'S.—To find F, we have

350 : 400 : : sin 40° : sin F,

from which log sin F = 9.866059, and F = 47° 16′ 28″, and its supplement 132° 43′ 32″. How are we to determine which of these to take? The *given* angle is 40°; hence, as far as that is concerned, either of the two will meet the conditions. There are, therefore, *two* angles, F = 47° 16′ 28″ and F = 132° 43′ 32″, which fulfill the conditions. We therefore solve two triangles, one having two of its sides 400 and 350, and the angles 40°, 47° 16′ 28″, and 92° 43′ 32″; and another triangle with the same given parts and the two required angles 132° 43′ 32″, and 7° 16′ 28″. This is readily illustrated geometrically. Thus, lay off DBF′ = 40°. Take BD = 400. Then from D as a centre, with radius 350 describe an arc cutting BF′. It is evident that if B is an acute angle the following cases may arise depending upon the value of DF:

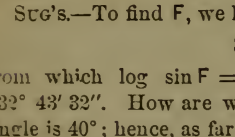


FIG. 16.

1st. If DF is less than the perpendicular p, the problem is impossible.

2nd. If DF = p, the triangle is right angled at F.

3rd. If DF > p and < BD there are two triangles, one with an acute angle at F′, as DF′B, and the other with an obtuse angle at F, as DFB, both of which fulfil the conditions of the problem.

4th. If DF > DB there is but one triangle which fulfills the conditions, viz., the one with an acute angle at F′.

The results in the above examples are, for triangle DF'B, angle DF B = 47° 16′ 28″, BDF′ = 92° 43′ 32″, and side BF′ = 543.89; for triangle DFB, angle DFB = 132° 43′ 32″, angle BDF = 7° 16′ 28″, and side BF = 68.94.

90. COR.—*In applying trigonometrical formulæ to the solution of triangles, if the part sought is found in terms of its* SINE, *the result is ambiguous, and we are to determine whether there really are two solutions to the problem in a geometrical sense, by certain geometrical considerations, or else by trying both values for the angle determined by its sine. This ambiguity arises only when an angle is determined by its sine, as will appear hereafter.*

6. Given two sides of a plane triangle 201 and 140, and the angle opposite the latter 36° 44′. Find the other parts.

Results.—There are two triangles.

Parts of the first, 120° 49′ 49″, 22° 26′ 11″, and 89.34;

Parts of the second, 59° 10′ 11″, 84° 5′ 49″, and 232.84.

7. Given two sides of a plane triangle 180, 100, and the angle opposite the former 127° 33′, to find the other parts.

There is but one triangle, and the parts are 26° 7′ 59″, 26° 19′ 1″, and 100.65.

8. Given two sides of a plane triangle 30.8 and 54.12, and the angle opposite the latter 36° 42′ 11″, to find the other parts. Why but one triangle?

9. Given two sides of a plane triangle 600 and 250, and the angle opposite the latter 42° 12′. Find the other parts.

SUG.—Attempting to get the angle opposite 600, we find log sin = 10.207400, which is impossible. It is in some such way that a trigonometrical solution shows a geometrical absurdity.

10. Given two sides of a plane triangle 1337.5 and 493, and the angle opposite the former 69° 46′. Find the other parts.

---

11. In a plane triangle, given two sides 1686 and 960, and the included angle 128° 04′, to find the other parts.

SOLUTION.—Let $a = 1686$, $b = 960$, and $C = 128° 04′$. (See ***89***, 3rd.) The sum of the angles A and B is $180° - 128° 04′ = 51° 56′$, and $\frac{1}{2}(A + B) = 25° 58′$. From (***86***) we have,

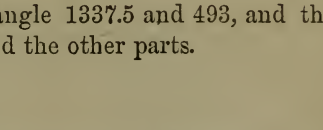


FIG. 17.

$$a + b : a - b :: \tan \tfrac{1}{2}(A + B) : \tan \tfrac{1}{2}(A - B), \text{ or}$$

$$2646 : 726 :: \tan 25° 58′ : \tan \tfrac{1}{2}(A - B).$$

Making the computations by logarithms, we find log tan $\frac{1}{2}(A - B) = 9.125887$ Hence, $\frac{1}{2}(A - B) = 7° 36' 40''$, the angle found in the table, or its supplement. But $\frac{1}{2}$ the difference of two angles of a triangle is less than 90°; consequently $\frac{1}{2}(A - B) = 7° 36' 40''$.

Now having $\frac{1}{2}(A + B) = 25° 58'$, and $\frac{1}{2}(A - B) = 7° 36' 40''$, we find $A = 33° 34' 40''$, and $B = 18° 21' 20''$.

The side c can be found by (*85*) as two opposite parts are now known. $c = 2400$.

12. In a plane triangle ABC, the side AB is 304, BC 280.3, and the included angle B is 100°. Find the angles A and C, and the side AC.

$A = 38° 3' 3''$; $C = 41° 56' 57''$; $AC = 447.856$.

13. In a plane triangle ABC, the side AB is 103, AC 126, and the included angle A is 56° 30'. Find the angles B and C, and the side BC.

$B = 72° 20' 15''$; $C = 51° 9' 45''$; $BC = 110.267$.

---

14. In a plane triangle ABC, given the three sides, $a = 3459$, $b = 4209$, and $c = 6030.4$, to find the angles.

Solution.—Applying (*87*), we have,

$$k = \sqrt{\frac{(\frac{1}{2}p - a)(\frac{1}{2}p - b)(\frac{1}{2}p - c)}{\frac{1}{2}p}}, \text{ or}$$

$$\log k = \tfrac{1}{2}\{\log(\tfrac{1}{2}p - a) + \log(\tfrac{1}{2}p - b) + \log(\tfrac{1}{2}p - c) - \log \tfrac{1}{2}p\}.$$

Also, $\tan \frac{1}{2}A = \frac{k}{\frac{1}{2}p - a}$, $\tan \frac{1}{2}B = \frac{k}{\frac{1}{2}p - b}$, and $\tan \frac{1}{2}C = \frac{k}{\frac{1}{2}p - c}$, or $\log \tan \frac{1}{2}A = \log k - \log(\frac{1}{2}p - a)$, $\log \tan \frac{1}{2}B = \log k - \log(\frac{1}{2}p - b)$, and $\log \tan \frac{1}{2}C = \log k - \log(\frac{1}{2}p - c)$.

COMPUTATION.

$a =$	3459		
$b =$	4209		
$c =$	6030.4		
$p =$	13698.4		
$\frac{1}{2}p =$	6849.2	(ar. comp.) log =	6.164360
$\frac{1}{2}p - a =$	3390.2	log =	3.530226
$\frac{1}{2}p - b =$	2640.2	log =	3.421637
$\frac{1}{2}p - c =$	818.8	log =	2.913178
			2)6.029401
		log k =	3.014700

$\log \tan \frac{1}{2}A = \log k - \log(\frac{1}{2}p - a) =$	9.484474	$\therefore A =$ 33° 56′ 10″.5
$\log \tan \frac{1}{2}B = \log k - \log(\frac{1}{2}p - b) =$	9.593063	$\therefore B =$ 42° 47′ 25″.3
$\log \tan \frac{1}{2}C = \log k - \log(\frac{1}{2}p - c) =$	10.101522	$\therefore C =$ 103° 16′ 24″.2

Proof, $A + B + C = 180° 00' 00''$.

15. In a plane triangle ABC, the side AB is 95.6, BC is 275, and AC 300. Find the angles A, B, and C.

$A = 65° 47' 55''$; $B = 95° 42' 52''$; $C = 18° 29' 13''$.

16. In a plane triangle BDF, the side BD is 500, DF is 403.7, and BF 395.75. Find the angles B, D, and F.

$B = 52° 0' 3''$; $D = 50° 34' 45''$; $F = 77° 25' 12''$.

OBLIQUE ANGLED TRIANGLES SOLVED BY MEANS OF RIGHT ANGLED TRIANGLES.

[NOTE.—Articles ***91-94*** inclusive, may be omitted in an elementary course, if thought desirable. Or ***91*** and its applications may be taken instead of ***86-88***. ***85*** should be included in any course. It is too elementary and important to be omitted.]

91. Prop.—*All cases of oblique angled plane triangles may be solved by the solution of right angled triangles.*

DEM.—Of the three given parts we may affirm that they are, 1st, All adjacent; 2nd, Two adjacent and one separated; or 3rd, All separated.

1st. *When the given parts are all adjacent; i. e.*, when they are *two sides* and the *included angle*, or *two angles* and the *included side*. To solve the first let fall a perpendicular from the extremity of one of the given sides upon the other given side, or upon that side produced. There will thus be formed two right angled triangles which can be computed, and from the parts of which the parts of the required triangle can be found. Thus, let A be the given angle, and b and c the given sides. In the right angled triangle ADC there are given A and b, whence AD, P, and angle ACD, can be computed. Then passing to triangle CDB, we know P, and DB (since we have c given and have computed AD). Hence, we can compute B, a, and DCB. Thus, the parts of ACB become known . . . . When the given parts are two angles and the included side, find the third angle by taking the supplement of the two given. Let fall a perpendicular from one extremity of the given side upon the opposite side. The two right angled triangles thus formed can then be computed. Thus, if A, b, and C are given, having found B, let fall CD. The triangle ACD has the angle A and side b known, whence its parts can be computed. Having computed P we can pass to the triangle CDB, and knowing P and B, can compute it. Thus the parts of ACB become known.

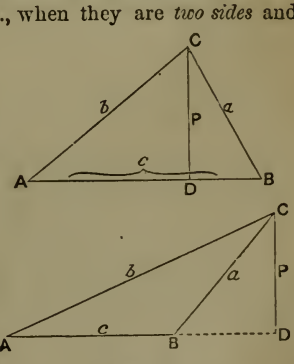


FIG. 18.

2nd. *When two of the given parts are adjacent and one separated; i. e.*, when two angles and a side opposite one are given; or two sides and an angle opposite one are given. The first of these cases is virtually the same as the last given. To solve the other, let fall a perpendicular from the angle between the given sides, and two right angled triangles will be formed which can readily be computed. Thus a, b, and A being given, and CD let fall from C, the triangle ACD can first be computed, and then CDB. This is the ambiguous case, but it is easily determined. Having computed P, if the given side a is less than P there is *no* solution; if $=$ to P, *one* solution (a right angled triangle); if $a >$ P and $< b$, there are two solutions, *i. e.*, it will go in *between* CD and AC, and also beyond CD; if $a >$ P and also $> b$ there is only *one* solution, as it will not go in between CD and AC.

3rd. *When the three given parts are all separated from each other.* This is the case in which the three sides are given to find the angles. It is readily solved by letting fall a perpendicular from the angle opposite the greatest side, upon that side, as CD upon AB. Then compute the segments AD (which call m), and DB (n), from the following relation (PART II, Ex. 12, page 162):

$$m + n \text{ (or } c) : b + a :: b - a : m - n.$$

Knowing m and n, the angles of the two right angled triangles ACD and CDB can be computed, and these make known the angles of ACB. Q. E D

[NOTE.—A few additional examples are here given which the pupil can use to illustrate the theory presented in (*91*). If more are needed the preceding can be used: these may also be used to apply the methods before given. Again, a very great variety and number of examples may be made from these by assigning different parts as known.]

EXERCISES.

1. In a plane triangle BDF, the side BF is 123.75, DF 500, and the included angle F 120°. Find the angles B and D, and the side BD.

B = 49° 12′ 4″; D = 10° 47′ 56″; BD = 572.006.

2. In a plane triangle ABC, the angle A is 70° 21′, the angle B 54° 22′, and the side BC 125. Find the angle C, and the sides AB and AC.

C = 55° 17′; AB = 109.1; AC = 107.88.

3. In a plane triangle ABC, the side AB is 98, the side BC 95.12, and the angle C 33° 21′. Find the angles A and B, and the side AC.

A = 32° 14′ 55″; B = 114° 24′ 5″; AC = 162.33.

4. In a plane triangle DAC, given AD = 450, AC = 309, and D = 27° 50′, to find the other parts.

C = 137° 9′ 36″, or 42° 50′ 24″; A = 15° 0′ 24″, or 109° 19′ 36″; DC = 171.36, or 624.5.

COMPUTATION.

It will give definiteness to the student's thought, if he first sketch the figure geometrically. Thus, lay off $D = 27° 50'$, and taking $AD = 450$, let fall the perpendicular AP.

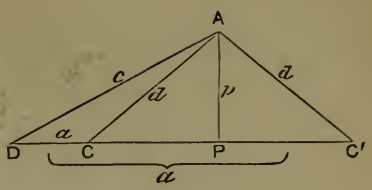


Fig. 19.

1st. *To compute p.*

$p = c \sin D = 450 \sin 27° 50'$,

$\log 450 = 2.653213$

$\log \sin 27° 50' = 9.669225$

$\log p = 2.322438.$ $\therefore p = 210.106.$

Knowing p, we see by inspection that AC can lie in both the positions AC and AC', and hence that there are two solutions.

2nd. *To compute* C, from the triangle ACP, in which d and p are now known.

$$\text{Sin C} = \frac{p}{d} = \frac{210.106}{309}$$

$\log 210.106 = 2.322438$

$\log 309 = 2.489958$

$\log \sin C = 9.832480.$ $\therefore$ C' $= 42° 50' 24''$, and C $= 137° 9' 33''$.

3rd. *To find the angle* A. DAC $= 180° - (D + ACD) = 180° - 164° 59' 36'' = 15° 0' 24''$. DAC' $= 180° - (D + AC'D) = 180° - 70° 40' 24'' = 109° 19' 36''$.

4th. *To find* DC. Compute DP and CP from the triangles APD and APC. DP $-$ CP $=$ DC, and DP $+$ CP $=$ DC'.

5. In a plane triangle ABC, the side AB is 460, BC is 340, and AC 280. Find the angles A, B, and C.

A $= 47° 23' 16''$; B $= 37° 18' 31''$; C $= 95° 18' 13''$.

6. The sides of a plane triangle are 40, 34, and 25 feet respectively; required the angles.

$38° 25' 20''$, $57° 41' 24''$, $83° 53' 16''$.

7. The sides of a plane triangle are 390, 350, and 270 feet respectively; required the angles.

$42° 22' 06''$, $60° 52' 33''$, and $76° 45' 21''$.

8. Given two sides of a plane triangle 450 and 540, and the included angle 80°, to find the remaining parts.

Angles, $56° 11'$, $43° 49'$; and the side, 640.08.

9. Given two sides of a plane triangle 76 and 109, and the included angle $101° 30'$, to find the remaining parts.

Angles, $30° 57' 30''$, $47° 32' 30''$; and the side, 144.8.

FUNCTIONS OF THE ANGLES OF A TRIANGLE IN TERMS OF THE SIDES.

92. Prop.—*Any side of a plane triangle equals the sum of the products of each of the other sides into the cosine of the angle which it makes with the first side.*

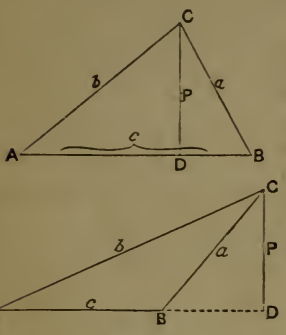


Fig. 20.

Dem.—In the first figure AB = AD + DB. But AD = $b \cos A$, and DB = $a \cos B$. $\therefore c = b \cos A + a \cos B$. In the second figure AB = AD − DB. But AD = $b \cos A$, and DB = $a \cos CBD = a(-\cos CBA) = -a \cos B$. $\therefore c = b \cos A - (-a \cos B) = b \cos A + a \cos B$. In like manner, we have $a = b \cos C + c \cos B$, and $b = a \cos C + c \cos A$. Collecting and arranging.

(1) $a = b \cos C + c \cos B$;
(2) $b = a \cos C + c \cos A$;
(3) $c = a \cos B + b \cos A$. Q. E. D.

93. Cor.—*The square of any side of a plane triangle equals the sum of the squares of the other two, minus twice their rectangle into the cosine of their included angle.*

Dem.—From (3) (***92***), we have by transposing and squaring,

$$a^2 \cos^2 B = c^2 + b^2 \cos^2 A - 2bc \cos A; \text{ and}$$

from (***85***) $a^2 \sin^2 B = b^2 \sin^2 A$.

Adding, $a^2 = c^2 + b^2 - 2bc \cos A$.
In like manner, $b^2 = a^2 + c^2 - 2ac \cos B$;
and $c^2 = a^2 + b^2 - 2ab \cos C$. Q. E. D.

94. Sch.—These *formulæ* afford another means for finding the angles of a plane triangle when the sides are given. Thus,

$$(1_1) \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc};$$

$$(2_1) \quad \cos B = \frac{a^2 + c^2 - b^2}{2ac};$$

$$(3_1) \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

These *formulæ* give directly the natural cosines of the angles in terms of the sides. To adapt them to logarithmic computation, we transform them as follows:

Subtracting each member of (1_1) from unity, $1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2bc} =$

$$\frac{a^2 - b^2 - c^2 + 2bc}{2bc} = \frac{a^2 - (b-c)^2}{2bc} = \frac{\{a + (b-c)\}\{a - (b-c)\}}{2bc} = \frac{(a+b-c)(a+c-b)}{2bc}$$

But $1 - \cos A = 2\sin^2 \frac{1}{2}A$ (*57*, O); and letting $p = a + b + c$, $\frac{1}{2}(a + b - c) = \frac{1}{2}p - c$, and $\frac{1}{2}(a + c - b) = \frac{1}{2}p - b$. Whence, substituting, $2\sin^2 \frac{1}{2}A = \frac{2(\frac{1}{2}p - c)(\frac{1}{2}p - b)}{2bc}$, or

$$(1_2)\quad \sin \tfrac{1}{2}A = \sqrt{\frac{(\frac{1}{2}p - c)(\frac{1}{2}p - b)}{bc}}.$$ In like manner,

$$(2_2)\quad \sin \tfrac{1}{2}B = \sqrt{\frac{(\frac{1}{2}p - c)(\frac{1}{2}p - a)}{ac}};$$ and

$$(3_2)\quad \sin \tfrac{1}{2}C = \sqrt{\frac{(\frac{1}{2}p - b)(\frac{1}{2}p - a)}{ab}}$$

In a manner altogether similar, by adding each member of (1·) to unity, and reducing, we get

$$(1_3)\quad \cos \tfrac{1}{2}A = \sqrt{\frac{\frac{1}{2}p(\frac{1}{2}p - a)}{bc}};$$ and from (2_1) and (3_1),

$$(2_3)\quad \cos \tfrac{1}{2}B = \sqrt{\frac{\frac{1}{2}p(\frac{1}{2}p - b)}{ac}};$$

$$(3_3)\quad \cos \tfrac{1}{2}C = \sqrt{\frac{\frac{1}{2}p(\frac{1}{2}p - c)}{ab}}.$$

Dividing (1_2) by (1_3), (2_2) by (2_3), and (3_2) by (3_3), we have.

$$(1_4)\quad \tan \tfrac{1}{2}A = \sqrt{\frac{(\frac{1}{2}p - c)(\frac{1}{2}p - b)}{\frac{1}{2}p(\frac{1}{2}p - a)}};$$

$$(2_4)\quad \tan \tfrac{1}{2}B = \sqrt{\frac{(\frac{1}{2}p - c)(\frac{1}{2}p - a)}{\frac{1}{2}p(\frac{1}{2}p - b)}};$$

$$(3_4)\quad \tan \tfrac{1}{2}C = \sqrt{\frac{(\frac{1}{2}p - b)(\frac{1}{2}p - a)}{\frac{1}{2}p(\frac{1}{2}p - c)}}.$$

EXERCISES.

[NOTE.—In order to render these *formulæ* familiar, and to give the student exercise in applying *formulæ*, a few examples are appended. If necessary, any which precede can be used.]

1. The sides of a plane triangle being 40, 34, and 25, find the angles.

SOLUTION.—*By natural functions.*

Let the sides be represented by a, b, and c in order, and the angles opposite by A, B, and C; then

$$\text{Cos A} = \frac{b^2 + c^2 - a^2}{2bc} = \frac{1156 + 625 - 1600}{1700} = .10647. \quad \therefore \text{A} = 83^\circ\ 53'\ 16''.$$

There is no ambiguity in this case, since the cosine is +, and hence the angle is $< 90°$.

The same angle is found by logarithmic computation, thus:

$a = 40$ . . . log = 1.602060	log $(\frac{1}{2}p - c)$ = 1.389166
$b = 34$ . . . log = 1.531479	log $(\frac{1}{2}p - b)$ = 1.190332
$c = 25$ . . . log = 1.397940	a. c. log b = 8.468521
$p = 99$ . . . log = 1.995635	a. c. log c = 8.602060
$\frac{1}{2}p = 49.5$ . . . log = 1.694605	$2\overline{)1.650079}$
$\frac{1}{2}p - a = 9.5$ . . . log = 0.977724	$\overline{1.825039}$
$\frac{1}{2}p - b = 15.5$ . . . log = 1.190332	
$\frac{1}{2}p - c = 24.5$ . . . log = 1.389166	log sin $\frac{1}{2}$A = 9.825039.

$\therefore \frac{1}{2}A = 41° 56' 38''$ and $A = 83° 53' 16''$.

In like manner the other angles may be found.

2. The sides of a plane triangle being 6, 5, and 4, find the angles.

The angles are 82° 49′ 09″, 55° 46′ 16″, and 41° 24′ 35″.

3. The sides of a plane triangle being 8601.5, 4082, and 7068, find the angles.

The angles are 54° 35′ 12″, 28° 4′ 44″, and 97° 20′ 4″.

4. The sides of a plane triangle being .5123864, .3538971, and .3090507, find the angles.

The angle opposite the last side is 36° 18′ 10″.2.

AREA OF PLANE TRIANGLES.

95. ***Prop.***—*The area of a plane triangle is equal to half the product of any two sides into the sine of the included angle.*

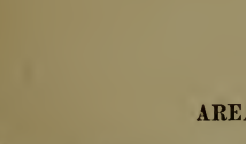


Fig. 21.

Dem.—Let ABC be any plane triangle, and b and c any two sides with A as the included angle. From the extremity of one of these two sides remote from A, let fall a perpendicular p, upon the other side Now,

$$\text{Area ACB} = \tfrac{1}{2}pc.$$

But, from ACD, $p = b \sin A$. $\therefore$ Area ACB $= \frac{1}{2}bc \sin A$. Q. E. D.

96. Cor.— *The area of a plane triangle is equal to the square root of the continued product of half its perimeter into half its perimeter minus each side separately.*

DEM.—From the proposition, and since $\sin A = 2\sin \frac{1}{2}A \cos \frac{1}{2}A$, we have,

$$\text{Area} = \tfrac{1}{2}bc \sin A = bc \sin \tfrac{1}{2}A \cos \tfrac{1}{2}A = bc \sqrt{\frac{(\frac{1}{2}p - c)(\frac{1}{2}p - b)}{bc}} \times \sqrt{\frac{\frac{1}{2}p(\frac{1}{2}p - a)}{bc}} =$$

$$\sqrt{\tfrac{1}{2}p(\tfrac{1}{2}p - a)(\tfrac{1}{2}p - b)(\tfrac{1}{2}p - c)}. \quad \text{Q. E. D.}$$

EXERCISES.

1. Given two sides of a plane triangle 125.81 and 57.65, and the included angle 57° 25′. Find the area.

Area = 3055.7.

2. Given the sides of a plane triangle 103.5 and 90, and the included angle 100°, to find the area.

Area = 4586.74.

3. How many square yards are there in a triangle whose sides are 30, 40, and 50 feet?

Area = $66\frac{2}{3}$.

4. Find the area of a triangle whose sides are 20, 30, and 40.

Area = 290.4737.

5. What is the area of a triangle whose sides are 30 and 40, and their included angle 28° 57′?

Area = 290.427.

6. What is the number of square yards in a triangle, of which the sides are 25 feet and 21.25 feet, and their included angle 45°?

Area = 20.8694.

7. Find the area of a triangle in which two of the angles are 80° and 60° respectively, and the included side 32 feet.

Area = 679.33 square feet.

8. Find the area of a triangular field having one of its sides 45 poles in length, and the two adjacent angles, respectively, 70° and 69° 40′.

Area = 1378.411 square poles.

9. Find the area of a triangular piece of ground having two angles respectively 73° 10′ and 90° 50′, and the side opposite the latter 75.3 poles.

Area = 748.03 square poles.

PRACTICAL APPLICATIONS.

[NOTE.—The following problems are inserted, not as any part of a treatise upon the subject of trigonometry as pure science, but as affording the student good mental exercise, and valuable and interesting information.]

1. To find the length (in miles) of a degree of longitude at Ann Arbor, Mich.

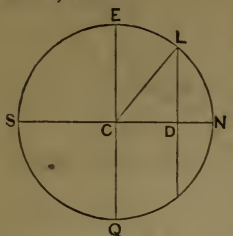


FIG. 22.

SOLUTION.—Let NESQ be a meridian section of the earth, EQ the equatorial diameter, and EL the latitude of Ann Arbor, 42° 16′ 48″.3. A degree of longitude at L is $\frac{1}{360}$ of the circumference of the circle whose radius is LD. CL the radius of the earth at this point = 3957.* Now in the right angled triangle LCD, we have CLD = ECL = 42° 16′ 48″.3, and CL = 3957; whence, LD = CL × cos 42° 16′ 48″.3, and LD = 2927.6. ∴ A degree = 51.1 miles. As a degree in longitude makes 4 minutes difference in time, 51.1 miles east or west on this parallel is equivalent to 4 minutes difference in time.

QUERY.—How does it appear from the above solution that the length of a degree of longitude varies as the cosine of the latitude?

2. To find the distance of a planet from the earth at any particular time.

SOLUTION.—To render the problem as simple as possible, we will suppose two observatories on the same meridian, at N, and N′; and that when the planet P is on the same meridian, the angles ZNP, and Z′N′P (the zenith distances) are measured. With these data and the radius of the earth, CN, CN′, known, the problem comes quite within the scope of the present study. The process is as follows: The arc NN′ being known, the angle NCN′ is known. Then in the triangle NCN′, two sides and the included angle are known, whence the other parts can be found.

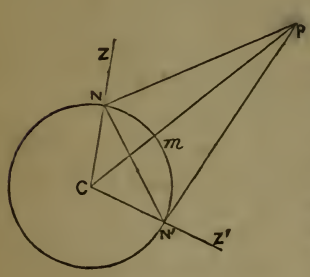


FIG. 23.

Now, knowing the angles PNC, PN′C, and CNN′, CN′N, we can find the angles PNN′, PN′N. This affords sufficient parts of the triangle PNN′ to determine the triangle, and we find PN, or PN′. Finally, in the triangle PNC, we know PN, NC, and the included angle; whence the other parts can be computed. But PC is the distance sought.

3. Suppose in case of the moon, the angles PNZ, and PN′Z′, being measured, are found to be respectively 44° 54′ 21″, and 48° 42′ 57″, the distance between the points of observation N and N′ is 92° 14′, and the radius of the earth is 3956.2 miles; find the distance to the moon.

Distance = 237,954.098 miles.

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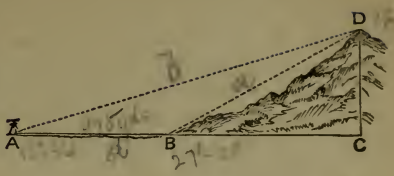
* The equatorial radius of the earth is 3962.8 miles; but in consequence of the flattening in the direction of the polar diameter it is less here.

4. Required the height of a hill D above a horizontal plane AB, the distance between A and B being equal to 975 yards, and the angles of elevation at A and B being respectively 15° 36′ and 27° 29′.



DC = 587.61 yards

5. Find the area of a regular hexagon, and also of a regular octagon, whose sides are each 10 feet.

Areas, 259.8, and 482.84 square feet

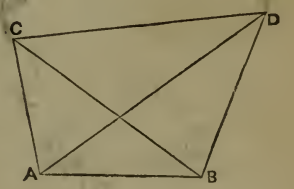
6. Find the area of a regular pentagon, and also of a regular decagon, whose sides are each 12 feet.

Areas, 247.74, and 1107.96 square feet.

7 Wishing to know the length of a certain pond of water, I measured a line 100 yards in length, and at each of its extremities observed the angles subtended by the other extremity and a couple of trees at the extremities of the pond. These angles were, at one end of the line, 32° and 98°, and at the other, 37° and 118°; what was the length of the pond?

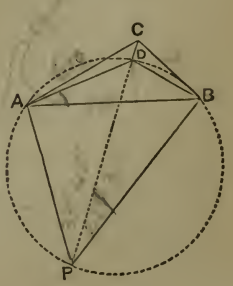
Draw the horizontal line AB equal to 100; make the angle BAD 32°, BAC 98°, ABC 37°, and ABD 118°. The intersections of the lines AC and BC, AD and BD, determine the extremities of the pond; the straight line CD is the length of the pond.

CD = 161.868 yards.



8. The distances AB, AC, and BC, between the points A, B, and C, are known; viz., AB = 800 yds., AC = 600 yds., and BC = 400 yds. From a fourth point P, the angles APC and BPC are measured; viz., APC = 33° 45′, and BPC = 22° 30′.

Required the distances AP, BP, and CP.



Distances, { AP = 710.193 yds. BP = 934.291 yds. CP = 1042.522 yds.

SUG'S.—Conceive the circumference passed through A, B, and P, and AD and DB drawn. In the triangle ADB, angle DAB = the given angle DPB, and DBA = APD. Hence, all the parts of triangle ADB can be found. Again, since the sides of the triangle ACB are given, its angles can be found. Then, since angle CAB − DAB = CAD, there are two sides and the included angle known in triangle ACD; whence angle ACD can be found. Thus we reach the triangle ACP, in which there are now known AC and the angles.

9. From the top of a mountain, three miles high, the angle of depression of a line tangent to the earth's surface is taken, and found to be 2° 13′ 27″. What is the diameter of the earth, considered as a sphere?

Ans. 7946.28 miles.

10. Taking the sun's mean apparent diameter as 32′ 3″.4, and his distance from the earth 91,430,000 miles, show that, if his centre were coincident with the earth's, his body would extend in all directions nearly 200,000 miles beyond the moon. (See Ex. 3.)

Sun's diameter = 852,574 miles.

11. Assuming the height of the Great Pyramid to be 486 feet, how far off may it be seen across the desert?

Ans., 27 miles.

12. What was the perpendicular height of a balloon, when its angles of elevation were 35° and 64°, as taken by two observers on the same level, at the same time, both on the same side of it, and in the same vertical plane; the distance between the two observers being 880 yards?

Ans., 935.757 yards.

13. Given two sides of a parallelogram 60 and 80, and a diagonal 100. Is this the longer or shorter diagonal? What is the other? What are the angles of the parallelogram?

14. A balloon being directly over one of two towns standing on the same horizontal plane, at a distance of eight miles from each other, the angle of depression to the more remote town was observed by the æronaut to be 10°. What was the height of the balloon?

Ans., 1.41 miles.

15. The most recent observations make the sun's horizontal parallax 8″.94, and the earth's equatorial radius 3962.8 miles. Show that the distance of the sun from the earth is nearly as given in Ex. 10, instead of 95 millions of miles, as it has been heretofore considered.

CHAPTER II.

SPHERICAL TRIGONOMETRY.

INTRODUCTION.

PROJECTION OF SPHERICAL TRIANGLES.

97. To Project a Spherical Triangle on a plane surface is to draw the triangle on that surface so that it will present the same appearance to the eye, situated at a particular point, as when drawn on the surface of a sphere.

98. The Simplest Method of projecting a spherical triangle is to project it *on the plane of one of its sides*, the eye being supposed situated in the axis of the sphere perpendicular to this plane, and at an infinite distance from it. The plane is called the *Plane of Projection;* and its intersection with the sphere is called the *Primitive Circle*, and is the base of the hemisphere on which the triangle is conceived as situated.

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99. Fundamental Propositions.—1st. *When the parts of a spherical triangle are each conceived as less than* 180°, *any such triangle can be represented on a hemisphere.*

2d. *The primitive Circle has its axis, and consequently its pole, projected at its centre.*

3d. *The semi-circumference of any circle of the sphere, perpendicular to the Primitive Circle, is projected in the chord representing the intersection of the circles; and, if the perpendicular circle be a great circle, its semi-circumference is projected in a diameter of the Primitive Circle.*

ILL'S.—These propositions are direct consequences of the fundamental conception. Thus, let ABA′B′ represent the base of the hemisphere on which the triangle is conceived as situated. This is the Primitive Circle, and the eye is supposed situated at an infinite distance, and in a line perpendicular to the plane of the paper at P. The pole of the Primitive Circle being in this line is projected (seen as) at P. As all great circles perpendicular to the Primitive Circle pass through its pole and include its axis, the eye is in all such planes, and any lines of these planes, as the semi-circumferences of the great circles in which they intersect the sphere, are projected (appear to the eye) as diameters of the Primitive Circle. Moreover, since the eye is *at infinity*, it is to be conceived as in the plane of any small circle which is perpendicular to the primitive, and which is therefore projected in a chord, as CC′.

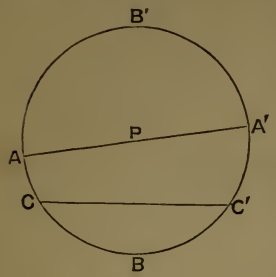

FIG. 24.

PROJECTION OF RIGHT ANGLED SPHERICAL TRIANGLES.

100. Prob. 1.—*To project a right angled spherical triangle on the plane of one of its sides, when the two sides about the right angle are given.**

SOLUTION.—Let the angles of the triangle be represented by A, B, and C, A being the right angle. Let the sides opposite these angles respectively be represented by a, b, and c; whence b and c are the given sides. Draw the primitive circle and the diameters BB′, NN′ at right angles to each other. From B lay off BA = c.† Let the right angle be at A; whence the side b is perpendicular to the primitive circle, and projected in the diameter AA′. To project the vertex C, conceive the semi-circumference, of which AA′ is the projection, to revolve on AA′ until it falls upon the semi-circumference A′BA, then will the point C fall at d. Hence make Ad = b. In like manner revolving the semi-circumference, of which AA′ is the projection, until it falls upon A′B′A, the point C will fall at d'. Hence make Ad' = b. The point C will describe the semi-circumference of a small circle perpendicular to the primitive circle, and whose projection is dd'. Now, as the

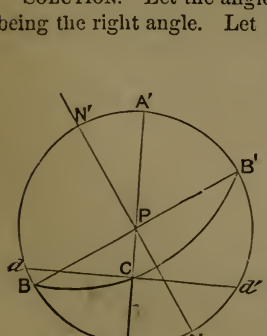

FIG. 25.

* In this treatise the discussions embrace only such triangles as have each part less than 180°.

† For the purposes for which we shall use these projections, an arc can be laid off with sufficient accuracy by observing its relation to 90°, 60°, 30°, or some aliquot part of the circumference, which is readily obtained on geometrical principles.

projection of the vertex of the triangle C is at the same time in AA′ and dd', it must be at their intersection. Finally, the hypotenuse a is projected in a curve passing through B, C, and B′, since two great circles intersect at the extremities of a diameter. This curve is really an ellipse, but for our present purpose it may be considered as the arc of a circle passing through B, C, and B′. Therefore, BAC is the projection required.

QUERIES.—Will a solution of this problem be possible for all values of b and c? How does it appear from the projection?

EXAMPLES.

1. Having given $b = 110°$, and $c = 60°$, to project the triangle. See *Fig.* 26.

2. Having given $b = 50°$, and $c = 130°$, to project the triangle.

3. Having given $b = 90°$, and $c = 30°$, to project the triangle.

4. Having given $b = 90°$, and $c = 90°$, to project the triangle.

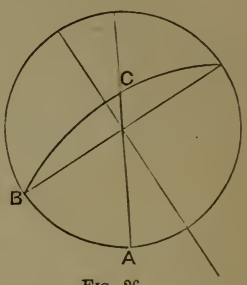

FIG. 26.

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101. Prob. 2.—*To project a right angled spherical triangle when the hypotenuse and one side are given.*

SOLUTION.—Using the common notation, let c represent the *known* side. Drawing the primitive circle and the conjugate* diameters BB′, NN′, lay off BA = c, and draw the diameter AA′. The projection of b will lie in AA′, and the projection of the vertex C will fall somewhere in this line. Now the arc a lies in a semi-circumference passing through B and B′. Conceive this semi-circumference to revolve on BB′, as an axis, till it coincides, first, with BNB′, and then with BN′B′. The point C will trace the semi-circumference of a small circle perpendicular to the primitive circle, and whose projection is dd'. Hence, making Bd = Bd′ = a, and drawing dd', the projection of the vertex C lies at the same time in AA′ and dd', and is therefore at their intersection. Passing an arc of a circle (strictly an ellipse) through BCB′, we have ABC, the projection desired.

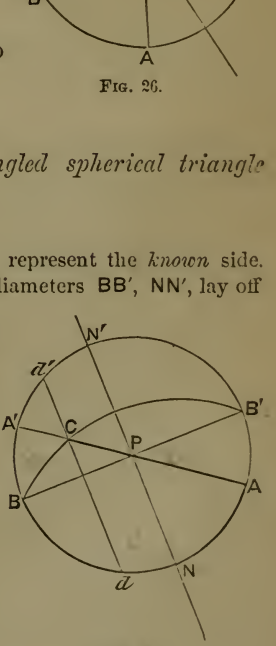

FIG. 27.

* Two diameters of a circle which are at right angles to each other are called *Conjugate Diameters.*

QUERIES.—Will a solution of this problem be possible for all values of a and c? If a had been greater than c in the above case, would the solution have been possible? Will dd' and AA′ always intersect, whatever may be the relative values of a and c?

EXAMPLES.

1. Having given $c = 75°$, and $a = 64°$, to project the triangle.

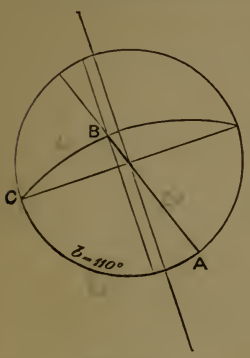

FIG. 28.

2. Having given $c = 45°$, and $a = 136°$, to project the triangle.

3. Having given $b = 110°$, and $a = 85°$, to project the triangle. See *Fig.* 28.

4. Having given $b = 110°$, and $a = 120°$, to project the triangle.

5. Having given $c = 90°$, and $a = 75°$, or $a = 120°$, to project the triangle.

QUERY.—If one side of a right angled spherical triangle is 90°, what must the hypotenuse be? Why?

---

102. Prob. 3.—*To project a right angled spherical triangle when an oblique angle and the hypotenuse, or the oblique angle and the adjacent side are given.*

SOLUTION.—Draw the primitive circle and the conjugate diameters BB′, NN′ as usual. To construct the given angle B, we observe that this angle is measured by an arc of the great circle which is projected in NN′. Hence, lay off $Nd = Nd' = B$, and draw dd'; then is NO the projection of the arc which measures B, and the projection of a lies in the arc passing through BOB′.* Having the angle B projected, if the hypotenuse a is the other given part, find the projection of C by taking $Be = Be' = a$, and drawing ee'. Complete the projection by drawing AC through P. When the adjacent side c is given, take $BA = c$, and draw AC as before. [The student will be able to give the reasons.]

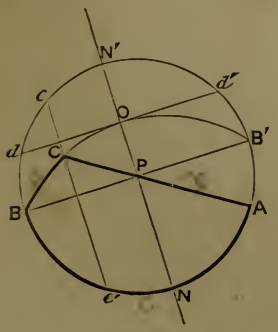

FIG. 29.

103. SCH.—As OP is the cosine of the angle ABC, the point O may be found by measuring

* As has been remarked, this arc is really a semi-ellipse. This fact, together with the method of constructing the semi-ellipse, and thus getting the correct projection of the hypot

from P (towards N if B < 90°, and towards N′ if B > 90°) a distance equal to the natural cosine of B.

QUERY.—Is the solution of this problem possible for all values of the hypotenuse or adjacent side, and the angle?

EXAMPLES.

1. Having given B = 65°, and the hypotenuse $a = 120°$, to project the triangle. See *Fig.* 30.

2. Having given C = 45°, and the adjacent side $b = 50°$, to project the triangle.

SUG'S.—Project the angle C as the angle B of the preceding, and lay off $b = 50°$ from its vertex on the circumference of the primitive circle.

3. Having given C = 170°, and hypotenuse $a = 160°$, to project the triangle.

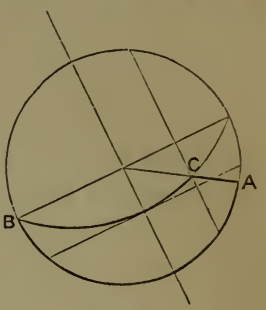

FIG. 30.

4. Having given B = 150°, and $c = 40°$, to project the triangle.

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104. Prob. 4.—*To project a right angled spherical triangle when an oblique angle and side opposite are given.*

SOLUTION.—Project the given angle at B, *Figs.* 31, 32, as in the last problem. Then, from any point in the circumference of the primitive circle, as N, take NO′, in the diameter passing through that point, equal to the projection of the given side. (This is done by taking $Nd = Nd' = b$, as in *Prob.* 1, and drawing dd'). Now, with P as a centre, and radius PO′, describe a circumference cutting BOB′. One extremity of the given side b will be projected in this circumference, since this circumference contains the projections of all the points in the surface of the hemisphere which are at a distance b from the circumference BNB′N′. But the vertex C is also projected in the

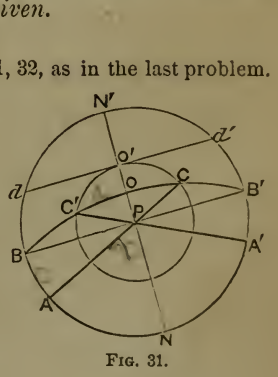

FIG. 31.

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cnuse, belong to a treatise on Conic Sections. In this case, BB′ and 2OP are the axes of the ellipse, and the curve can be constructed by taking BP as a radius, and striking arcs from O as a centre, cutting BB′. These intersections are the foci of the required ellipse. Then take a string equal in length to BB′, and, fastening its ends to the foci, place a pencil against the string, and keeping the string tight, carry the pencil around the curve.

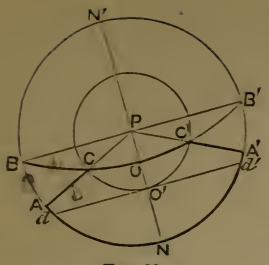

Fig. 32.

arc BOB′; hence, it must be at the intersections C, C′. Drawing the projections of the perpendiculars CA, and C′A′, the projection is complete.

Queries.—When will there be two triangles fulfilling the given conditions? When but one? When none? When there is but one triangle what kind of a triangle is it? If $B > 90°$, must b be greater or less than B in order to have two solutions? If $B < 90°$, how is it? If $B > 90°$, can b be less? If $B < 90°$, can $b > 90°$?

EXAMPLES.

1. Given $C = 120°$, $c = 150$, to project the triangle. See *Fig.* 33.
2. Given $C = 80°$, $c = 60°$, to project the triangle. See *Fig.* 34.

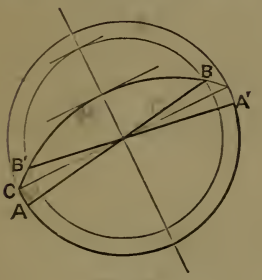

Fig. 33.

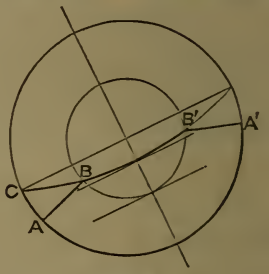

Fig. 34.

3. Given $B = 70°$, $b = 70°$, to project the triangle. See *Fig.* 35.
4. Given $B = 64°$, $b = 75°$, to project the triangle. See *Fig.* 36.

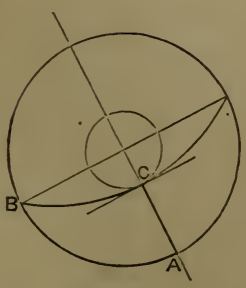

Fig. 35.

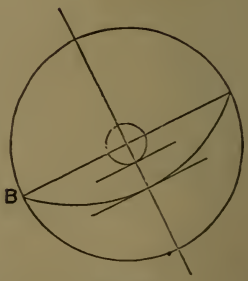

Fig. 36.

5. Given $B = 160°$, $b = 110°$, to project the triangle.

105. Scii.—When the given parts are the two oblique angles, the projection is most readily effected by first computing one of the sides. The projection in this case will be considered in connection with the numerical solution of the case, in the next section.

PROJECTION OF OBLIQUE ANGLED SPHERICAL TRIANGLES.

106. Prob. 1.—*To project a spherical triangle when two sides and the included angle are given.*

Solution.—Let a, c, and B denote the given parts. Project the given angle B at some point in the circumference of the primitive circle, as B. Lay off one of the given sides, as c, from B on BNB′. Let BA = c. Determine the extremity C of the projection of the other given side a, as in *Prob.* 2, etc., and drawing the diameter AA′, pass the arc through ACA′; BCA is the projection sought.

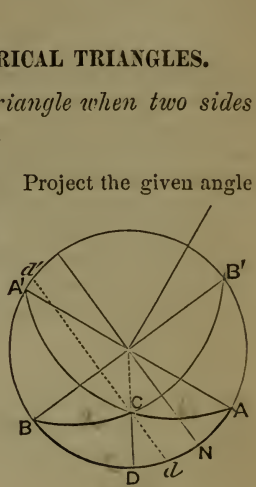

Fig. 37.

Query.—Is this projection always possible, whatever the relative magnitude of the given parts?

EXAMPLES.

1. Given A = 130°, c = 85°, and b = 100°, to project the triangle.
2. Given C = 40°, a = 37°, and b = 80°, to project the triangle.

---

107. Prob. 2.—*To project a spherical triangle when two sides and an angle opposite one of them are given.*

Solution.—Let the given parts be a, b, and B. Make the projection upon the plane of the *unknown side* c. Thus, drawing the primitive circle and the conjugate diameters BB′, NN′, conceive c as projected from B on the arc BNB′, and project the given angle B as in preceding problems. On the arc BB′, take BC = the projection of the given *adjacent* side a. To determine the projection of the opposite side b, describe a circle about P, as a centre, with a radius PC. Through C draw PD, and taking Dd = Dd′ = b draw dd′. Through the intersections o, o′ of dd′ with the circumference of the small circle, draw the radii PA, PA′. Finally, passing arcs through the points A, C, A″, and A′, C, A‴, BAC, and BA′C are the projections of triangles which fulfill the given conditions. The projections of B and

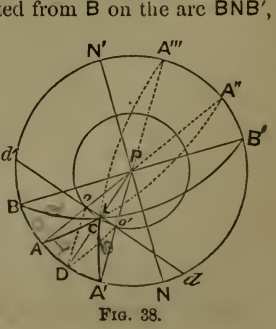

Fig. 38.

a were made upon principles previously established; and it only remains to show that AC, and A′C are projections of b. Since by construction DL is the projection of an arc equal to b, the projections of arcs of great circles connecting D with o and o' are projections of arcs equal to b. But the figure $DoP = ACP$, and $Do'P = A'CP$; therefore $AC = A'C = Do$ = the projection of b.

108. Sch.—It is evident that this problem has *one* solution, *two* solutions, or *no* solution, according to the value of b as compared with a and B. Thus, if the projection of b = DC, o and o' coincide, there is but *one* solution, and the triangle is right angled at A (which in this case falls at D). Also, if the projection of b is intermediate in value between DC and *only one* of the arcs BC, B′C, there is only *one* solution. If, however, as in the figure given, the projection of b is intermediate in value between DC and *both* BC and B′C, there are *two* solutions. Finally, if b is given of such value that the chord dd' does not touch the small circle, there is *no* solution. The latter case occurs when $B < 90°$, if the projection of $b <$ DC; and when $B > 90°$, if the projection of $b >$ DC, as will appear from *Figs.* 38, 39.

We may observe, also, that there are *two* solutions when o and o' both fall on the same side of BB′ as c; *one* solution when o and o' coincide, and when they fall on opposite sides of BB′; and *no* solution when o and o' are imaginary, *i. e.*, when dd' does not touch the small circle, or when both fall on the opposite side of BB′ from c.

EXAMPLES.

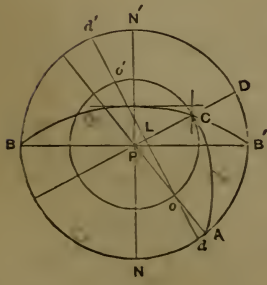

Fig. 39.

1. Given $B = 110°$, $a = 120°$, and $b = 83°$, to project the triangle. See *Fig.* 39.

2. Given $B = 110°$, $a = 120°$, and $b = 130°$, to project the triangle.

3. Given $C = 64°$, $a = 120°$, and $c = 75°$, to project the triangle.

4. Given $C = 80°$, $b = 60°$, and $c = 40°$, to project the triangle.

5. Given $C = 112°$, $b = 75°$, and $c = 150°$, to project the triangle.

109. Prob. 3.—*To project a spherical triangle when the three sides are given.*

SOLUTION.—Drawing the primitive circle and the conjugate diameters, as usual, take BA $= c$, the side on the plane of which it is proposed to project the triangle. Take Be $=$ Be' $= a$ and draw ee'; then as before shown, the projection of the vertex C lies in ee'. In like manner taking Ad $=$ Ad' $= b$, and drawing dd' the projection of C lies in dd'. The intersection of the chords ee' and dd' is therefore the projection of the vertex C. Finally, passing arcs through ACA′ and BCB′, the projection is complete.

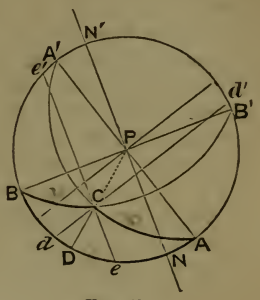

FIG. 40.

QUERIES.—How does the projection show the impossibility when the sum of the three sides is greater than 360°? How when the sum of two sides is less than the third side?

EXAMPLES.

1. Given $a = 100°$, $b = 80°$, and $c = 68°$, to project the triangle.

2. Given $a = 108°$, $b = 120°$, and $c = 25°$, to project the triangle. See *Fig.* 41.

3. Given $a = 120°$, $b = 65°$, and $c = 40°$, to project the triangle.

4. Given $a = 150°$, $b = 140°$, and $c = 170°$, to project the triangle.

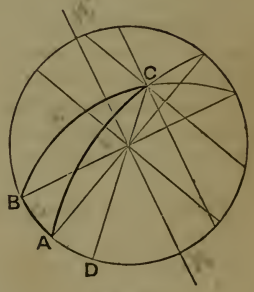

FIG. 41.

SECTION I.

SOLUTION OF RIGHT ANGLED SPHERICAL TRIANGLES.

110. Spherical Trigonometry treats of the relations between the trigonometrical functions of the sides and angles of spherical triangles, and of the solution of such triangles by means of these relations.

111. SCH.—In the present treatise we shall confine our attention to triangles none of whose sides or angles exceed 180°.

112. The Six Parts of a spherical triangle are the three sides and the three angles: any three of these being given, the others may be found.

113. Sch.—The last statement involves the assertion that the three angles of a spherical triangle determine the sides, whereas we are accustomed to say in *Plane* Trigonometry, that, at least *one* given part of a plane triangle must be a *side*, in order that the triangle may be determined. There is really no such difference as these two statements imply. For example, if we have the angles of a plane triangle given, we know the ratios of the sides to each other, since the sides are to each other as the sines of the angles opposite; but we cannot determine the *absolute values* of the sides. This is in accordance with the statement that mutually equiangular plane triangles are *similar* figures (not necessarily *equal*). Now these are exactly the facts in the case of spherical triangles, *if we do not limit them to the same or equal spheres.* Thus, the angles of a spherical triangle being given as 48° 30′, 125° 20′, and 62° 54′, we solve the triangle by the rules of spherical trigonometry and find that the sides are 56° 39′ 30″, 114° 29′ 58″, and 83° 12′ 6″. But, so long as the radius of the sphere is unknown, these results are merely the *relative* values of the sides, not their absolute lengths. Moreover, consider two concentric spheres whose radii are m and n. Now, any triangle being constructed on the one, if planes are passed through its sides intersecting at the common centre, their intersection with the surface of the other sphere will form a triangle mutually equiangular with the first, and any one side of the one triangle is to the corresponding side of the other, as the radii of the spheres; hence the homologous sides are proportional. We see, therefore, that to determine absolutely a spherical triangle, it is necessary to know one of the sides in linear extent as well as angular measure, or, what is equivalent, the radius of the sphere must be known.

114. The Species of a part of a spherical triangle is determined by its relation to 90°. Two parts, both greater or both less than 90°, are of the *same species;* two parts, one of which is greater and the other less than 90°, are of *different species.* Thus, two parts which are 58°, and 63°, respectively, are of the same species; two which are 160°, and 115°, are of the same species; two which are 120°, and 48°, are of different species.

115. Napier's Five Circular Parts are the two sides of a *right angled* spherical triangle about the right angle, and the *complements* of the hypotenuse and of the oblique angles. These terms are merely conventional, and are applied exclusively to right angled triangles.

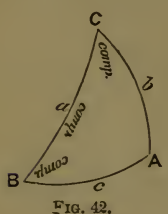

Fig. 42.

Ill.—In a spherical triangle ABC, right angled at A, the sides b and c, the complement of hypotenuse a, and the complements of the angles B and C, are the circular parts. We may designate them b, c, comp a, comp B, and comp C.

[It is essential that this nomenclature and the statements of the two following paragraphs be clearly understood, and firmly fixed in the mind, in order that the phraseology of the fundamental rules may be intelligible.]

116. When five things occur in succession, as it were in a circle, like the circular parts *b*, *c*, comp B, comp *a*, and comp C, in the preceding figure (no account being made of A), it will be observed, that, taking any three at pleasure, one of the three may always be selected which lies adjacent to each of the others, or separated from each of them. Of the three parts thus considered, the ***Middle Part*** is the one which has the other two adjacent to or separated from it; while the latter are called the ***Extremes,*** *adjacent* or *opposite,* as the case may be.

Ill's.—Let the three parts under consideration be comp *a*, *b*, and *c;* comp *a* is the middle part, and *b* and *c* are the *opposite* extremes. If *b*, *c*, and comp C are under consideration, *b* is the middle part, and *c* and comp C are the *adjacent* extremes.

117. Sch.—In solving a right angled spherical triangle, there are always three parts under consideration at once, viz., the two given parts and the part sought, no mention being made of the right angle. Now, the first question to be settled in order to a solution is, *Which of the three parts under consideration is the middle part, and are the extremes opposite or adjacent?* Beginners are very liable to make mistakes by failing to use the *complements* of the proper parts; or by not correctly distinguishing the middle part, and the character of the extremes, as opposite or adjacent. The student should practise upon such simple exercises as the following until the questions can be answered *instantly*, and *without mistake.*

EXERCISES.

1. In *Fig.* 42, given *a* and *c*, to find C. What are the circular parts under consideration? Which is the middle part? Are the extremes adjacent or opposite?

Answers.—The circular parts are comp *a*, *c*, and comp C. *c* is the middle part, and the extremes are opposite.

2. Having C, and *a* given, to find *b*. What are the circular parts? Which is the middle part? Are the extremes adjacent or opposite?

3. Having *c*, and *b* given, to find B. What are the circular parts? Which is the middle part? Are the extremes adjacent or opposite?

4. Having *a*, and *b* given, to find C. What are the circular parts? Which is the middle part? Are the extremes adjacent or opposite?

5. Having B and C given, to find *b*. What are the circular parts? Which is the middle part? Are the extremes adjacent or opposite?

6. What are the opposite extremes when b is the middle part? What the adjacent extremes? Which are the opposite and which the adjacent extremes when c is the middle part? When comp B is the middle part? When comp C is the middle part? When comp a is the middle part?

7. What part is middle part to comp C and c as adjacent extremes? As opposite extremes?

Ans., b, and *none*.

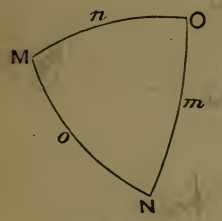

FIG. 43.

8. In *Fig.* 43, M being the right angle, what are the circular parts? Given O and m, to find o. What are the circular parts under consideration? Are the extremes adjacent or opposite?

9. What are the opposite extremes to comp O? What the adjacent? To comp m? To o? To n?

NAPIER'S RULES.

118. Rule I. Prop.—*In any right angled spherical triangle, the sine of the middle part equals the product of the cosines of the opposite extremes.*

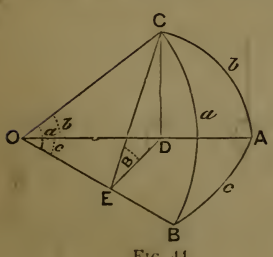

FIG. 44.

DEM.—In the spherical triangle BAC, right angled at A, taking b, c, comp B, comp C, and comp a in succession as middle parts, we are to prove that

$$\sin b = \cos(\text{comp } a) \times \cos(\text{comp B}), \quad \text{or} \quad \sin b = \sin a \sin B; \qquad (1)$$

$$\sin c = \cos(\text{comp } a) \times \cos(\text{comp C}), \quad \text{or} \quad \sin c = \sin a \sin C; \qquad (2)$$

$$\sin(\text{comp B}) = \cos b \times \cos(\text{comp C}), \quad \text{or} \quad \cos B = \cos b \sin C; \qquad (3)$$

$$\sin(\text{comp C}) = \cos c \times \cos(\text{comp B}), \quad \text{or} \quad \cos C = \cos c \sin B \qquad (4)$$

$$\sin(\text{comp } a) = \cos b \times \cos c, \quad \text{or} \quad \cos a = \cos b \cos c. \qquad (5)$$

To demonstrate these relations, let O be the centre of the sphere, and draw the radii OA, OB, and OC. The angles BOC, AOC, and AOB, are measured respectively by a, b, and c, the sides of the triangle; hence these angles at the centre and their measuring arcs may be used interchangeably. From one of

the oblique angles, as C, let fall a perpendicular upon the radius OA. From the foot of this perpendicular draw DE perpendicular to OB, and join C and E. Now CDE is a right angle (PART II., *426*), CE is perpendicular to OB (PART II., *399*), and DEC is equal to angle B of the triangle (PART II., *558*).

$$CD = \sin b,\ OD = \cos b,\ CE = \sin a, \text{ and } OE = \cos a.$$

From the triangle CDE, right angled at D, we have

$$CD = CE \times \sin CED, \quad \text{or} \quad \sin b = \sin a \sin B. \tag{1}$$

Generalized, (1) becomes, *The sine of either side about the right angle = the sine of the hypotenuse into the sine of the angle opposite the side.* Hence, from analogy to (1), we may write

$$\sin c = \sin a \sin C. \tag{2}$$

Or (2) may be proved in the same manner as (1), by letting fall from B a perpendicular upon OA, from its foot drawing DE perpendicular to OC, and joining E and B. Then $BD = \sin c$, $OD = \cos c$, $BE = \sin a$, $OE = \cos a$, and angle BED = angle C. From the triangle BDE, we have

$$BD = BE \times \sin BED, \text{ or } \sin c = \sin a \sin C. \tag{2}$$

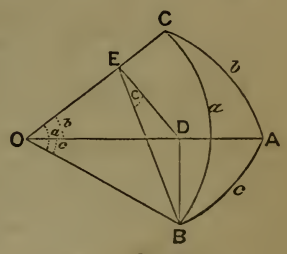

FIG. 45

To prove (3), we have from the triangle CDE, Fig. 44,

$$\cos CED = \frac{ED}{CE}, \quad \text{or} \quad \cos B = \frac{ED}{\sin a}.$$

But from triangle OED, right angled at E, $ED = OD \times \sin DOE = \cos b \sin c =$ [from (2)], $\cos b \sin a \sin C$. Substituting this value of ED, we have

$$\cos B = \frac{\cos b \sin a \sin C}{\sin a} = \cos b \sin C. \tag{3}$$

We may write (4) from (3) by analogy, as (2) was from (1); or, better, let the student produce it from Fig. 45, as (3) was produced from Fig. 44.

Finally, to produce (5), consider the triangle ODE, in either figure, right angled at E. This gives

$$OE = OD \times \cos DOE, \quad \text{or} \quad \cos a = \cos b \cos c. \tag{5}$$

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119. Rule 2. Prop.—*In any right angled spherical triangle, the sine of the middle part equals the product of the tangents of the adjacent extremes.*

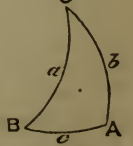

FIG. 46.

DEM.—In the spherical triangle BAC, right angled at A, taking *b*, *c*, comp B, comp C, and comp *a*, in succession as middle parts, we are to prove that,

$$\sin b = \tan c \times \tan(\text{comp } \mathsf{C}), \text{ or } \sin b = \tan c \cot \mathsf{C}; \quad (1)$$

$$\sin c = \tan b \times \tan(\text{comp } \mathsf{B}), \text{ or } \sin c = \tan b \cot \mathsf{B}; \quad (2)$$

$$\sin(\text{comp } \mathsf{B}) = \tan c \times \tan(\text{comp } a), \text{ or } \cos \mathsf{B} = \cot a \tan c; \quad (3)$$

$$\sin(\text{comp } \mathsf{C}) = \tan b \times \tan(\text{comp } a), \text{ or } \cos \mathsf{C} = \cot a \tan b; \quad (4)$$

$$\sin(\text{comp } a) = \tan(\text{comp } \mathsf{B}) \times \tan(\text{comp } \mathsf{C}), \text{ or } \cos a = \cot \mathsf{B} \cot \mathsf{C}. \quad (5)$$

Taking the *formulæ* of RULE 1st, and in the second member of each substituting the value of each factor as found in some other of the set, we readily write the following

$$\sin b = \sin a \sin \mathsf{B} = \frac{\sin c \cos \mathsf{C}}{\sin \mathsf{C} \cos c} = \frac{\sin c}{\cos c} \times \frac{\cos \mathsf{C}}{\sin \mathsf{C}} = \tan c \cot \mathsf{C}; \quad (1)$$

$$\sin c = \sin a \sin \mathsf{C} = \frac{\sin b \cos \mathsf{B}}{\sin \mathsf{B} \cos b} = \frac{\sin b}{\cos b} \times \frac{\cos \mathsf{B}}{\sin \mathsf{B}} = \tan b \cot \mathsf{B}; \quad (2)$$

$$\cos \mathsf{B} = \cos b \sin \mathsf{C} = \frac{\cos a \sin c}{\cos c \sin a} = \frac{\cos a}{\sin a} \times \frac{\sin c}{\cos c} = \cot a \tan c; \quad (3)$$

$$\cos \mathsf{C} = \cos c \sin \mathsf{B} = \frac{\cos a \sin b}{\cos b \sin a} = \frac{\cos a}{\sin a} \times \frac{\sin b}{\cos b} = \cot a \tan b; \quad (4)$$

$$\cos a = \cos b \cos c = \frac{\cos \mathsf{B} \cos \mathsf{C}}{\sin \mathsf{C} \sin \mathsf{B}} = \frac{\cos \mathsf{B}}{\sin \mathsf{B}} \times \frac{\cos \mathsf{C}}{\sin \mathsf{C}} = \cot \mathsf{B} \cot \mathsf{C}. \quad (5)$$

Q. E. D.

120. SCH. 1.—It will be a good exercise for the student to demonstrate RULE 2d from the annexed figures, as RULE 1st was from *Figs.* 44 and 45. The 5th is not as readily obtained from the figure as the others. The student may trace the following relations, some in one figure, and some in the other.

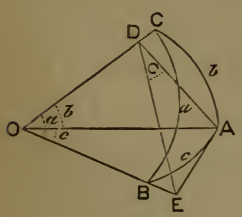

FIG. 47.

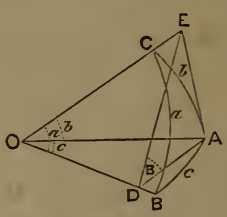

FIG. 48.

$\text{Cos } a = \frac{\text{OD}}{\text{OE}} = \frac{\cos b}{\sec c} = \cos b \cos c$. But, $\cot \mathsf{C} = \frac{\text{AD}}{\text{AE}} = \frac{\sin b}{\tan c}$, and $\cot \mathsf{B} = \frac{\text{AD}}{\text{AE}} = \frac{\sin c}{\tan b}$; whence, $\cot \mathsf{B} \cot \mathsf{C} = \frac{\sin b \sin c}{\tan b \tan c} = \cos b \cos c$. $\therefore \cos a = \cot \mathsf{B} \cot \mathsf{C}$.

121. SCH. 2.—It is of much importance, especially for the purposes of Spherical Astronomy, that the student observe that the relations expressed in the above formulæ, and in fact all the relations between the sides and angles of spherical triangles, are also the relations between the facial and diedral angles of triedrals. Thus, if a, b, and c represent the facial angles, and A, B, and C the opposite diedrals, all these relations can be established, and in exactly the same manner as above, *without any allusion to the spherical triangle.* [The student should do it.]

DETERMINATION OF SPECIES.

122. In the solution of spherical triangles the determination of the species of a part sought becomes of essential importance, since *any* part of such a triangle may have any value between 0° and 180°. Hence, when we have learned the numerical value of any function of a part, we have yet to determine whether the part itself is less or greater than 90°, *i. e.*, the species of the part. This may always be effected by some one of the following propositions.

---

123. Prop.—*If the part sought is found in terms of its cos, tan, or cot, its species can be determined by the algebraic signs of the functions in the formula used.*

DEM.—In each of the *formulæ* arising from the application of Napier's rules, there are three functions, the arcs corresponding to two of which are always known, hence the algebraic signs of their functions are known, and the signs of these two determine the sign of the third or unknown function. Now, when a cos, tan, or cot is +, the corresponding angle is less than 90° (if less than 180°); and when one of these functions is −, the corresponding angle is greater than 90°; *i. e.*, in a triangle, it is between 90° and 180°.

---

124. When the part sought is found in terms of its sine, the species cannot be determined by the signs of the formula, since the part being less than 180° its sine is always +. The three following propositions dispose of such cases.

---

125. Prop.—*An oblique angle of a right angled spherical triangle and its opposite side are always of the same species.*

DEM.—From Napier's first rule we have, $\cos B = \cos b \sin C$. But $\sin C$ is necessarily +; therefore, $\cos B$ and $\cos b$ always have the same sign, and B and b are of the same species. In like manner, we see from $\cos C = \cos c \sin B$, that C and c are of the same species. Q. E. D.

---

126. Prop.—*When the hypotenuse of a right angled spherical triangle is less than* 90°, *the other two sides (and consequently the*

oblique angles) are of the same species with each other. But when the hypotenuse is greater than 90°, the other two sides (and consequently the oblique angles) are of different species from each other.

DEM.—From Napier's first rule we have, $\cos a = \cos b \cos c$. Now, if $a < 90°$, $\cos a$ is $+$; hence $\cos b$ and $\cos c$ must have *like* signs, and b and c be both less or both greater than 90° But if $a > 90°$ (and less than 180°, as it is), $\cos a$ is $-$; hence, $\cos b$ and $\cos c$ must have *different* signs, and b and c be one greater and the other less than 90°. Finally, since the oblique angles are of the same species as their opposite sides, they are of the same species with each other when $a < 90°$, and of different species from each other when $a > 90°$.

---

127. Prop.—*When a side and its opposite angle are given in a right angled spherical triangle, there is* NO *solution if the sine of the side is greater than the sine of its opposite angle; there is* ONE *solution and the triangle is bi-rectangular if the sine of the side equals the sine of its opposite angle; and there are* TWO *solutions if the sine of the side is less than the sine of its opposite angle.*

DEM.—We have $\sin b = \sin a \sin \mathrm{B}$, or $\sin a = \dfrac{\sin b}{\sin \mathrm{B}}$. Now, $\sin b > \sin \mathrm{B}$ makes $\sin a > 1$, which is manifestly impossible. $\sin b = \sin \mathrm{B}$ makes $b = \mathrm{B}$, since they are of the same species. But when the arc included by the sides of an oblique angle of a right angled spherical triangle is equal to the angle, the vertex of the angle is the pole of the arc. Hence, in this case the other sides of the triangle are each 90°. Finally, if $\sin b < \sin \mathrm{B}$, a has two values, one greater and the other less than 90°. Hence there are two triangles.

128. SCH.—These relations between an angle and its opposite side may be observed directly from a figure. When $\mathrm{B} < 90°$, the measure of it, that is $b = \mathrm{B}$, is the greatest included perpendicular which can be drawn to one side of the lune BAB′C. Hence, in this case, b cannot exceed B, which implies that $\sin b$ cannot be $> \sin \mathrm{B}$, as when the arcs are less than 90°, the greater arc has the greater sine. If $\sin b = \sin \mathrm{B}$, $b = \mathrm{B}$, and $\mathrm{BA} = \mathrm{BC} = 90°$, and the side b can occupy but one position in the lune, thus giving rise to but one triangle BAC, which satisfies the conditions (or two equal triangles BAC and B′AC). If $b < \mathrm{B}$, which, when B is less than 90°, implies that $\sin b < \sin \mathrm{B}$, the side can occupy two positions in the lune, b' and b'' giving rise to two triangles, BA′C′ and BA″C″, both of which fulfill the conditions.

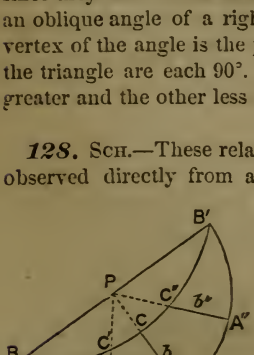

FIG. 49.

Again, when $B > 90°$, the measure of it, *i. e.*, $b = B$, is the *least* included perpendicular that can be drawn to one side of the lune. Hence, in this case we cannot have $b < B$, which implies that $\sin b$ cannot be $> \sin B$, since the greater arc has the less sine. If $\sin b = \sin B$, $b = B$, and $BA = BC = 90°$, and the side b can occupy but one position in the lune, thus giving rise to but one triangle BAC, which satisfies the conditions (or two equal triangles BAC and B'AC). If $b > B$, which implies that $\sin b < \sin B$, the side b can occupy two positions in the lune, as b' and b'', thus giving rise to two triangles BA'C', and BA''C'', both of which satisfy the conditions.

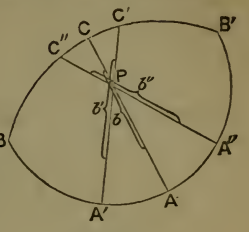

FIG. 50.

EXERCISES.

1. In a right angled spherical triangle BAC, A being the right angle, $B = 80° 40'$, and $a = 105° 34'$, to project the triangle and compute the other parts.

PROJECTION.*—Projecting the triangle upon the plane of the side c (***102***), we have, BCA, *Fig.* 51. [The student should give the process.]

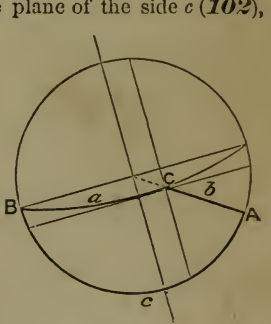

FIG. 51.

SOLUTION.—It is immaterial which of the required parts we seek first. We will seek c. Now the three circular parts under consideration are c, comp a, and comp B. Comp B is middle part, and the extremes are adjacent; hence, by Napier's second rule we have,

$$\cos B = \tan c \cot a.$$

or
$$\tan c = \frac{\cos B}{\cot a} = \frac{\cos 80° 40'}{\cot 105° 34'}.$$

Now $\cos 80° 40'$ is $+$, and $\cot 105° 34'$ is $-$. Therefore $\tan c$ is $-$, and $c > 90°$.

Computing by logarithms,

$\log \cos 80° 40' =$	9.209992
$- \log \cot 105° 34' =$	9.444947
$= \log \tan c =$	$\bar{1}.765045$
Add 10 for tab. tan	10.
	9.765045. $\therefore c = 149° 47' 37''$.

---

* It is recommended that the projection be given *always* before the trigonometrical solution. It is an excellent exercise, and gives clearness of perception.

To find b, $\sin b = \sin a \sin \mathsf{B} = \sin 105° 34' \times \sin 80° 40'$. This makes b known by means of its sine, whence the signs of the formula do not determine the species of b. But b is of the same species as B (***124***), and therefore less than 90°.

Computing by logarithms,

$$\begin{aligned} &\log \sin 105° 34' = 9.983770 \\ +\ &\log \sin 80° 40' = 9.994212 \\ &\text{Deducting } 10 = 9.977982 = \log \sin b. \ \therefore\ b = 71° 54' 33''. \end{aligned}$$

To find C, $\cos a = \cot \mathsf{B} \cot \mathsf{C}$, or $\cot \mathsf{C} = \dfrac{\cos a}{\cot \mathsf{B}} = \dfrac{\cos 105° 34'}{\cot 80° 40'}$. Whence $\cot \mathsf{C}$ is $-$, and $\mathsf{C} > 90°$.

Computing by logarithms,

$$\begin{aligned} &\log \cos 105° 34' = 9.428717 \\ -\ &\log \cot 80° 40' = 9.215780 \\ &\text{Adding } 10 = 10.212937 = \log \cot \mathsf{C}. \ \therefore\ \mathsf{C} = 148° 30' 54''. \end{aligned}$$

Sch.—It is expedient to find each part directly from the parts given in the example, in order that an error in finding one may not extend itself through the whole solution.

2. Given $a = 86° 51'$, and $\mathsf{B} = 18° 03' 32''$, to project the triangle and compute the other parts.

$$c = 86° 41' 14'', \ b = 18° 01' 50'', \ \mathsf{C} = 88° 58' 25''.$$

3. Given $b = 155° 27' 54''$, and $c = 29° 46' 08''$, to project the triangle and compute the other parts. See *Fig.* 52.

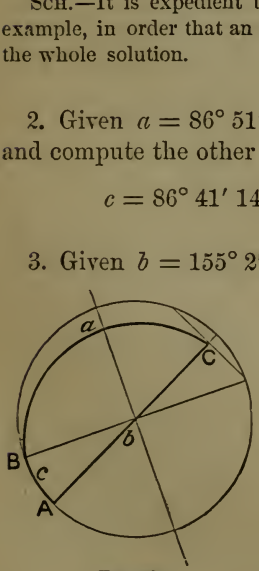

Fig. 52.

$$a = 142° 09' 13'', \ \mathsf{C} = 54° 01' 16'', \ \mathsf{B} = 137° 24' 21''.$$

4. Given $c = 73° 41' 35''$, and $\mathsf{B} = 99° 17' 33''$, to project the triangle and compute the other parts.

$$\mathsf{C} = 73° 54' 46'', \ b = 99° 40' 30'', \ a = 92° 42' 17''.$$

5. Given $\mathsf{B} = 47° 13' 43''$, and $\mathsf{C} = 126° 40' 24''$, to project the triangle and compute the other parts.

$$b = 32° 08' 56'', \ a = 133° 32' 26'', \ c = 144° 27' 03''.$$

PROJECTION.—In order to project this case, *i. e.*, when the two oblique angles are given (***105***), it is most convenient to compute the base before projecting. It is also expedient, when *two angles* are given, to project the *larger* at a point in the circumference of the primitive circle, as at C, especially if the smaller be quite small. In this case, projecting the angle C at C, Fig. 53, *conceive* BA as drawn through P (or, if desired, sketch it hypothetically), and then compute b, from the relation $\cos B = \sin C \cos b$, or $\cos b = \frac{\cos B}{\sin C}$. Having found $b = 32° 08' 56''$, take $CA = b$, and draw AB through P.

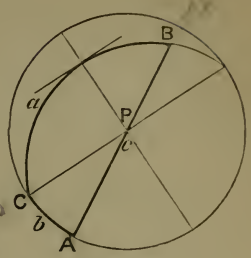

FIG. 53.

6. Given $B = 100°$, and $b = 112°$, to project the triangle and compute the other parts.

PROJECTION.—See Fig. 54.

NUMERICAL SOLUTION.—*To find* c, we have

$$\sin c = \tan b \cot B = \tan 112° \cot 100°.$$

Computing by logarithms,

$$\begin{aligned} \log \tan 112° &= 10.393590 \\ + \log \cot 100° &= 9.246319 \\ \text{Rejecting } 10 &= 9.639909 = \log \sin c. \end{aligned}$$

$\therefore c = 25° 52' 33''.4$, or $154° 07' 26''.6$, *i. e.*, BA, or BA′.

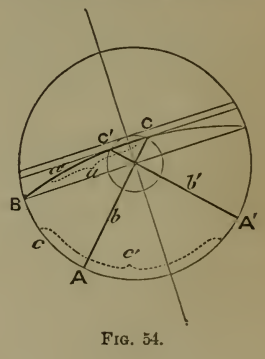

FIG. 54.

To find a, we have

$$\sin b = \sin a \sin B; \text{ whence } \sin a = \frac{\sin b}{\sin B} = \frac{\sin 112°}{\sin 100°}.$$

Computing by logarithms,

$$\begin{aligned} \log \sin 112° &= 9.967166 \\ - \log \sin 100° &= 9.993351 \end{aligned}$$

Adding $10 = 9.973815 = \log \sin a$. $\therefore a = 70° 18' 10''.7$, or $109° 41' 49''.3$, *i. e.*, BC′, or BC.

To find C, we have

$$\cos B = \cos b \sin C; \text{ whence } \sin C = \frac{\cos B}{\cos b} = \frac{\cos 100°}{\cos 112°}.$$

Computing by logarithms,

$$\begin{aligned} \log \cos 100° &= 9.239670 \\ - \log \cos 112° &= 9.573575 \end{aligned}$$

Adding $10 = 9.666095 = \log \sin C$. $\therefore C = 27° 36' 58''.8$, or $152° 23' 01''.2$, *i. e.*, BCA, or BC′A′.

Thus we see that each of the two triangles BCA and BC'A' fulfills the conditions of the problem.

7. Given one side of a right angled spherical triangle 160°, and the opposite angle 150°, to project the triangle and compute the other parts.

Results.—There are two triangles. The other sides of the first are 136° 50′ 23″, and 39° 04′ 51″; and the angle opposite the latter side is 67° 09′ 43″. The corresponding parts of the other triangle are 43° 09′ 37″, 140° 55′ 09″, and 112° 50′ 17″.

8. In the spherical triangle DEF, right angled at E, given an oblique angle 58°, and the side opposite 64°, to project the triangle and compute the other parts.

9. In a right angled spherical triangle given an oblique angle 165°, and the opposite side 112°, to project the triangle and compute the other parts.

10. In a right angled spherical triangle given one side 65° 23′ 12″, and the opposite angle 65° 23′ 12″, to project the triangle and compute the other parts.

11. Given C = 60° 47′ 24″.3, B = 57° 16′ 20″.2, and A = 90°, to project and compute.

$a = 68° 56′ 28″.9$, $c = 54° 32′ 32″.1$, and $b = 51° 43′ 36″.1$.

12. Given $c = 116°$, $b = 16°$, and the included angle 90°, to project and compute.

QUADRANTAL TRIANGLES.

129. A Quadrantal Triangle is a spherical triangle one of whose sides is a quadrant, or 90°. Such a triangle is readily solved by passing to its polar, solving it, and then passing back. The polar triangle to a quadrantal triangle, being right angled, is solved by Napier's rules.

Ex. 1. Given $a = 90°$, B = 75° 42′, and $c = 18° 37′$, to compute the other parts.

Sug's.—Representing the supplemental parts of the polar triangle by A′, B′, C′, a', b', and c', we have A′ = 180° − a = 90°, b' = 180° − B = 104° 18′, and

$C' = 180° - c = 161° 23'$, from which to find B', a', and c'. This being right angled, we find, by applying Napier's rules, $B' = 94° 31' 21''$, $a' = 76° 25' 11''$, and $c' = 161° 55' 20''$. Hence in the primitive triangle we have $b = 180° - B' = 85° 28' 39''$, $A = 180° - a' = 103° 34' 49''$, and $C = 180° - c' = 18° 04' 40''$.

Ex. 2. Given $a = 90°$, $C = 42° 10'$, and $A = 115° 20'$, to find the other parts.

$$B = 54° 44' 24'', \; b = 64° 36' 40'', \; c = 47° 57' 47''.$$

SECTION II.

OF OBLIQUE ANGLED SPHERICAL TRIANGLES.

130. All cases of oblique angled spherical triangles can be solved by Napier's rules and the following proposition.

131. Prop.—*In any spherical triangle, if a perpendicular be let fall from either vertex upon the opposite side or side produced, the tangent of half the sum of the segments* of that side is to the tangent of half the sum of the other two sides, as the tangent of half the difference of those sides is to the tangent of half the difference of the segments.*

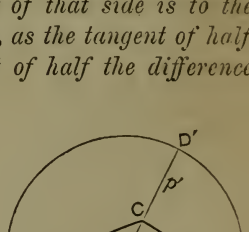

Fig. 55.

Dem.—In the triangle BAC let fall the perpendicular p, from C upon the opposite side. Let $BD = s$, and $DA = s'$. By Napier's first rule, $\cos a = \cos p \cos s$, and $\cos b = \cos p \cos s'$. Dividing the former by the latter, $\frac{\cos a}{\cos b} = \frac{\cos s}{\cos s'}$; whence, by composition and division,

$$\frac{\cos b - \cos a}{\cos a + \cos b} = \frac{\cos s' - \cos s}{\cos s + \cos s'}.$$

But by (***61***),

$$\frac{\cos b - \cos a}{\cos a + \cos b} = \tan \tfrac{1}{2}(a + b) \tan \tfrac{1}{2}(a - b),$$

and

$$\frac{\cos s' - \cos s}{\cos s + \cos s'} = \tan \tfrac{1}{2}(s + s') \tan \tfrac{1}{2}(s - s'),$$

* When the perpendicular falls without the base, as in Fig. 56, this term is to be understood as meaning the distances from the foot of the perpendicular to each extremity of the base, as BD and AD. This, in fact, is the general statement—applying as well to the case when the perpendicular falls on the base.

$$\therefore \tan \tfrac{1}{2}(a+b)\tan \tfrac{1}{2}(a-b) = \tan \tfrac{1}{2}(s+s')\tan \tfrac{1}{2}(s-s');$$

or, $\tan \frac{1}{2}(s+s') : \tan \frac{1}{2}(a+b) :: \tan \frac{1}{2}(a-b) : \tan \frac{1}{2}(s-s')$. Q. E. D.

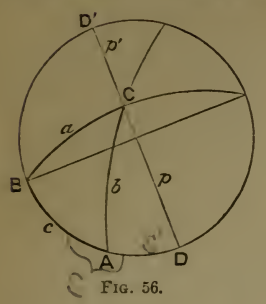

FIG. 56.

132. SCH. 1.—Since from a point in the surface of a hemisphere *two* perpendiculars can always be drawn to the circumference of the great circle which forms its base, and since the feet of these perpendiculars are 180° apart, and no side of a spherical triangle can equal 180°, the foot of one perpendicular will always fall within the base or upon one extremity of it, and the other without the base; or both will fall without the base. If we take the foot of the perpendicular which falls within the base, or the *nearer* one when both fall without, the *sum* of the distances from the foot of the perpendicular to the extremities of the base is always less than 180°, *i. e.*, $s + s' < 180°$. When the perpendicular falls within, $s + s'$ makes up one side of the triangle, and hence is less than 180°. If both perpendiculars fall without, let D, *Fig.* 56, be the foot of the nearer one. Now DB + BD′ = 180°; but by hypothesis DA < BD′, ∴ DB + DA < 180°. When DA = BD′, DB + DA = 180°.

133. SCH. 2.—As in spherical triangles the greater segment is not always adjacent to the greater side, it becomes necessary to determine the position of the segments. This can be done by the signs of the proportion

$$\tan \tfrac{1}{2}(s+s') : \tan \tfrac{1}{2}(a+b) :: \tan \tfrac{1}{2}(a-b) : \tan \tfrac{1}{2}(s-s').$$

1st. Tan $\frac{1}{2}(s+s')$ is always +, since, if D falls in the base, $s + s' < 180°$; and if D falls without, by taking the nearer perpendicular, $s + s'$ is still $< 180°$ (***132***). $\therefore \frac{1}{2}(s+s') < 90°$, and $\tan \frac{1}{2}(s+s')$ is +.

2d. When $a + b < 180°$, $\tan \frac{1}{2}(a+b)$ is +; and when $a + b > 180°$. $\tan \frac{1}{2}(a+b)$ is −.

3d. When $a > b$, $a - b$ is a positive arc less than 180°, hence $\tan \frac{1}{2}(a-b)$ is +; and when $a < b$, $(a - b)$ is a negative arc and less than 180°, hence $\tan \frac{1}{2}(a-b)$ is −.

4th. The signs of these terms being determined, that of $\tan \frac{1}{2}(s-s')$ becomes known. Now, as $\frac{1}{2}(s-s')$ cannot be numerically greater than 90°, $\tan \frac{1}{2}(s-s')$ is + when $s > s'$, and − when $s < s'$.

5th. When $s + s' = 180°$, $\tan \frac{1}{2}(s+s') = \infty$. Now as $a - b < 180°$, $\tan \frac{1}{2}(a-b)$ cannot be ∞, nor can $\tan \frac{1}{2}(s-s') = 0$ when the perpendicular falls without. Hence to make the proportion possible, $\tan \frac{1}{2}(a+b)$ must be ∞, or $a + b = 180°$. In this case we project on the plane of a or b. If $a+b = 180°$, and $a+c = 180°$, we project on the plane of a. If $a+b = 180°$, $a+c = 180°$, and $b+c = 180°$, the triangle is trirectangular.

134. SCH. 3.—If either segment is greater than the whole base, the perpendicular falls without the triangle. In this case the shorter segment lies in an opposite direction from its angle to that considered in the demonstration, and hence is to be considered −; and the algebraic sum of the segments is still equal to the side upon which the perpendicular is let fall.

135. Prop.—*In a spherical triangle, the sines of the sides are to each other as the sines of their opposite angles.*

DEM.—By Napier's first rule we have from either *Fig.* 55 or *Fig.* 56,

$$\sin p = \sin a \sin \mathrm{B}, \quad \text{and} \quad \sin p = \sin b \sin \mathrm{A}.$$

$$\therefore \sin a \sin \mathrm{B} = \sin b \sin \mathrm{A}, \quad \text{or} \quad \sin a : \sin b :: \sin \mathrm{A} : \sin \mathrm{B}. \quad \text{Q. E. D.}$$

SCH.—This proposition is not introduced here because it is *necessary* for the *solution* of spherical triangles, but because of its essential importance. It is often *convenient* to use it in the solution of a triangle, but never necessary, as will appear hereafter. *It affords a ready method of determining a part* OPPOSITE *a given part, provided the species of the part be determined by other considerations.*

SOLUTION OF OBLIQUE ANGLED SPHERICAL TRIANGLES BY NAPIER'S RULES FOR RIGHT ANGLED SPHERICAL TRIANGLES.

136. The examples which arise in the solution of oblique angled spherical triangles are all comprised under the three following problems, each of which consists of two cases:—

1. When the given parts are all adjacent to each other.
2. When two of the given parts are adjacent and one separate.
3. When the given parts are all separate from each other.

---

137. Prob. 1.—*Given three adjacent parts of an oblique angled spherical triangle, to solve the triangle.*

CASE 1st.—*Given two sides and the included angle.*

SOLUTION.—Project the triangle on the plane of *one of the given sides* (***106***), and let fall a perpendicular from the angle opposite upon this side or upon

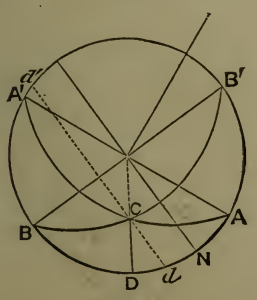

FIG. 57.

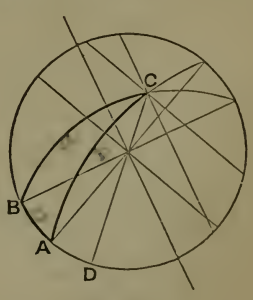

FIG. 58.

this side produced, as the case may be. There are thus formed two right angled triangles, as BDC and DCA, each of which can be solved by Napier's

rules, by first solving the one containing the given angle. Thus, in the triangle BDC right angled at D, a and B are supposed known; whence CD, BD, and the angle BCD, can be computed. As BA $= c$ is known, the segment DA can be found, it being the difference between c and the arc BD. When the solution of this triangle gives BD $> c$, it is evident that the perpendicular falls without the triangle, which will agree with the projection. Passing to the triangle ADC, right angled at D, we now know CD and DA; whence the other parts can be found. Finally, the angle BCA of the required triangle = BCD + DCA when the perpendicular falls within the triangle, and BCD − DCA when the perpendicular falls without.

CASE 2d.—*Given two angles and the included side.*

SOLUTION.—The solution of this case is effected by passing to the polar triangle, projecting and solving it by Case 1st, and then passing back.

138. SCH.—A slight saving of labor is effected by using (***135***) in the solution. Thus, in the triangle BCD, compute CD and BD as before, and (not computing angle BCD) then passing to the triangle DCA, compute b and A. Finally, compute C (the entire angle) from the proportion

$$\sin b : \sin c :: \sin \mathrm{B} : \sin \mathrm{C}.$$

---

139. Prob. 2.—*In an oblique angled spherical triangle, given two parts adjacent to each other and one separated from both of them, to solve the triangle.*

CASE 1st.—*Given two sides and an angle opposite one of them.*

SOLUTION.—Project the triangle on the plane of the *unknown side*, with the given angle at B; and let fall the perpendicular from the angle C opposite the unknown side. Compute the triangle BDC. Having computed this triangle,

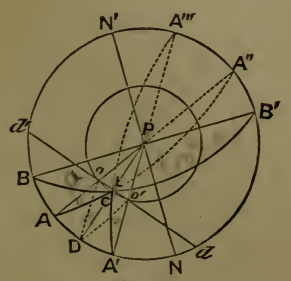

FIG. 59.

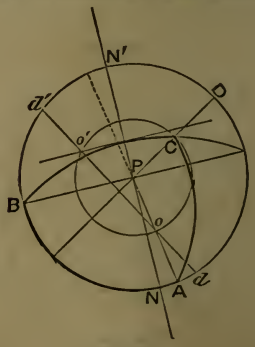

FIG. 60.

compare the side opposite the given angle, as b, with the perpendicular and the arcs BC and CB', *i. e.*, with p, a, and $180° - a$. If $b = p$ there is but

one solution and the triangle is right angled, A falling at D. If b is intermediate in value between p and *both* a and $180° - a$, it can occupy two positions as in *Fig.* 59, and there are two solutions. If b is intermediate in value between p and only one of the arcs a or $180° - a$, there is but one solution. When $B < 90°$ the perpendicular is *less* than any oblique arc; hence in this case, if $b < p$, there is *no* solution. But if $B > 90°$, the perpendicular is greater than the oblique arcs; hence in this case, if $b > p$, there is *no* solution. [These results should be obtained independently of the results given by the projection, and one be made a check upon the other.] The solution is now completed by computing the parts of DCA, and adding or subtracting the segments BD and AD, and the angles BCD and ACD, as the case may require.

CASE 2d.—*Given two angles and a side opposite one of them.*

SOLUTION.—Pass to the polar triangle; solve it, and then pass back.*

140. SCH.—The relation established in (***135***) may also be used in the solution of this problem. Thus, having projected the triangle, computed p, and determined whether there are one or two solutions, to find A, when a, b, and B are given, we have, $\sin b : \sin a :: \sin B : \sin A$. Then computing the third side c (or sides), by means of the right angled triangles BCD and DCA as before, we may use the proportion (***135***) to find the angle BCA and BCA′. But the use of this proportion gives no advantage except in cases in which there is only one solution.

---

141. Prob. 3.—*In an oblique angled spherical triangle, given three parts all separated from each other, to solve the triangle.*

CASE 1st.—*Given the sides to find the angles.*

SOLUTION.—Project the triangle on the plane of one of its sides, as c. From the proportion,

$$\tan \tfrac{1}{2}(s + s') : \tan \tfrac{1}{2}(a + b) :: \tan \tfrac{1}{2}(a - b) : \tan \tfrac{1}{2}(s - s'),$$

---

* This case can be projected and solved in a manner altogether similar to the first, without passing to the polar triangle. Thus, let B, a, and A be the given parts. Project the triangle on the plane of c, as in the figure. Project B in the usual way, and make BC = the projection of a. Through C draw DD′, and make BDO = the projection of A. Drawing the small circle with radius PC, draw diameters through the intersections O and O′. Then will A and A′ be the vertices of the triangle required. The student may prove that the figures POD and PCA are equal, and also PO′D and PCA′, and hence that angle BDO = A = A′.

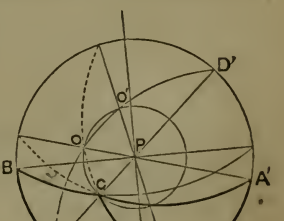

FIG. 61.

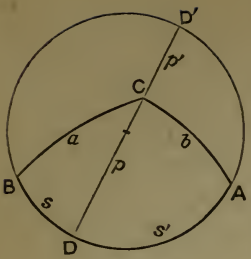

Fig. 62.

find $\frac{1}{2}(s - s')$. Then half the sum of the segments + half the difference gives the greater segment, and half the sum − half the difference gives the less. Determine from the signs of the terms whether s is greater or less than s': and also determine whether the perpendicular lies within or without the triangle (***134***). Observe that these results correspond to those given by the projection. Finally, in each of the two right angled triangles BCD and DCA, there are two sides given; whence the angles can be found by Napier's rules. If the perpendicular falls within, C = BCD + DCA, and A, of the required triangle = DAC. If the perpendicular falls without, C = BCD − DCA, and A of the triangle = 180° − DAC.

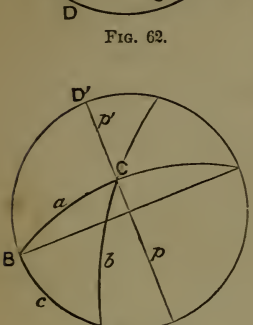

Fig. 63.

CASE 2d.—*Given the angles to find the sides.*

SOLUTION.—Pass to the polar triangle; solve it, and then pass back.

142. SCH.—Here, again, (***135***) affords a slightly more expeditious solution. Having projected the triangle, found and located the segments, and computed one angle, as B, by the methods given above, the other angles may be found from the proportions,

$$\sin b : \sin a :: \sin \mathrm{B} : \sin \mathrm{A},$$

and

$$\sin b : \sin c :: \sin \mathrm{B} : \sin \mathrm{C}.$$

EXERCISES.

1. Given $b = 120° 30' 30''$, $c = 70° 20' 20''$, and $\mathrm{A} = 50° 10' 10''$, to project and solve the triangle.

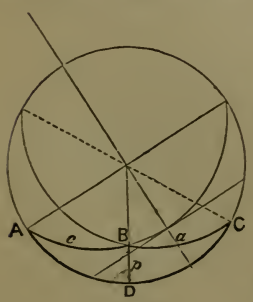

Fig. 64.

PROJECTION.—See *Fig.* 64.

TRIGONOMETRICAL SOLUTION.—1st. *To solve the triangle* ABD, in which the two known parts are situated.

(*a*) To find p, $\sin p = \sin c \sin \mathrm{A}$.

$$\log \sin 70° 20' 20'' = 9.973912$$
$$+ \log \sin 50° 10' 10'' = 9.885329$$
$$\text{Rejecting } 10 = \overline{9.859241} = \log \sin p$$

$\therefore p = 46° 19' 01''$, the species being determined by the opposite angle (***125***). [Observe that the result corresponds with the projection].

(*b*) To find AD, $\cos A = \cot c \tan AD$, or $\tan AD = \frac{\cos A}{\cot c}$.

$\log \cos 50° 10' 10'' = 9.806532$
$- \log \cot 70° 20' 20'' = 9.553016$
Adding $10 = \overline{10.253516} = \log \tan AD$. $\therefore AD = 60° 50' 49''$,
the species being determined by the signs of the formula.

(*c*) To find angle ABD, $\cos c = \cot A \cot ABD$, or $\cot ABD = \frac{\cos c}{\cot A}$.

$\log \cos 70° 20' 20'' = 9.526929$
$- \log \cot 50° 10' 10'' = 9.921204$
Adding $10 = \overline{9.605725} = \log \cot ABD$. $\therefore ABD = 68° 01' 53''$
the species being determined by the signs of the formula.

2d. *To solve the triangle* DBC.

(*a*) To find DC. Since $AD < b$, the foot of the perpendicular falls in the base, and $DC = AC - AD = b - AD = 120° 30' 30'' - 60° 50' 49'' = 59° 39' 41''$.

(*b*) To find a, $\cos a = \cos p \cos DC$

$\log \cos 46° 19' 01'' = 9.839270$
$+ \log \cos 59° 39' 41'' = 9.703386$
Rejecting $10 = \overline{9.542656} = \log \cos a$. $\therefore a = 69° 34' 56''$, the species being determined by the signs of the formula.

(*c*) To find C, $\sin p = \sin a \sin C$, or $\sin C = \frac{\sin p}{\sin a}$.

$\log \sin p = 9.859241$
$- \log \sin 69° 34' 56'' = 9.971820$
Adding $10 = \overline{9.887421} = \log \sin C$. $\therefore C = 50° 30' 08''$, the species being determined by the side opposite.

(*d*) To find angle DBC, $\sin p = \tan DC \cot DBC$, or $\cot DBC = \frac{\sin p}{\tan DC}$.

$\log \sin p = 9.859241$
$- \log \tan 59° 39' 41'' = 10.232653$
Adding $10 = \overline{9.626588} = \log \cot DBC$. $\therefore DBC = 67° 03' 36''$,
the species being determined by the signs of the formula.

Finally, $B = ABD + DBC = 68° 01' 53'' + 67° 03' 36'' = 135° 05' 29''$.

Scn.—We might have omitted the computation of angle ABD in the first part, and DBC in the second, and have found instead the entire angle B from $\sin a : \sin b :: \sin A : \sin B$. To compute this requires the looking out of but two logarithms, since $\sin a$ is given in the second part (*c*), and $\sin A$ in the first part (*a*).

2. Given $a = 97° 35'$, $b = 27° 08' 22''$, and $A = 40° 51' 18''$, to project and compute the triangle. Between what limits must the

value of a be assigned in order that there may be two solutions? Beyond what limiting values of a is a solution impossible?

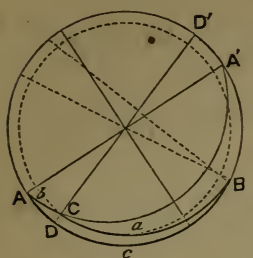

Fig. 65.

PROJECTION. See *Fig.* 65.

TRIGONOMETRICAL SOLUTION.—To find p, $\sin p = \sin b \sin \mathsf{A}$.

$$\begin{aligned} \log \sin 27^\circ\ 08'\ 22'' &= 9.659115 \\ + \log \sin 40^\circ\ 51'\ 18'' &= 9.815675 \\ \text{Rejecting } 10 &= \overline{9.474790} = \log \sin p. \end{aligned}$$

$\therefore p = 17^\circ\ 21'\ 40''$. [Reason for the species.]

We now observe that there is *one* and *only one* solution, since the arc a ($97^\circ\ 35'$) cannot lie between CD ($17^\circ\ 21'\ 40''$) and b ($27^\circ\ 08'\ 22''$), but *can* lie between CD and CA′ ($180^\circ - b = 152^\circ\ 51'\ 38''$).

To find AD, $\cos \mathsf{A} = \cot b \tan \mathsf{AD}$ or $\tan \mathsf{AD} = \dfrac{\cos \mathsf{A}}{\cot b}$.

$$\begin{aligned} \log \cos 40^\circ\ 51'\ 18'' &= 9.878733 \\ - \log \cot 27^\circ\ 08'\ 22'' &= 10.290226 \\ \text{Adding } 10 &= \overline{9.588507} = \log \tan \mathsf{AD}. \end{aligned}$$

$\therefore \mathsf{AD} = 21^\circ\ 11'\ 30''.6$.

To find angle ACD, $\cos b = \cot \mathsf{A} \cot \mathsf{ACD}$, or $\cot \mathsf{ACD} = \dfrac{\cos b}{\cot \mathsf{A}}$.

$$\begin{aligned} \log \cos 27^\circ\ 08'\ 22'' &= 9.949340 \\ - \log \cot 40^\circ\ 51'\ 18'' &= 10.0630575 \\ \text{Adding } 10 &= \overline{9.8862825} = \log \cot \mathsf{ACD}. \end{aligned}$$

$\therefore \mathsf{ACD} = 52^\circ\ 25'\ 01''$. [Reason for the species.]

To find B, $\sin p = \sin a \sin \mathsf{B}$, or $\sin \mathsf{B} = \dfrac{\sin p}{\sin a}$.

$$\begin{aligned} \log \sin p \text{ (as above)} &= 9.474790 \\ - \log \sin 97^\circ\ 35' &= 9.996185 \\ \text{Adding } 10 &= \overline{9.478605} = \log \sin \mathsf{B}. \end{aligned}$$

$\therefore \mathsf{B} = 17^\circ\ 31'\ 09''$. [Reason for the species.]

To find DB, $\cos a = \cos p \cos \mathsf{DB}$, or $\cos \mathsf{DB} = \dfrac{\cos a}{\cos p}$.

$$\begin{aligned} \log \cos 97^\circ\ 35' &= 9.120469 \\ - \log \cos 17^\circ\ 21'\ 40'' &= 9.979750 \\ \text{Adding } 10 &= \overline{9.140719} = \log \cos \mathsf{DB}. \end{aligned}$$

$\therefore \mathsf{DB} = 97^\circ\ 56'\ 51''.3$. [Reason for the species.]

$\mathsf{AB} = \mathsf{AD} + \mathsf{DB} = 21^\circ\ 11'\ 30''.6 + 97^\circ\ 56'\ 51''.3 = 119^\circ\ 08'\ 21''.9$.

To find DCB, $\sin p = \tan \mathsf{DB} \cot \mathsf{DCB}$, or $\cot \mathsf{DCB} = \dfrac{\sin p}{\tan \mathsf{DB}}$.

$$\begin{aligned} \log \sin p \text{ (as above)} &= 9.474790 \\ - \log \tan 97^\circ\ 56'\ 51''.3 &= 10.855090 \\ \text{Adding } 10 &= \overline{8.619700} = \log \cot \mathsf{DCB}. \end{aligned}$$

$\therefore \mathsf{DCB} = 92^\circ\ 23'\ 7''.7$. [Reason for the species.]

$\mathsf{ACB} = \mathsf{C} = \mathsf{ACD} + \mathsf{DCB} = 52^\circ\ 25'\ 01'' + 92^\circ\ 23'\ 7''.7 = 144^\circ\ 48'\ 8''.7$.

Finally, we observe that, if any value were assigned to a between b (27° 08′ 22″) and CD (17° 21′ 40″) there would be *two* solutions; since for such values the side a could lie on both sides of CD. But, for any value of a less than CD (17° 21′ 40″), there would be *no* solution; since CD is the shortest distance from C to the arc ABA′. Also, for any value of a greater than the arc CA′ (152° 51′ 38″), there would be *no* solution, as such an arc would fall between CA′ and CD′ (if not > CD′), and consequently would make $c > 180°$.

Sch.—Such examples as this and the preceding can be more expeditiously solved by using p in each equation in solving the triangles ACD and DCB. By this means and using (***135***) to determine the side c, the solution can be effected with only 12 logarithms. Thus in Ex. 2.

1st. To find p, $\sin p = \sin b \sin A$, requires 3 logarithms

2d. To find ACD, $\cos ACD = \cot b \tan p$, requires 3 "

3d. To find DCB, $\cos DCB = \cot a \tan p$, requires 2 " (log tan p being known).

4th. To find B, $\sin p = \sin a \sin B$, requires 2 (log sin p being known).

5th. To find c, $\sin A : \sin C :: \sin a : \sin c$, requires 2 " (log's of sin A and sin a being known.)

Total 12 logarithms

3. Given $a = 76° 35′ 36″$, $b = 50° 10′ 30″$, and $c = 40° 00′ 10″$, to project and solve the triangle.

Projection.—See *Fig.* 66.

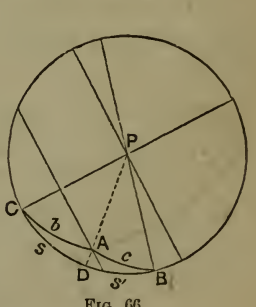

Fig. 66.

Trigonometrical Solution.—1st. To find the segments CD and DB, we have,

$$\tan \tfrac{1}{2}(s + s'),$$

or

$$\tan \tfrac{1}{2}a : \tan \tfrac{1}{2}(b + c) :: \tan \tfrac{1}{2}(b - c) : \tan \tfrac{1}{2}(s - s').$$

Computing by logarithms.

a. c. $\log \tan \tfrac{1}{2}a = \log \tan 38° 17′ 48″ = 0.102561$

$+ \log \tan \tfrac{1}{2}(b + c) = \log \tan 45° 05′ 20″ = 10.001347$

$+ \log \tan \tfrac{1}{2}(b - c) = \log \tan 5° 05′ 10″ = 8.949406$

Rejecting 10 $= 9.053314 = \log \tan \tfrac{1}{2}(s - s')$.

$\therefore \tfrac{1}{2}(s - s') = 6° 27′ 02″$.

In order to determine whether s or s' is the greater,* we observe the signs of the proportion, and finding $\tan \tfrac{1}{2}(s - s')$ positive, know that $s > s'$.

Hence, $s = \tfrac{1}{2}(s + s') + \tfrac{1}{2}(s - s') = 38° 17′ 48″ + 6° 27′ 02″ = 44° 44′ 50″$,

and $s' = \tfrac{1}{2}(s + s') - \tfrac{1}{2}(s - s') = 38° 17′ 48″ - 6° 27′ 02″ = 31° 50′ 46″$.

The angles sought are now readily found by computing the two right angled triangles ADC and ADB.

---

* Though the projection generally determines such facts as this, the species of parts, and the number of solutions in ambiguous cases, *the student should not rely upon it*, but determine each such fact upon purely trigonometrical considerations, merely using the projection to give clearness to the conception, and as a rough check against errors.

Or, having computed C from the triangle ACD, we may find A and B more expeditiously by using the proportions,

$$\sin c : \sin b :: \sin C : \sin B,$$

and

$$\sin c : \sin a :: \sin C : \sin A.$$

The angles are $C = 34° 15' 03''$, $A = 121° 36' 12''$, and $B = 42° 15' 13''$.

4. Given $A = 128° 45'$, $C = 30° 35'$, and $a = 68° 50'$, to solve the triangle. What values of A give two solutions? What none?

$c = 37° 28' 20''$, $b = 40° 09' 04''$, and $B = 32° 37' 58''$.

5. Given $A = 129° 05' 28''$, $B = 142° 12' 42''$, and $C = 105° 08' 10''$, to solve the triangle.

$a = 135° 49' 20''$, $b = 146° 37' 15''$, and $c = 60° 04' 54''$.

6. Given $a = 68° 46' 02''$, $b = 37° 10'$, and $C = 43° 37' 38''$, to project and solve the triangle.

$A = 116° 22' 22''$, $B \leqslant 35° 29' 54''$, and $c = 45° 52' 34''$.

7. Given $a = 40° 16'$, $b = 47° 44'$, and $A = 52° 34'$, to project and solve the triangle. What values of a give but one solution? What none?

There are two triangles.—In the 1st, $c = 53° 19' 20''$, $B = 65° 16' 35''$, and $C = 79° 52' 21''$. In the 2d, $c = 14° 18' 22''$, $B = 114° 43' 25''$, and $C = 17° 39' 22''$.

8. Given $a = 62° 38'$, $b = 10° 13' 19''$, and $C = 150° 24' 12''$, to project and solve the triangle.

$A = 27° 31' 44''$, $B = 5° 17' 58''$, and $c = 71° 37' 06''$.

9. Given $a = 56° 40'$, $b = 83° 13'$, and $c = 114° 30'$, to project and solve the triangle.

$A = 48° 31' 18''$, $B = 62° 55' 44''$, and $C = 125° 18'' 56''$.

10. Given $A = 50° 12'$, $B = 58° 08'$, and $a = 62° 42'$, to solve the triangle. What values of A give but one solution? What none?

There are two solutions.—1st, $b = 79° 12' 10''$, $c = 119° 03' 26''$, and $C = 130° 54' 28''$. 2d, $b = 100° 47' 50''$, $c = 152° 14' 18''$, and $C = 156° 15' 06''$.

11. Given $A = 36° 25'$, $B = 42° 17' 10''$, and $C = 95° 10' 05''$, to project and solve the triangle.

12. Given $a = 124° 53'$, $b = 31° 19'$, and $c = 171° 48' 22''$, to solve the triangle.

13. Given $a = 150° 17' 23''$, $b = 43° 12'$, and $c = 82° 50' 12''$, to solve the triangle.

14. Given $a = 115° 20' 10''$, $b = 57° 30' 06''$, and $A = 126° 37' 30''$, to solve the triangle.

15. Given $A = 109° 55' 42''$, $B = 116° 38' 33''$, and $C = 120° 43' 37''$, to solve the triangle.

16. Given $A = 50°$, $b = 60°$, and $a = 40°$, to solve the triangle.

17. Given $a = 50° 45' 20''$, $b = 69° 12' 40''$, and $A = 44° 22' 10''$, to project and solve the triangle.

There are two solutions.—1st, $B = 57° 34' 51''$, $C = 115° 57' 51''$, and $c = 95° 18' 16''$. 2d, $B = 122° 25' 09''$, $C = 25° 44' 32''$, and $c = 28° 45' 05''$.

18. Given $b = 99° 40' 48''$, $c = 100° 49' 30''$, and $A = 65° 33' 10''$, to project and solve the triangle.

$a = 64° 23' 15''$, $B = 95° 38' 04''$, and $C = 97° 26' 29''$.

19. Given $A = 48° 30'$, $B = 125° 20'$, and $C = 62° 54'$, to solve the triangle.

$a = 56° 39' 30''$, $b = 114° 29' 58''$, and $c = 83° 12' 06''$.

20. Given $C = 54° 15' 03''$, $B = 40° 18' 13''$, and $a = 70° 30' 30''$, to solve the triangle.

21. Given $A = 47° 54' 21''$, $c = 61° 04' 56''$, and $a = 40° 31' 20''$, to project and solve the triangle.

22. Given $A = 50° 10' 10''$, $b = 69° 34' 35''$, and $a = 120° 30' 30''$, to project and solve the triangle.

SECTION III.

GENERAL FORMULÆ.

[NOTE.—This section is designed for such as make mathematics a specialty. The preceding sections are thought sufficient for the general student.]

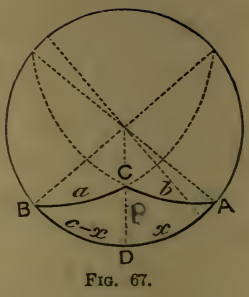

FIG. 67.

143. Prop.—*In a Spherical Triangle the cosine of any side is equal to the product of the cosines of the other two sides, plus the product of the sines of those sides into the cosine of their included angle; that is,*

$$\left.\begin{aligned} &(1)\ \cos a = \cos b \cos c + \sin b \sin c \cos \mathrm{A};\\ &(2)\ \cos b = \cos a \cos c + \sin a \sin c \cos \mathrm{B};\\ &(3)\ \cos c = \cos a \cos b + \sin a \sin b \cos \mathrm{C}. \end{aligned}\right\} \text{A.}$$

DEM.—From *Fig.* 67, we have,

$$\cos a = \cos (c - x) \cos p$$

$$= \frac{\cos (c - x) \cos b}{\cos x} \quad \left[\text{since } \cos p = \frac{\cos b}{\cos x}\right]$$

$$= \frac{\cos b \cos c \cos x + \cos b \sin c \sin x}{\cos x} \quad [\text{expanding } \cos (c - x)]$$

$$= \cos b \cos c + \cos b \sin c \tan x \quad \left[\text{since } \frac{\sin x}{\cos x} = \tan x\right]$$

$$= \cos b \cos c + \sin b \sin c \cos \mathrm{A} \quad \left[\text{since } \cos \mathrm{A} = \cot b \tan x = \frac{\cos b \tan x}{\sin b}\right].$$

In a similar manner (2) and (3) may be produced.

144. COR. 1.—*From set* A, *by passing to the polar triangle, we have,*

$$\left.\begin{aligned} &(1)\ \cos \mathrm{A} = -\cos \mathrm{B} \cos \mathrm{C} + \sin \mathrm{B} \sin \mathrm{C} \cos a;\\ &(2)\ \cos \mathrm{B} = -\cos \mathrm{A} \cos \mathrm{C} + \sin \mathrm{A} \sin \mathrm{C} \cos b;\\ &(3)\ \cos \mathrm{C} = -\cos \mathrm{A} \cos \mathrm{B} + \sin \mathrm{A} \sin \mathrm{B} \cos c. \end{aligned}\right\} \text{B.}$$

DEM.—Letting a', b', c', A′, B′, and C′ represent the parts of the polar triangle,

we have $a = 180° - A'$, $b = 180° - B'$, $c = 180° - C'$, $A = 180° - a'$, $B = 180° - b'$, and $C = 180° - c'$. Whence, substituting in (1) A, we have,

$$\cos(180° - A') = \cos(180° - B')\cos(180° - C') + \sin(180° - B')\sin(180° - C')\cos(180° - a'),$$

or, $$\cos A' = -\cos B'\cos C' + \sin B'\sin C'\cos a',$$

since $\cos(180° - A') = -\cos A'$, etc.; and $\sin(180° - B') = \sin B'$, etc.

Finally, dropping the accents, since the results are general, and treating (2) and (3) of set A in the same way, we have set B.

145. Cor. 2.—*From* A *and* B *we readily find the angles in terms of the sides, and the sides in terms of the angles. Thus, from* A,

$$\left.\begin{aligned} (1)\ \cos A &= \frac{\cos a - \cos b\cos c}{\sin b\sin c};\\ (2)\ \cos B &= \frac{\cos b - \cos a\cos c}{\sin a\sin c};\\ (3)\ \cos C &= \frac{\cos c - \cos a\cos b}{\sin a\sin b}. \end{aligned}\right\} \text{A}'.$$

From B,

$$\left.\begin{aligned} (1)\ \cos a &= \frac{\cos A + \cos B\cos C}{\sin B\sin C};\\ (2)\ \cos b &= \frac{\cos B + \cos A\cos C}{\sin A\sin C};\\ (3)\ \cos c &= \frac{\cos C + \cos A\cos B}{\sin A\sin B}. \end{aligned}\right\} \text{B}'.$$

---

146. Prop.—*Formulæ* A′, *and* B′, *adapted to logarithmic computation, become,*

$$\left.\begin{aligned} (1)\ \sin\tfrac{1}{2}A &= \sqrt{\frac{\sin(\tfrac{1}{2}s - b)\sin(\tfrac{1}{2}s - c)}{\sin b\sin c}};\\ (2)\ \sin\tfrac{1}{2}B &= \sqrt{\frac{\sin(\tfrac{1}{2}s - a)\sin(\tfrac{1}{2}s - c)}{\sin a\sin c}};\\ (3)\ \sin\tfrac{1}{2}C &= \sqrt{\frac{\sin(\tfrac{1}{2}s - a)\sin(\tfrac{1}{2}s - b)}{\sin a\sin b}}. \end{aligned}\right\} \text{A}''.$$

And
$$\left.\begin{aligned} &(1)\ \sin\tfrac{1}{2}a = \sqrt{-\frac{\cos\tfrac{1}{2}\mathrm{S}\cos(\tfrac{1}{2}\mathrm{S}-\mathrm{A})}{\sin\mathrm{B}\sin\mathrm{C}}}\ ;\\ &(2)\ \sin\tfrac{1}{2}b = \sqrt{-\frac{\cos\tfrac{1}{2}\mathrm{S}\cos(\tfrac{1}{2}\mathrm{S}-\mathrm{B})}{\sin\mathrm{A}\sin\mathrm{C}}}\ ;\\ &(3)\ \sin\tfrac{1}{2}c = \sqrt{-\frac{\cos\tfrac{1}{2}\mathrm{S}\cos(\tfrac{1}{2}\mathrm{S}-\mathrm{C})}{\sin\mathrm{A}\sin\mathrm{B}}}. \end{aligned}\right\}\ \mathrm{B}''.$$

DEM.—Subtracting each member of (1) A′ from 1, we have,

$$1-\cos\mathrm{A} = 1-\frac{\cos a-\cos b\cos c}{\sin b\sin c} = \frac{\cos b\cos c+\sin b\sin c-\cos a}{\sin b\sin c}.$$

$$\therefore\ 2\sin^2\tfrac{1}{2}\mathrm{A} = \frac{\cos(b-c)-\cos a}{\sin b\sin c},\ \text{since}\ 1-\cos\mathrm{A} = 2\sin^2\tfrac{1}{2}\mathrm{A}\ (\boldsymbol{62,5}),$$

and $\cos b\cos c+\sin b\sin c = \cos(b-c)$ (**55**, D).

Now letting $y = b-c$, and $x = a$, we see from (**59**, D′) that $\cos(b-c)-\cos a = 2\sin\tfrac{1}{2}(a+b-c)\sin\tfrac{1}{2}(a+c-b)$.

Hence,
$$2\sin^2\tfrac{1}{2}\mathrm{A} = \frac{2\sin\tfrac{1}{2}(a+b-c)\sin\tfrac{1}{2}(a+c-b)}{\sin b\sin c},$$

or,
$$\sin\tfrac{1}{2}\mathrm{A} = \sqrt{\frac{\sin\tfrac{1}{2}(a+b-c)\sin\tfrac{1}{2}(a+c-b)}{\sin b\sin c}}$$

Finally, putting $s = a+b+c$, whence $\tfrac{1}{2}(a+b-c) = \tfrac{1}{2}s-c$,

and
$$\tfrac{1}{2}(a+c-b) = \tfrac{1}{2}s-b,$$

we have,
$$\sin\tfrac{1}{2}\mathrm{A} = \sqrt{\frac{\sin(\tfrac{1}{2}s-b)\sin(\tfrac{1}{2}s-c)}{\sin b\sin c}}.$$

In like manner, (2) and (3) of set A′ reduce to (2) and (3) of set A″.

Again, subtracting each member of (1) set B′, from 1, we have, 98

$$1-\cos a = 1-\frac{\cos\mathrm{A}+\cos\mathrm{B}\cos\mathrm{C}}{\sin\mathrm{B}\sin\mathrm{C}} = \frac{\sin\mathrm{B}\sin\mathrm{C}-\cos\mathrm{B}\cos\mathrm{C}-\cos\mathrm{A}}{\sin\mathrm{B}\sin\mathrm{C}}.$$

$$\therefore\ 2\sin^2\tfrac{1}{2}a = \frac{-\cos(\mathrm{B}+\mathrm{C})-\cos\mathrm{A}}{\sin\mathrm{B}\sin\mathrm{C}} = -\frac{\cos(\mathrm{B}+\mathrm{C})+\cos\mathrm{A}}{\sin\mathrm{B}\sin\mathrm{C}}$$

$$= -\frac{2\cos\tfrac{1}{2}(\mathrm{A}+\mathrm{B}+\mathrm{C})\cos\tfrac{1}{2}(\mathrm{B}+\mathrm{C}-\mathrm{A})}{\sin\mathrm{B}\sin\mathrm{C}}.$$

Now putting $S = A + B + C$, whence $\frac{1}{2}(B + C - A) = \frac{1}{2}S - A$, we have,

$$\sin \tfrac{1}{2}a = \sqrt{-\frac{\cos \frac{1}{2}S \cos(\frac{1}{2}S - A)}{\sin B \sin C}}.$$

In the same manner, (2) and (3) of B″ are deduced from (2) and (3) of B′.

147. Cor. 1.—*Passing to the polar triangle,* A″ *and* B″ *become*

$$\left.\begin{aligned}
&(1)\ \cos \tfrac{1}{2}a = \sqrt{\frac{\cos(\frac{1}{2}S - B)\cos(\frac{1}{2}S - C)}{\sin B \sin C}};\\
&(2)\ \cos \tfrac{1}{2}b = \sqrt{\frac{\cos(\frac{1}{2}S - A)\cos(\frac{1}{2}S - C)}{\sin A \sin C}};\\
&(3)\ \cos \tfrac{1}{2}c = \sqrt{\frac{\cos(\frac{1}{2}S - A)\cos(\frac{1}{2}S - B)}{\sin A \sin B}}.
\end{aligned}\right\} B'''$$

And

$$\left.\begin{aligned}
&(1)\ \cos \tfrac{1}{2}A = \sqrt{\frac{\sin \frac{1}{2}s \sin(\frac{1}{2}s - a)}{\sin b \sin c}};\\
&(2)\ \cos \tfrac{1}{2}B = \sqrt{\frac{\sin \frac{1}{2}s \sin(\frac{1}{2}s - b)}{\sin a \sin c}};\\
&(3)\ \cos \tfrac{1}{2}C = \sqrt{\frac{\sin \frac{1}{2}s \sin(\frac{1}{2}s - c)}{\sin a \sin b}}.
\end{aligned}\right\} A'''$$

148. Sch.—Formulæ A‴ and B‴ can be obtained directly from A′ and B′, in a manner altogether similar to that in which A″ and B″ were deduced, by *adding* each member of the equations in sets A′ and B′ to 1, instead of subtracting, and observing that $1 + \cos x = 2\cos^2 \frac{1}{2}x$.

149. Cor. 2.—*Dividing the formulæ of set* A″ *by the corresponding ones of set* A‴; *and, in a similar manner, those of* B‴ *by those of* B″, *and putting* $\sqrt{\frac{\sin(\frac{1}{2}s - a)\sin(\frac{1}{2}s - b)\sin(\frac{1}{2}s - c)}{\sin \frac{1}{2}s}} = k$. *and* $\sqrt{\frac{\cos(\frac{1}{2}S - A)\cos(\frac{1}{2}S - B)\cos(\frac{1}{2}S - C)}{-\cos \frac{1}{2}S}} = K$, *we have,*

$$\left.\begin{aligned}
&(1)\ \tan \tfrac{1}{2}A = \sqrt{\frac{\sin(\frac{1}{2}s - b)\sin(\frac{1}{2}s - c)}{\sin \frac{1}{2}s \sin(\frac{1}{2}s - a)}} = \frac{k}{\sin(\frac{1}{2}s - a)};\\
&(2)\ \tan \tfrac{1}{2}B = \sqrt{\frac{\sin(\frac{1}{2}s - a)\sin(\frac{1}{2}s - c)}{\sin \frac{1}{2}s \sin(\frac{1}{2}s - b)}} = \frac{k}{\sin(\frac{1}{2}s - b)};\\
&(3)\ \tan \tfrac{1}{2}C = \sqrt{\frac{\sin(\frac{1}{2}s - a)\sin(\frac{1}{2}s - b)}{\sin \frac{1}{2}s \sin(\frac{1}{2}s - c)}} = \frac{k}{\sin(\frac{1}{2}s - c)}.
\end{aligned}\right\} A^{IV}$$

$$(1)\ \cot\tfrac{1}{2}a = \sqrt{\frac{\cos(\tfrac{1}{2}S - B)\cos(\tfrac{1}{2}S - C)}{-\cos\tfrac{1}{2}S\cos(\tfrac{1}{2}S - A)}} = \frac{K}{\cos(\tfrac{1}{2}S - A)};$$

$$(2)\ \cot\tfrac{1}{2}b = \sqrt{\frac{\cos(\tfrac{1}{2}S - A)\cos(\tfrac{1}{2}S - C)}{-\cos\tfrac{1}{2}S\cos(\tfrac{1}{2}S - B)}} = \frac{K}{\cos(\tfrac{1}{2}S - B)};$$

$$(3)\ \cot\tfrac{1}{2}c = \sqrt{\frac{\cos(\tfrac{1}{2}S - A)\cos(\tfrac{1}{2}S - B)}{-\cos\tfrac{1}{2}S\cos(\tfrac{1}{2}S - C)}} = \frac{K}{\cos(\tfrac{1}{2}S - C).}$$ } B[IV]

Sch.—In these *formulæ* k is the tangent of the arc with which the inscribed circle is described, and K is the cotangent of the arc with which the circum-

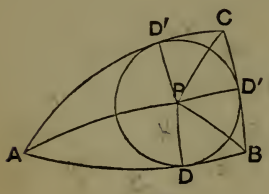

Fig. 68.

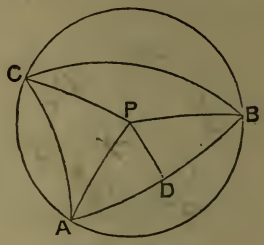

Fig. 69.

scribed circle is described. Thus, using the common notation, we have in *Fig.* 68, $AD = AD' = \frac{1}{2}s - a$, and angle $PAD = \frac{1}{2}A$; whence

$$\sin AD = \cot PAD \times \tan PD = \frac{\tan PD}{\tan PAD},$$

or $$\tan\tfrac{1}{2}A = \frac{\tan PD}{\sin(\tfrac{1}{2}s - a)} = \frac{k}{\sin(\tfrac{1}{2}s - a)},\ [(1)\ A^{IV}].\quad \therefore k = \tan PD.$$

From *Fig.* 69, we have, $AD = \frac{1}{2}c$, and angle $PAD = \frac{1}{2}S - C$.

Hence, $$\cos(\tfrac{1}{2}S - C) = \cot AP \times \tan\tfrac{1}{2}c, \text{ or } \tan\tfrac{1}{2}c = \frac{\cos(\tfrac{1}{2}S - C)}{\cot AP},$$

or $$\cot\tfrac{1}{2}c = \frac{\cot AP}{\cos(\tfrac{1}{2}S - C)} = \frac{K}{\cos(\tfrac{1}{2}S - C)},\ [(3)\ B^{IV}].\quad \therefore K = \cot AP.$$

GAUSS'S EQUATIONS.

150. Prob.—*To deduce Gauss's Equations, which are*

$$(1)\ \frac{\sin\tfrac{1}{2}(A + B)}{\cos\tfrac{1}{2}C} = \frac{\cos\tfrac{1}{2}(a - b)}{\cos\tfrac{1}{2}c};$$

$$(2)\ \frac{\sin\tfrac{1}{2}(A - B)}{\cos\tfrac{1}{2}C} = \frac{\sin\tfrac{1}{2}(a - b)}{\sin\tfrac{1}{2}c};$$

$$(3)\quad \frac{\cos\frac{1}{2}(\mathrm{A}+\mathrm{B})}{\sin\frac{1}{2}\mathrm{C}} = \frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}c};$$

$$(4)\quad \frac{\cos\frac{1}{2}(\mathrm{A}-\mathrm{B})}{\sin\frac{1}{2}\mathrm{C}} = \frac{\sin\frac{1}{2}(a+b)}{\sin\frac{1}{2}c}.$$

SOLUTION.—From A, page 25, we have,

$\sin(\frac{1}{2}\mathrm{A}+\frac{1}{2}\mathrm{B})$, or $\sin\frac{1}{2}(\mathrm{A}+\mathrm{B}) = \sin\frac{1}{2}\mathrm{A}\cos\frac{1}{2}\mathrm{B} + \cos\frac{1}{2}\mathrm{A}\sin\frac{1}{2}\mathrm{B}$.

Substituting in the second member the values of $\sin\frac{1}{2}\mathrm{A}$, $\cos\frac{1}{2}\mathrm{B}$, $\cos\frac{1}{2}\mathrm{A}$, and $\sin\frac{1}{2}\mathrm{B}$, from A'' and A''', there results,

$$\sin\tfrac{1}{2}(\mathrm{A}+\mathrm{B}) = \frac{\sin(\frac{1}{2}s-b)}{\sin c}\sqrt{\frac{\sin\frac{1}{2}s\sin(\frac{1}{2}s-c)}{\sin a\sin b}} + \frac{\sin(\frac{1}{2}s-a)}{\sin c}\sqrt{\frac{\sin\frac{1}{2}s\sin(\frac{1}{2}s-c)}{\sin a\sin b}}$$

$$= \frac{\sin(\frac{1}{2}s-b)+\sin(\frac{1}{2}s-a)}{\sin c}\sqrt{\frac{\sin\frac{1}{2}s\sin(\frac{1}{2}s-c)}{\sin a\sin b}}. \quad [\text{See (3) A'''}].$$

$$= \frac{\sin(\frac{1}{2}s-b)+\sin(\frac{1}{2}s-a)}{\sin c}\cos\tfrac{1}{2}\mathrm{C}.$$

But $\sin(\frac{1}{2}s-b)+\sin(\frac{1}{2}s-a) = \sin(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}c-b)+\sin(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}c-a)$

$= \sin[\frac{1}{2}c+\frac{1}{2}(a-b)]+\sin[\frac{1}{2}c-\frac{1}{2}(a-b)]$

$= 2\sin\frac{1}{2}c\cos\frac{1}{2}(a-b)$. (**59**, A').

Also, $\sin c = 2\sin\frac{1}{2}c\cos\frac{1}{2}c$.

Substituting these values, the preceding becomes,

$$\sin\tfrac{1}{2}(\mathrm{A}+\mathrm{B}) = \frac{2\sin\frac{1}{2}c\cos\frac{1}{2}(a-b)}{2\sin\frac{1}{2}c\cos\frac{1}{2}c}\cos\tfrac{1}{2}\mathrm{C}.$$

$$\therefore \frac{\sin\frac{1}{2}(\mathrm{A}+\mathrm{B})}{\cos\frac{1}{2}\mathrm{C}} = \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}c}. \quad (1)$$

In like manner starting with

$\sin(\frac{1}{2}\mathrm{A}-\frac{1}{2}\mathrm{B})$, or $\sin\frac{1}{2}(\mathrm{A}-\mathrm{B}) = \sin\frac{1}{2}\mathrm{A}\cos\frac{1}{2}\mathrm{B} - \cos\frac{1}{2}\mathrm{A}\sin\frac{1}{2}\mathrm{B}$,

there results, $$\frac{\sin\frac{1}{2}(\mathrm{A}-\mathrm{B})}{\cos\frac{1}{2}\mathrm{C}} = \frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}c} \quad (2)$$

Starting with $\cos(\frac{1}{2}\mathrm{A}+\frac{1}{2}\mathrm{B})$, or $\cos\frac{1}{2}(\mathrm{A}+\mathrm{B}) = \cos\frac{1}{2}\mathrm{A}\cos\frac{1}{2}\mathrm{B} - \sin\frac{1}{2}\mathrm{A}\sin\frac{1}{2}\mathrm{B}$,

there results, $$\frac{\cos\frac{1}{2}(\mathrm{A}+\mathrm{B})}{\sin\frac{1}{2}\mathrm{C}} = \frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}c} \quad (3)$$

Starting with $\cos(\frac{1}{2}\mathrm{A}-\frac{1}{2}\mathrm{B})$, or $\cos\frac{1}{2}(\mathrm{A}-\mathrm{B}) = \cos\frac{1}{2}\mathrm{A}\cos\frac{1}{2}\mathrm{B} + \sin\frac{1}{2}\mathrm{A}\sin\frac{1}{2}\mathrm{B}$,

there results, $$\frac{\cos\frac{1}{2}(\mathrm{A}-\mathrm{B})}{\sin\frac{1}{2}\mathrm{C}} = \frac{\sin\frac{1}{2}(a+b)}{\sin\frac{1}{2}c} \quad (4)$$

NAPIER'S ANALOGIES.

151. Prob.—*To deduce Napier's Analogies, which are*

$$(1)\quad \frac{\tan \frac{1}{2}(\mathsf{A}+\mathsf{B})}{\cot \frac{1}{2}\mathsf{C}} = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)};$$

$$(2)\quad \frac{\tan \frac{1}{2}(\mathsf{A}-\mathsf{B})}{\cot \frac{1}{2}\mathsf{C}} = \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)};$$

$$(3)\quad \frac{\tan \frac{1}{2}(a+b)}{\tan \frac{1}{2}c} = \frac{\cos \frac{1}{2}(\mathsf{A}-\mathsf{B})}{\cos \frac{1}{2}(\mathsf{A}+\mathsf{B})};$$

$$(4)\quad \frac{\tan \frac{1}{2}(a-b)}{\tan \frac{1}{2}c} = \frac{\sin \frac{1}{2}(\mathsf{A}-\mathsf{B})}{\sin \frac{1}{2}(\mathsf{A}+\mathsf{B})}.$$

Solution.—To deduce (1), divide the 1st of Gauss's Equations by the 3d. To deduce (2), divide the 2d of Gauss's by the 4th. To deduce (3), divide the 4th of Gauss's by the 3d. To deduce (4), divide the 2d of Gauss's by the 1st.

152. Sch.—In using these formulæ the species must be carefully attended to. Thus in (1), $\cot \frac{1}{2}\mathsf{C}$ and $\cos \frac{1}{2}(a-b)$ are necessarily +; hence $\tan \frac{1}{2}(\mathsf{A}+\mathsf{B})$ and $\cos \frac{1}{2}(a+b)$ are of the same sign with each other. In (2), $\cot \frac{1}{2}\mathsf{C}$ and $\sin \frac{1}{2}(a+b)$ are both +; hence, $\tan \frac{1}{2}(\mathsf{A}-\mathsf{B})$ and $\sin \frac{1}{2}(a-b)$ are of the same sign with each other. And similar inspections may be made upon (3) and (4).

EXERCISES.

153. The proposition that "The sines of the angles are to each other as the sines of their opposite sides" (***135***), Napier's Analogies (***151***), and formulæ A^{IV}, B^{IV} (***149***) are sufficient, in themselves, to effect the solution of all cases of oblique spherical triangles; and for practical purposes they generally require less labor than Napier's Rules. We give a few solutions and refer the student to the preceding Exercises for further practice.

1. Given $a = 100°$; $\mathsf{C} = 5°$ and $b = 10°$, to solve the triangle. (*Prob.* 1, Case 1st, ***137.***)

1st. To find A and B we have,

$$\cos \tfrac{1}{2}(a+b) : \cos \tfrac{1}{2}(a-b) :: \cot \tfrac{1}{2}\mathsf{C} : \tan \tfrac{1}{2}(\mathsf{A}+\mathsf{B});$$

and $$\sin \tfrac{1}{2}(a+b) : \sin \tfrac{1}{2}(a-b) :: \cot \tfrac{1}{2}\mathsf{C} : \tan \tfrac{1}{2}(\mathsf{A}-\mathsf{B}) \quad [\textbf{\textit{150}}\ (1)\ (2)]$$

Computing by logarithms, we have,

$$\begin{aligned}
\text{ar. co. log cos } [\tfrac{1}{2}(a+b) = 55°] &= 0.241409 \\
+ \log \cos [\tfrac{1}{2}(a-b) = 45°] &= 9.849485 \\
+ \log \cot [\tfrac{1}{2}\,\mathsf{C} = 2°\,30'] &= \underline{11.359907} \\
\text{Rejecting } 10 &= 11.450801 = \log \tan \tfrac{1}{2}(\mathsf{A}+\mathsf{B}). \\
&\therefore \tfrac{1}{2}(\mathsf{A}+\mathsf{B}) = 87°\,58'\,18''.
\end{aligned}$$

$$\begin{aligned}
\text{ar. co. log sin } [\tfrac{1}{2}(a+b) = 55°] &= 0.086635 \\
+ \log \sin [\tfrac{1}{2}(a-b) = 45°] &= 9.849485 \\
+ \log \cot [\tfrac{1}{2}\mathsf{C} = 2°\,30'] &= \underline{11.359907} \\
\text{Rejecting } 10 &= 11.296027 = \log \tan \tfrac{1}{2}(\mathsf{A}-\mathsf{B}). \\
&\therefore \tfrac{1}{2}(\mathsf{A}-\mathsf{B}) = 87°\,06'\,16''.
\end{aligned}$$

The signs of all the terms being $+$, $\frac{1}{2}(\mathsf{A}+\mathsf{B})$ and $\frac{1}{2}(\mathsf{A}-\mathsf{B})$ are both less than 90°.

$$\tfrac{1}{2}(\mathsf{A}+\mathsf{B}) + \tfrac{1}{2}(\mathsf{A}-\mathsf{B}) = \mathsf{A} = 87°\,58'\,18'' + 87°\,06'\,16'' = 175°\,04'\,34''$$

$$\tfrac{1}{2}(\mathsf{A}+\mathsf{B}) - \tfrac{1}{2}(\mathsf{A}-\mathsf{B}) = \mathsf{B} = 87°\,58'\,18'' - 87°\,06'\,16'' = 0°\,52'\,02''.$$

2d. To find c. This may be found from the proportion,

$$\sin \mathsf{A} : \sin \mathsf{C} :: \sin a : \sin c,$$

or from the 3d or 4th of Napier's Analogies. We use the last, though the first is equally expeditious.

$$\sin \tfrac{1}{2}(\mathsf{A}-\mathsf{B}) : \sin \tfrac{1}{2}(\mathsf{A}+\mathsf{B}) :: \tan \tfrac{1}{2}(a-b) : \tan \tfrac{1}{2}c.$$

$$\begin{aligned}
\text{ar. co. log sin } [\tfrac{1}{2}(\mathsf{A}-\mathsf{B}) = 87°\,06'\,16''] &= 0.000555 \\
+ \log \sin [\tfrac{1}{2}(\mathsf{A}+\mathsf{B}) = 87°\,58'\,18''] &= 9.999728 \\
+ \log \tan [\tfrac{1}{2}(a-b) = 45°] &= \underline{10.000000} \\
\text{Rejecting } 10 &= 10.000283 = \log \tan \tfrac{1}{2}c. \\
&\therefore c = 90°\,02'\,14''.
\end{aligned}$$

2. Given $\mathsf{A} = 135°\,05'\,28''.6$, $\mathsf{C} = 50°\,30'\,08''.6$, and $b = 69°\,34'\,56''.2$, to solve the triangle.

1st. To find a and c. The 3d and 4th of Napier's Analogies give,

$$\cos \tfrac{1}{2}(\mathsf{A}+\mathsf{C}) : \cos \tfrac{1}{2}(\mathsf{A}-\mathsf{C}) :: \tan \tfrac{1}{2}b : \tan \tfrac{1}{2}(a+c);$$

and

$$\sin \tfrac{1}{2}(\mathsf{A}+\mathsf{C}) : \sin \tfrac{1}{2}(\mathsf{A}-\mathsf{C}) :: \tan \tfrac{1}{2}b : \tan \tfrac{1}{2}(a-c).$$

Computing by logarithms, we have

$$\begin{aligned}
\text{ar. co. log cos } [\tfrac{1}{2}(\mathsf{A}+\mathsf{C}) = 92°\,47'\,48''.6] &= 1.3116286\ * \\
+ \log \cos [\tfrac{1}{2}(\mathsf{A}-\mathsf{C}) = 42°\,17'\,40''] &= 9.8690535 \\
+ \log \tan [\tfrac{1}{2}b = 34°\,47'\,28''.1] &= \underline{9.8418527} \\
\text{Rejecting } 10 &= 11.0225348 = \log \tan \tfrac{1}{2}(a+c). \\
&\therefore \tfrac{1}{2}(a+c) = 95°\,25'\,25''.
\end{aligned}$$

$\frac{1}{2}(a+c) > 90°$, since $\cos \frac{1}{2}(\mathsf{A}+\mathsf{C})$ is $-$, $\cos \frac{1}{2}(\mathsf{A}-\mathsf{C})$ is $+$, and $\tan \frac{1}{2}b$ is $+$, making $\tan \frac{1}{2}(a+c)$ $-$.

---

* These logarithms are taken from 7-place tables, in order to obtain the tenths of seconds accurately.

ar. co. log sin $[\frac{1}{2}(A + C) = 92° 47' 48''.6] = 0.0005176$
$+ \log \sin [\frac{1}{2}(A - C) = 42° 17' 40'' \ ] = 9.8279768$
$+ \log \tan [\frac{1}{2}b = 34° 47' 28''.1] = 9.8418527$
Rejecting 10 $= \overline{9.6703471} = \log \tan \frac{1}{2}(a - c)$.
$\therefore \frac{1}{2}(a - c) = 25° 05' 05''$.

$\frac{1}{2}(a - c) < 90°$, since the signs of the terms are all $+$.
$\frac{1}{2}(a + c) + \frac{1}{2}(a - c) = a = 120° 30' 30''$, and $\frac{1}{2}(a + c) - \frac{1}{2}(a - c) = c = 70° 20' 20''$.

2d. To find B. Either of the 1st two of Napier's Analogies will give B. Thus (1) becomes,

$$\cos \tfrac{1}{2}(a - c) : \cos \tfrac{1}{2}(a + c) :: \tan \tfrac{1}{2}(A + C) : \cot \tfrac{1}{2}B;$$

and (2) $$\sin \tfrac{1}{2}(a - c) : \sin \tfrac{1}{2}(a + c) :: \tan \tfrac{1}{2}(A - C) : \cot \tfrac{1}{2}B.$$

But as $\frac{1}{2}(a + c)$ is so near 90°, it will be better to use the second of these than the first. Or we may with equal accuracy use,

$$\sin c : \sin b :: \sin C : \sin B.$$

ar. co. log sin $(c = 70° 20' 20'') = 0.0260878$
$+ \log \sin (b = 69° 34' 56''.2) = 9.9718202$
$+ \log \sin (C = 50° 30' 08''.6) = 9.8874210$
Rejecting 10 $= \overline{9.8853290} = \log \sin B$. $\therefore B = 50° 10' 10''$.

3. Given $a = 50° 45' 20''$, $b = 69° 12' 40''$, and $A = 44° 22' 10''$, to solve the triangle.

1st. To find B. $$\sin a : \sin b :: \sin A : \sin B.$$

ar. co. log sin $(a = 50° 45' 20'') = 0.1110044$
$+ \log \sin (b = 69° 12' 40'') = 9.9707626$
$+ \log \sin (A = 44° 22' 10'') = 9.8446525$
Rejecting 10 $= \overline{9.9264195} = \log \sin B$. $\therefore B = 57° 34' 51''.4$,
and $122° 25' 08''.6$. There are two solutions, since a is intermediate in value between p and both b and $180° - b$.*

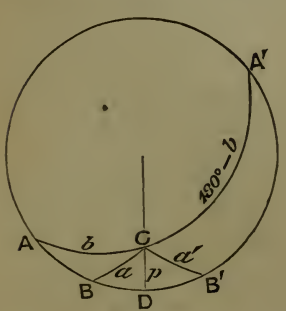

FIG. 70.

* The determination of the species of B, or what is the same thing, the number of solutions, can usually be effected by a simple inspection without any computation whatever. Thus, $\sin p = \sin b \sin A$, the logarithms of which are given above, as is $\log \sin a$. Now, as both a and p are $< 90°$, and $\log \sin p < \log \sin a$, $p < a$. But $a < b$, and also less than $180° - b$. All this can be seen at a glance.

To find C and c of the larger triangle in which B = 57° 34′ 51″.4

Napier's 1st gives

ar. co. log cos $[\frac{1}{2}(b - a) =$ 9° 13′ 40″] = 0.0056570
+ log cos $[\frac{1}{2}(b + a) =$ 59° 59′] = 9.6991887
+ log tan $[\frac{1}{2}(B + A) =$ 50° 58′ 30″.7] = 10.0912464
Rejecting 10 = 9.7960921 = log cot $\frac{1}{2}C$.
$\therefore$ C = 115° 57′ 50″.7.

Napier's 3d gives

ar. co. log cos $[\frac{1}{2}(B - A) =$ 6° 36′ 20″.7] = 0.0028928
+ log cos $[\frac{1}{2}(B + A) =$ 50° 58′ 30″.7] = 9.7991039
+ log tan $[\frac{1}{2}(b + a) =$ 59° 59′] = 10.2382689
Rejecting 10 = 10.0402656 = log tan $\frac{1}{2}c$.
$\therefore$ c = 95° 18′ 16″.4,

3d. To find C and c of the smaller triangle in which B = 122° 25′ 08″.6. Using the same *formulæ* as before.

ar. co. log cos $[\frac{1}{2}(b - a) =$ 9° 13′ 40″] = 0.0056570
+ log cos $[\frac{1}{2}(b + a) =$ 59° 59′] = 9.6991887
+ log tan $[\frac{1}{2}(B + A) =$ 83° 23′ 39″.3] = 10.9362703
Rejecting 10 = 10.6411160 = log cot $\frac{1}{2}C$.
$\therefore$ C = 25° 44′ 31″.6.

ar. co. log cos $[\frac{1}{2}(B - A) =$ 39° 01′ 29″.3] = 0.1096506
+ log cos $[\frac{1}{2}(B + A) =$ 83° 23′ 39″.3] = 9.0608369
+ log tan $[\frac{1}{2}(b + a) =$ 59° 59′] = 10.2382689
Rejecting 10 = 9.4087564 = log tan $\frac{1}{2}c$.
$\therefore$ c = 28° 45′ 05″.2.

154. Sch.—When Napier's Analogies are used for solving *Prob. 2nd* (***139***), the most expeditious and elegant method of resolving the ambiguity, is by means of the analogies themselves. Thus, in the above example, after having found that B = 57° 34′ 51″.4, or 122° 25′ 08″.6, or both, a simple inspection of the analogy next used will determine the number of solutions.

Napier's 1st may be written

$$\cot \tfrac{1}{2}C = \frac{\cos \frac{1}{2}(b + a)}{\cos \frac{1}{2}(b - a)} \tan \tfrac{1}{2}(B + A).$$

Now $\frac{1}{2}C < 90°$, hence cot $\frac{1}{2}$C is +. If, therefore, neither of the values of B renders cot $\frac{1}{2}$C −, there are *two* solutions. If one value renders cot $\frac{1}{2}$C +, and the other −, there is *one* solution and it corresponds to the value of B which makes cot $\frac{1}{2}$C +. If both values of B render cot $\frac{1}{2}$C −, there is *no* solution. In the last example, we see that cos $[\frac{1}{2}(b + a) = 59° 59']$, and cos $[\frac{1}{2}(b - a) = 9° 13' 40'']$ are both +. Also tan $[\frac{1}{2}(B + A) = 50° 58' 30''.7$, or 83° 23′ 39″.3, or both] is + for both values of B. Therefore there are *two* solutions.

4. Given $\mathsf{A} = 95° 16'$, $\mathsf{B} = 80° 42' 10''$, and $a = 57° 38'$, to solve the triangle.

1st. To find b, $\quad \sin \mathsf{A} : \sin \mathsf{B} :: \sin a : \sin b.$

ar. co. log sin ($\mathsf{A} = 95° 16'$) $= 0.001837$
+ log sin ($\mathsf{B} = 80° 42' 10''$) $= 9.994257$
+ log sin ($a = 57° 38'$) $= 9.926671$
Rejecting $10 = \overline{9.922765} = \log \sin b.$
$\therefore b = 56° 49' 57''$, or $123° 10' 03''$, or both.

2d. To find c, $\tan \frac{1}{2}c = \dfrac{\cos \frac{1}{2}(\mathsf{A} + \mathsf{B})}{\cos \frac{1}{2}(\mathsf{A} - \mathsf{B})} \tan \frac{1}{2}(a + b)$. Now for $b = 56° 49' 57''$, $\tan \frac{1}{2}c$ is $+$; but for $b = 123° 10' 03''$ $\tan \frac{1}{2}c$ is $-$; hence there is but one solution, and that corresponds to the smaller value of b.

ar. co. log cos [$\frac{1}{2}(\mathsf{A} - \mathsf{B}) = 7° 16' 55''$] $= 0.003517$
+ log cos [$\frac{1}{2}(\mathsf{A} + \mathsf{B}) = 87° 59' 05''$] $= 8.546124$
+ log tan [$\frac{1}{2}(a + b) = 57° 13' 58''$] $= 10.191352$
Rejecting $10 = \overline{8.740993} = \log \tan \frac{1}{2}c.$
$\therefore c = 6° 18' 19''.$

3d. To find C, we may use (1) or (2) of Napier's Analogies, or

$$\sin a : \sin c :: \sin \mathsf{A} : \sin \mathsf{C},$$

the last of which is the most expeditious.

ar. co. log sin ($a = 57° 38'$) $= 0.073329$
+ log sin ($c = 6° 18' 19''$) $= 9.040705$
+ log sin ($\mathsf{A} = 95° 16'$) $= 9.998163$
Rejecting $10 = \overline{9.112197} = \log \sin \mathsf{C}.$ $\therefore \mathsf{C} = 7° 26' 22'$

This value is taken for C instead of its supplement, since C is opposite the smallest side c.

5. Given $a = 70° 14' 20''$, $b = 49° 24' 10''$, and $c = 38° 46' 10''$; to solve the triangle.

COMPUTATION.

a	$= 70° 14' 20''$		
b	$= 49° 24' 10''$		
c	$= 38° 46' 10''$		
s	$= 158° 24' 40''$		
$\frac{1}{2}s$	$= 79° 12' 20''$	ar. co. log sin $=$	0.007753
$\frac{1}{2}s - a$	$= 8° 58' 00''$	" " $=$	9.192734
$\frac{1}{2}s - b$	$= 29° 48' 10''$	" " $=$	9.696370
$\frac{1}{2}s - c$	$= 40° 26' 10''$	" " $=$	9.811977
			$2)\overline{18.708834}$
		$\therefore \log k =$	$\overline{9.354417}$

$\log \tan \frac{1}{2}\mathsf{A} = \log k - \log \sin (\frac{1}{2}s - a) + 10 = 10.161683.$ $\therefore \mathsf{A} = 110^\circ\ 51'\ 16''.$
$\log \tan \frac{1}{2}\mathsf{B} = \log k - \log \sin (\frac{1}{2}s - b) + 10 = 9.658047.$ $\therefore \mathsf{B} = 48^\circ\ 56'\ 04''.$
$\log \tan \frac{1}{2}\mathsf{C} = \log k - \log \sin (\frac{1}{2}s - c) + 10 = 9.542440.$ $\therefore \mathsf{C} = 38^\circ\ 26'\ 48''.$

6. Given $\mathsf{A} = 109^\circ\ 55'\ 42''$, $\mathsf{B} = 116^\circ\ 38'\ 33''$, and $\mathsf{C} = 120^\circ 43'\ 37''$, to solve the triangle.

COMPUTATION.

$\mathsf{A} =$	$109^\circ\ 55'\ 42''$		
$\mathsf{B} =$	$116^\circ\ 38'\ 33''$		
$\mathsf{C} =$	$120^\circ\ 43'\ 37''$		
$\mathsf{S} =$	$347^\circ\ 17'\ 52''$		
$\frac{1}{2}\mathsf{S} =$	$173^\circ\ 38'\ 56''$	ar. co. log cos =	0.002683
$\frac{1}{2}\mathsf{S} - \mathsf{A} =$	$63^\circ\ 43'\ 14''$	" " =	9.646158
$\frac{1}{2}\mathsf{S} - \mathsf{B} =$	$57^\circ\ 00'\ 23''$	" " =	9.736035
$\frac{1}{2}\mathsf{S} - \mathsf{C} =$	$52^\circ\ 55'\ 19''$	" " =	9.780247
			2)19.165123
		$\therefore \log \mathsf{K} =$	9.582561

$\log \cot \frac{1}{2}a = \log \mathsf{K} - \log \cos (\frac{1}{2}\mathsf{S} - \mathsf{A}) + 10 = 9.936403.$ $\therefore a = 98^\circ\ 21'\ 38''.$
$\log \cot \frac{1}{2}b = \log \mathsf{K} - \log \cos (\frac{1}{2}\mathsf{S} - \mathsf{B}) + 10 = 9.846526.$ $\therefore b = 109^\circ\ 50'\ 20''.$
$\log \cot \frac{1}{2}c = \log \mathsf{K} - \log \cos (\frac{1}{2}\mathsf{S} - \mathsf{C}) + 10 = 9.802314.$ $\therefore c = 115^\circ\ 13'\ 28''.$

SCH. 1.—The student can use the exercises in the preceding section to familiarize the methods here given. In doing so, it will be well for him to seek the most expeditious solutions. He will find that

Examples under PROB. 1 require 11 logarithms by Napier's Analogies and (***135***), and 12 logarithms by Napier's Rules and (***135***).

Examples under PROB. 2, when there is but one solution, require 10 logarithms by Napier's Analogies and (***135***), and 12 logarithms by Napier's Rules and (***135***). When there are two solutions, 15 logarithms are required by Napier's Analogies and (***135***), and only 14 by Napier's Rules alone, or by these rules and (***135***).

Examples under PROB. 3 require but 7 logarithms by the method given in this section and 13 by the previous method.

SCH. 2.—In cases in which the angles or sides are near the limits 0°, 90°, or 180°, so that the functions used in the particular solution change very rapidly in proportion to the arc, it is often possible to select one among the several methods which will give more accurate results than the others. There are also other methods which are better adapted to such cases than those here given. For these, as well as for much other interesting matter, and especially for the discussion of the General Spherical Triangle, American students have an excellent resource in the treatise of Professor Chauvenet of Washington University, St. Louis.

SECTION IV.

AREA OF SPHERICAL TRIANGLES.

155. Prob.—*Having the angles of a spherical triangle given, to find the area.*

SOLUTION.—[The solution is given in PART II. (***613***), and we simply reproduce the result in order to give completeness to this section.] The area is equal to the ratio of the spherical excess to 90°, or $\frac{1}{2}\pi$, into the trirectangular triangle. That is, letting the sum of the angles be S°, the area K, and the radius of the sphere 1, whence the area of the trirectangular triangle is $\frac{1}{2}\pi$, we have

$$\mathsf{K} = \frac{\mathsf{S} - \pi}{\frac{1}{2}\pi} \times \tfrac{1}{2}\pi = \mathsf{S} - \pi.$$

In the latter expression S is the sum of the angles in terms of the radius, *i. e.*, $\mathsf{S} = \frac{\mathsf{S}^\circ}{57^\circ.29578}$, or approximately, $\mathsf{S} = \frac{\mathsf{S}^\circ}{57^\circ.3}$ (***9***).

EXERCISES.

1. What is the area of a spherical triangle whose angles are 100°, 58°, and 62°, on a sphere whose diameter is 6 feet?

SOLUTION. $\mathsf{K} = \mathsf{S} - \pi = \frac{220^\circ}{57^\circ.3} - 3.14159 = .698$, the area of a similar triangle on a sphere whose radius is 1. Hence, the area of the required triangle is $.698 \times 3^2 = 6.282$. [The method given in PART II. (***613***) is more expeditious, but it is our purpose to illustrate the form here given.]

2. What is the area of a spherical triangle whose angles are 170°, 135°, and 115°, on a sphere whose radius is 10 feet?

Ans. 418.875 square feet.

3. What is the area of a spherical triangle whose angles are 150°, 110°, and 60°, on a sphere whose radius is 3 feet?

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156. Prob.—*Having the sides of a spherical triangle given, to find the area.*

SOLUTION.—The angles may be found by (***148***), and then the area by (***155***).

But a more direct method is to find the spherical excess by means of *Lhuilier's* formula, which we will now produce.

$$\tfrac{1}{4}K = \tfrac{1}{4}(A + B + C - \pi)$$

Whence $\tan \tfrac{1}{4}K = \tan \tfrac{1}{4}[A + B + C - \pi] = \tan [\tfrac{1}{4}(A + B) - \tfrac{1}{4}(\pi - C)]$

$$= \frac{\sin \tfrac{1}{2}(A + B) - \sin \tfrac{1}{2}(\pi - C)}{\cos \tfrac{1}{2}(A + B) + \cos \tfrac{1}{2}(\pi - C)} \qquad \text{(7, page } \textit{31}\text{)}$$

$$= \frac{\sin \tfrac{1}{2}(A + B) - \cos \tfrac{1}{2}C}{\cos \tfrac{1}{2}(A + B) + \sin \tfrac{1}{2}C}$$

$$= \frac{[\cos \tfrac{1}{2}(a - b) - \cos \tfrac{1}{2}c] \cos \tfrac{1}{2}C}{[\cos \tfrac{1}{2}(a + b) + \cos \tfrac{1}{2}c] \sin \tfrac{1}{2}C} \qquad \text{(}\textit{150}\text{, 1st and 3d)}$$

$$= \frac{\cos \tfrac{1}{2}(a - b) - \cos \tfrac{1}{2}c}{\cos \tfrac{1}{2}(a + b) + \cos \tfrac{1}{2}c} \sqrt{\frac{\sin \tfrac{1}{2}s \sin (\tfrac{1}{2}s - c)}{\sin (\tfrac{1}{2}s - a) \sin (\tfrac{1}{2}s - b)}} \qquad \text{(}\textit{146, 147}\text{)}$$

$$= \frac{\sin \tfrac{1}{4}(a + c - b) \sin \tfrac{1}{4}(b + c - a)}{\cos \tfrac{1}{4}(a + b + c) \cos \tfrac{1}{4}(a + b - c)} \sqrt{\frac{\sin \tfrac{1}{2}s \sin (\tfrac{1}{2}s - c)}{\sin (\tfrac{1}{2}s - a) \sin (\tfrac{1}{2}s - b)}} \qquad \text{(}\textit{59}\text{, C', D')}$$

$$= \sqrt{\frac{\sin^2 \tfrac{1}{2}(\tfrac{1}{2}s - b) \sin^2 \tfrac{1}{2}(\tfrac{1}{2}s - a) \sin \tfrac{1}{2}s \sin (\tfrac{1}{2}s - c)}{\cos^2 \tfrac{1}{4}s \cos^2 \tfrac{1}{2}(\tfrac{1}{2}s - c) \sin (\tfrac{1}{2}s - a) \sin (\tfrac{1}{2}s - b)}} \qquad (\text{as } s = a + b + c)$$

$$= \sqrt{\frac{\sin^2 \tfrac{1}{2}(\tfrac{1}{2}s - b) \sin^2 \tfrac{1}{2}(\tfrac{1}{2}s - a) \sin \tfrac{1}{4}s \cos \tfrac{1}{4}s \sin \tfrac{1}{2}(\tfrac{1}{2}s - c) \cos \tfrac{1}{2}(\tfrac{1}{2}s - c)}{\cos^2 \tfrac{1}{4}s \cos^2 \tfrac{1}{2}(\tfrac{1}{2}s - c) \sin \tfrac{1}{2}(\tfrac{1}{2}s - a) \cos \tfrac{1}{2}(\tfrac{1}{2}s - a) \sin \tfrac{1}{2}(\tfrac{1}{2}s - b) \cos \tfrac{1}{2}(\tfrac{1}{2}s - b)}} \qquad \text{(}\textit{56}\text{).}$$

$$\therefore \text{Tan } \tfrac{1}{4}K =^{*} \sqrt{\tan \tfrac{1}{4}s \tan \tfrac{1}{2}(\tfrac{1}{2}s - a) \tan \tfrac{1}{2}(\tfrac{1}{2}s - b) \tan \tfrac{1}{2}(\tfrac{1}{2}s - c)}. \qquad \text{(A)}$$

Having found K, the spherical excess, or what is the same thing, the area of a similar triangle on a sphere whose radius is 1, we have but to multiply K by the square of the radius in any given case.

EXERCISES.

1. Given $a = 98°$, $b = 110°$, and $c = 115°$, to find the area of a spherical triangle, on a sphere whose radius is 4000 miles.

COMPUTATION.

log tan ($\tfrac{1}{4}s$	$= 80° 45') = 10.788185$
+ log tan [($\tfrac{1}{4}s - \tfrac{1}{2}a$)	$= 31° 45'] = 9.791563$
+ log tan [($\tfrac{1}{4}s - \tfrac{1}{2}b$)	$= 25° 45'] = 9.683356$
+ log tan [($\tfrac{1}{4}s - \tfrac{1}{2}c$)	$= 23° 15'] = 9.633098$
	$2)39.896202$
Rejecting 10	$= 9.948101 = \log \tan \tfrac{1}{4}K.$

$\therefore K = 166° 20' 20''$

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* The + is always taken; otherwise, $\tfrac{1}{4}K$ being $> 90°$, K would be $> 360°$ which is impossible. (PART III., 256).

Whence area $= \frac{166° 20' 20''}{90°} \cdot \frac{1}{2}\pi (4000)^2 = \frac{166.34}{90} \cdot \frac{1}{2}\pi (4000)^2 = 46{,}450{,}440$, nearly.

2. Given $a = 70° 14' 20''$, $b = 49° 24' 10''$, and $c = 38° 46' 10''$, to find the area of a spherical triangle on a sphere whose diameter is 8 feet. *Ans.* 5.1, nearly.

***157. Prob.**—Having two sides and their included angle given in a spherical triangle, to find the area.*

Solution.—Compute the other two angles by Napier's Analogies, and find the area from the angles. [The formula $\cot \frac{1}{2}K = \frac{\cot \frac{1}{2}a \cot \frac{1}{2}b + \cos C}{\sin C}$ gives the spherical excess in terms of two sides and their included angle; but it is of no practical value for finding the area, as it is not adapted to logarithmic computation. For the manner of producing it and several other forms for K, see Todhunter's Spherical Trigonometry, (***103***)].

PRACTICAL APPLICATIONS.

[Note.—The three following problems are given merely to indicate to the student some departments of investigation in which Spherical Trigonometry is of essential service. The two sciences to which this branch of Pure Mathematics is indispensable, are Geodesy, or the mathematical measurement of the earth, and Astronomy.]

***Prob. 1.**—To find the shortest distance on the earth's surface between two points whose latitudes and longitudes are known.*

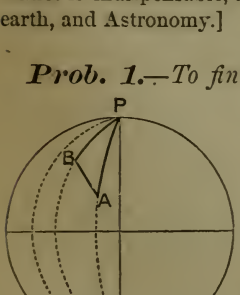

Fig. 71.

Sug's.—The shortest distance on the surface between two points is the arc of a great circle joining the points. Hence, the Problem is: Given two sides (the co-latitudes) and the included angle (the difference in longitude), to find the third side.

Ex. 1. Berlin is situated in Lat. 52° 31' 13'' N., Lon. 13° 23' 52'' E., and Alexandria, Egypt, in Lat. 31° 13' N., Lon. 29° 55' E. What is the shortest distance in miles on the earth's surface between them, the earth being considered a sphere whose radius is 3962 miles?

Ans. 1691.96 miles.

Ex. 2. A ship starts from Valparaiso, Chili, Lat. 33° 02′ S., Lon. 71° 43′ W., and sails on the arc of a great circle in a northwesterly direction 3840 miles, when her longitude is found to be 120° W. What is her latitude? *Ans.* 52′ 48″ S.

Ex. 3. A ship starts from Rio Janeiro, Brazil, Lat. 22° 54′ S., Lon. 42° 45′ W., and sails in a northeasterly direction on the arc of a great circle 624.44 miles, when her latitude is found to be 50° N. What is her longitude? *Ans.* 2° 01′ 14″ W.

Prob. 2.—*To find the time of day from the altitude of the sun.*

Sug's.—Let NESQ represent the projection of the concave of the heavens upon the plane of the meridian of observation. The equator of this concave sphere is simply the intersection of the plane of the earth's equator with this imaginary concave sphere, and its axis is the prolongation of the earth's axis, the poles being the points N and S where the axis pierces the imaginary concave. EQ is the projection of this celestial equator, NS the axis or the projection of a great circle perpendicular to the meridian of the observer (NESQ) and to the equator, HO the projection of the horizon, and ZZ′ the projection of the prime vertical (that is, a great circle of the heavens passing through the zenith of the observer and the east and west points in his horizon).

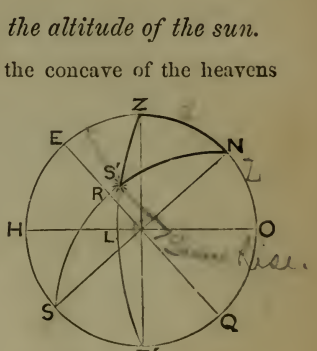

Fig. 72.

Now let S′ be the place of the sun at the time of observation. RS′, the sun's declination, is known from the almanac; LS′, the sun's altitude, is measured with the sextant (or other instrument); and EZ is the latitude of the observer. Hence, in the spherical triangle ZNS′ we know the three sides, viz., NS′ = the co-declination of the sun, ZS′ = the co-altitude of the sun, and ZN = the co-latitude of the observer. We may therefore compute the angle ZNS′, which reduced to time gives the time before or after noon as the case may be.

Ex. 1. On April 21st the master of a ship at sea in latitude 39° 06′ 20″ N., observed the altitude of the sun's centre at a certain time in the forenoon and found it to be 30° 10′ 40″, and looking in the almanac found the sun's declination at that time to be 12° 03′ 10″ N. What was the time of day?

COMPUTATION.

90° − 30° 10′ 40″ = 59° 49′ 20″	a. c. log sin 94° 19′ 55″ = 0.001242
90° − 12° 03′ 10″ = 77° 56′ 50″	a. c. log sin 34° 30′ 35″ = 0.246765
90° − 39° 06′ 20″ = 50° 53′ 40″	log sin 16° 23′ 05″ = 9.450381
2)188° 39′ 50″	log sin 43° 26′ 15″ = 9.837312
94° 19′ 55″	2)19.535700
	log tan 30° 22′ 03″.3 = 9.767850

Therefore ½ the hour angle NNS′ = 30° 22′ 03″.3, and the hour angle is 60° 44′ 07″. This reduced to time at 4 minutes to a degree, gives 4 h. 2 m. 56 s. before noon, or 7 h. 57 m. 4 s. A. M.

Ex. 2. In latitude 40° 21′ N., when the declination of the sun is 3° 20′ S., and its altitude 36° 12′, what is the time of day?

Ans. 9 h. 42 m. 40 s. A. M.

Ex. 3. In latitude 21° 02′ S., when the sun's declination was 18° 32′ N., and the altitude in the afternoon 40° 08′, what was the time of day? *Ans.* 2 h. 3 m. 57″ P. M.

Prob. 3.—*To find the time of sunrising and sunsetting at any given place on a given day.*

Sᴜɢ's.—The projection being the same as before, let M′RS′M represent the apparent diurnal path of the sun. Since S′M is described in 6 hours, the time taken to describe RS′ is the time before 6 o'clock, at which the sun rises, *i. e.*, passes the horizon HO. But the time requisite to describe RS′, is the same part of 24 hours (360° angular measure) that the angle CNL (= arc CL) is of 360°. Hence, the arc CL, in time, is the time before 6 o'clock at which the sun rises. In a similar manner, C*l*, in time, is seen to be the time *after* 6 o'clock when the sun is south of the equator. The solution of the problem, therefore, consists in finding CL. Now, in the triangle RLC, right angled at L, LR = the sun's declination at the time, and angle RCL = ECH = the co-latitude of the place.* From these data CL is readily found.

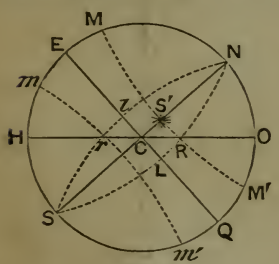

Fig. 73.

Ex. 1. Required the time of sunrise at latitude 42° 33′ N., when the sun's declination is 13° 28′ N.

COMPUTATION.

cot 47° 27′	= 9.962813
tan 13° 28′	= 9.379239
sin 12° 41′ 52″	= 9.342052

(12° 41′ 52″) × 4 gives the time before 6 o'clock as 50′ 47″. ∴ The sun rises at 5 h. 09 m. 13 s.

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* This may be seen thus: Suppose a person to start from the equator at E and travel north. When he is at E, the south point of his horizon (H) is at S; and for every degree he goes north, the south pole (S) sinks a degree below his horizon. Hence, HCS = his latitude, and ECH = co-latitude.

Ex. 2. Required the time of sunrise at latitude 57° 02′ 54″ N., when the sun's declination is 23° 28′ N.

Sun rises at 3 h. 11 m. 49 s.

Ex. 3. How long is the sun above the horizon in latitude 58° 12′ N., when the sun's declination is 18° 41′ S., that is about January 25th? *Ans.* 7 h. 35 m. 36 s.

Ex. 4. What is the length of the longest day at Ann Arbor, Mich., Lat. 42° 16′ 48″.3, the sun's greatest declination being 23° 27′?

Ans. 15 h. 05 m. 50 s.

[NOTE.—In such problems as the foregoing, several small corrections have to be made in order to entire accuracy, such, for example, as that for refraction in taking the altitude, and for the time required for the sun's disk to pass the horizon. But they would be out of place here.]

THE END.

INTRODUCTION

TO THE

TABLE OF LOGARITHMS.

[NOTE.—If the student understands the nature and use of logarithms so as to be able to use the common tables, it will not be necessary that he should read this introduction. Otherwise a careful study of it will be needful before reading Section 4 of the Plane Trigonometry.]

1. A Logarithm is the exponent by which a fixed number is to be affected in order to produce any required number. The fixed number is called the *Base* of the System.

ILL. Let the *Base* be 3: then the logarithm of 9 is 2; of 27, 3; of 81, 4; of 19623, 9; for $3^2 = 9$; $3^3 = 27$; $3^4 = 81$; and $3^9 = 19683$. Again, if 64 is the base, the logarithm of 8 is $\frac{1}{2}$, or .5, since $64^{\frac{1}{2}}$, or $64^{.5} = 8$; *i. e.*, $\frac{1}{2}$, or .5 is the exponent by which 64, the base, is to be affected in order to produce the number 8. So also, 64 being the base, $\frac{1}{3}$, or .333 +, is the logarithm of 4, since $64^{\frac{1}{3}}$, or $64^{.333+} = 4$; *i. e.*, $\frac{1}{3}$, or .333 +, is the exponent by which 64, the base, is to be affected in order to produce the number 4. Once more, since $64^{\frac{2}{3}}$, or $64^{.666+} = 16$, $\frac{2}{3}$, or .666 +, is the logarithm of 16, if the base is 64. Finally, $64^{-\frac{1}{2}}$, or $64^{-.5} = \frac{1}{8}$, or .125; hence $-\frac{1}{2}$, or $-.5$, is the logarithm of $\frac{1}{8}$, or .125, when the base is 64. In like manner, with the same base, $-\frac{1}{3}$, or $-.333+$, is the logarithm of $\frac{1}{4}$, or .25.

EXAMPLES.

1. If 2 is the base what is the logarithm of 4? of 8? of 32? of 128? of 1024?

SOLUTION. 7 is the logarithm of 128, if 2 is the base, since 7 is the exponent by which 2 is to be affected in order to produce the number 128.

2. If 5 is the base, what is the logarithm of 625? of 15625? of 125? of 25?

3. If 10 is the base, what is the logarithm of 100? of 1000? of 10,000? of 10,000,000?

4. If 2 is the base, what is the logarithm of $\frac{1}{4}$, or .25? of $\frac{1}{8}$, or .125? of $\frac{1}{32}$, or .03125? *Ans.* to the last, -5.

5. If 8 is the base, of what number is $\frac{2}{3}$, or .666 + the logarithm? of what number is $\frac{4}{3}$, or 1.333 +, the logarithm? of what number is 2 the logarithm? of what number is $2\frac{1}{3}$, or 2.333 + ? of what number $3\frac{2}{3}$, or 3.666 + ? *Ans.* to the last, 2048.

SCH. Since any number with 0 for its exponent is 1, the logarithm of 1 is 0 in all systems. Thus $10^0 = 1$, whence 0 is the logarithm of 1, in a system in which the base is 10.

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2. A System of Logarithms is a scheme by which all numbers can be represented, either exactly or approximately, by exponents by which a fixed number (the base) can be affected.

3. There are *Two Systems* of Logarithms in common use, called, respectively, the *Briggean* or *Common* System, and the *Napierian* or *Hyperbolic* System. The base of the former is 10, and of the latter 2.71828 +. In speaking of logarithms of numbers, the common logarithm is always signified, if no specification is made.

4. The logarithms of all numbers, except the exact powers of the base, indicate a power of a root, and are consequently fractional and usually only approximations. It is customary to write them in the form of decimal fractions. The integral part is called the ***Characteristic,*** and the fractional part the ***Mantissa.*** The characteristic can always be told by a simple inspection of the number itself; hence only the mantissa is commonly given in the table.

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5. Prop.—*The characteristic of the common logarithm of any number greater than unity, is one less than the number of integral figures in the given number.*

ILL. The logarithm of 4685 is more than 3, because $10^3 = 1000$, and less than 4, because $10^4 = 10{,}000$; hence it is 3 + a fraction. The same method may be pursued to determine the characteristic of the logarithm of any other number greater than unity, and the truth of the proposition be observed. Thus the logarithm of 25645.827 is 4, since the number lies between the 4th and 5th powers of the base, 10.

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6. Prop.—*The mantissa of a decimal fraction, or of a mixed number, is the same as the mantissa of the number considered as integral.*

DEM. Suppose it is known log 2845672 = 6.454185. This means that $10^{6.454185} = 2845672$. Dividing by 10 successively, we have

$10^{5.454185} = 284567.2$, or log 284567.2* = 5.454185,
$10^{4.454185} = 28456.72$, or log 28456.72 = 4.454185,
$10^{3.454185} = 2845.672$, or log 2845.672 = 3.454185,
$10^{2.454185} = 284.5672$, or log 284.5672 = 2.454185,
$10^{1.454185} = 28.45672$, or log 28.45672 = 1.454185,
$10^{0.454185} = 2.845672$, or log 2.845672 = 0.454185.

Now if we continue the operation of division, only writing $0.454185 - 1$, $\bar{1}.454185$, meaning by this that the characteristic is negative and the mantissa positive, and the subtraction not performed, we have

$10^{\bar{1}.454185} = 0.2845672$, or log 0.2845672 = $\bar{1}.454185$,
$10^{\bar{2}.454185} = 0.02845672$, or log 0.02845672 = $\bar{2}.454185$,
$10^{\bar{3}.454185} = 0.002845672$, or log 0.002845672 = $\bar{3}.454185$,

etc., etc. Q. E. D.

7. COR. *The characteristic of a number consisting entirely of a decimal fraction, is negative, and one more than the number of* 0's *immediately following the decimal point,* as appears from the last demonstration, or from the fact that $1^{-1} = \frac{1}{10} = .1$; $10^{-2} = \frac{1}{100} = .01$; $10^{-3} = \frac{1}{1000} = .001$; etc., etc.

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8. One of the most important uses of logarithms is to facilitate the multiplication, division, involution, and extraction of roots of large numbers. These processes are performed upon the following principles:

9. *Prop. 1.*—*The sum of the logarithms of two numbers is the logarithm of their product.*

DEM. Let a be the base of the system. Let m and n be any two numbers whose logarithms are x and y respectively. Then by definition $a^x = m$, and $a^y = n$. Multiplying these equations together we have $a^{x+y} = mn$. Whence $x + y$ is the logarithm of mn. Q. E. D.

10. *Prop. 2.*—*The logarithm of the quotient of two numbers is the logarithm of the dividend minus the logarithm of the divisor.*

DEM. Let a be the base of the system, and m and n any two numbers whose logarithms are, respectively, x and y. Then by definition we have $a^x = m$, and $a^y = n$. Dividing, we have $a^{x-y} = \frac{m}{n}$. Whence $x - y$ is the logarithm of $\frac{m}{n}$. Q. E. D.

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* This is the common abbreviation indicating the logarithm of a number, and should be read "logarithm of 284567.2," not "*log* 284567.2," which is grossly inelegant.

11. Prop. 3.—*The logarithm of a power of a number is the logarithm of the number multiplied by the index of the power.*

DEM. Let a be the base, and x the logarithm of m. Then $a^x = m$; and raising both to any power, as the zth, we have $a^{xz} = m^z$. Whence xz is the logarithm of the zth power of m. Q. E. D.

12. Prop. 4.—*The logarithm of any root of a number is the logarithm of the number divided by the number expressing the degree of the root.*

DEM. Let a be the base, and x the logarithm of m. Then $a^x = m$. Extracting the zth root we have $a^{\frac{x}{z}} = \sqrt[z]{m}$. Whence $\frac{x}{z}$ is the logarithm of $\sqrt[z]{m}$. Q. E. D.

TABLE OF LOGARITHMS.

13. In order to apply the above principles practically, we need what is called a *Table of Logarithms.* That is, a table from which we can readily obtain the logarithm of any number, or the number corresponding to any logarithm. Such a table is given on pages 11 to 28. For methods of computing it, the student is referred to algebra. Its nature and manner of use will be learned from the two following problems:

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14. Prob.—*To find the logarithm of a number from the table.*

SOLUTION. *The logarithm of any number between* 1 *and* 100 is seen directly from the table on page 11. The column marked N contains the numbers, and the adjacent column to the right gives the logarithm of the corresponding number to 6 places of fractions. Thus, $\log 7 = 0.845098$; $\log 33 = 1.518514$.

The mantissa of the logarithm of any number expressed with three figures is found in the column headed 0, on some one of the pages from 12 to 26 inclusive. The given number being found in the column marked N, the mantissa of its logarithm stands opposite. The characteristic can be determined by (*5*), (*6*), (*7*). When but four figures are found opposite the number in the 0 column, the two left-hand figures of the mantissa are the same as in the next mantissa above, which has six. Thus, $\log 443 = 2.646404$.

To find the logarithm of a number consisting of four figures. Let it be required to find the logarithm of 2936. Looking for 293 (the first three figures) in the column of numbers, and then passing to the right until reaching the column headed 6, the fourth figure, we find 7756, to which prefixing the figures 46, which belong to all the logarithms following them till some others are indicated, we have for the mantissa of the logarithm of 2936, .467756. But, as 3 is the

logarithm of 1000, and 4 of 10,000, log 2936 is 3 and this decimal, or log 2936 = 3.467756.

To find the logarithm of a number consisting of more than four figures. Let it be required to find the logarithm of 2845672. Finding the decimal part of logarithm of the first four figures 2845, as before, we find it to be .454082. Now the logarithm of 2846 is 153 (millionths, really) more than that of 2845, as found in the right-hand column, marked D. Hence, assuming that if an increase of the number by 1000 makes an increase in its logarithm of 153, an increase of 672 in the number will make an increase in the logarithm of $\frac{672}{1000}$, or .672 of 153, or 103, omitting lower orders, and adding this to .454082, we have .454185 as the mantissa of log 2845672. The integral part is 6, since 2845672 lies between the 6th and 7th powers of 10. Hence, log 2845672 = 6.454185. Q. E. D.

Sch. 1. If in seeking the logarithm of any number, any of the heavy dots noticed in the table are passed, their places are to be filled with 0's, and the first two figures of the decimal of the logarithm taken from the 0 column in the line below. Thus, log 3166 is 3.500511. This arrangement of the table is merely a convenient method of saving space.

Sch. 2. The column marked D is called the column of Tabular Differences; and any number in it is the difference between the mantissas found in columns 4 and 5, which is usually the same as between any two consecutive logarithms in the same horizontal line. The assumption made in using this difference, viz., that the logarithms increase in the same ratio as the numbers, is only approximately true, but still is accurate enough for ordinary use.

EXERCISES.

1. Find the logarithm of 2200. . . . , . . . *Logarithm,* 3.342423.
2. Find the logarithm of 24.36. . . . . *Logarithm,* 1.386677.
3. Find the logarithm of 2.698. . . . . *Logarithm,* 0.431042.
4. Find the logarithm of 3585.9 . . . . . *Logarithm,* 3.554598.
5. Find the logarithm of 42.6634 . . . . . *Logarithm,* 1.630056.
6. Find the logarithm of 331.957 . . . . . *Logarithm,* 2.521082.
7. Find the logarithm of 2519.38 . . . . . *Logarithm,* 3.401294.
8. Find the logarithm of .538329 . . . . . *Logarithm,* $\bar{1}$.731047.
9. Find the logarithm of .087346 . . . . . *Logarithm,* $\bar{2}$.941243.
10. Find the logarithm of .007389 . . . . . *Logarithm,* $\bar{3}$.868586.

15. Sch. 3. It will be observed that the tabular difference varies rapidly at the beginning of the table; hence, for numbers between 10000 and 11000 it is better to use the last two pages of the table.

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16. Prob 2.—*To find a number corresponding to a given logarithm.*

SOLUTION. Let it be required to find the number corresponding to the logarithm 5.515264. Looking in the table for the *next less* mantissa, we find .515211, the number corresponding to which is 3275 (no account now being taken as to whether it is integral, fractional, or mixed; as in any case the figures will be the same). Now from the *tabular difference*, in column D, we find that an increase of 133 (millionths, really) upon this logarithm (.515211), would make an increase of 1 in the number, making it 3276. But the given logarithm is only 53 greater than this, hence it is assumed (though only approximately correct) that the increase of the number is $\frac{53}{133}$ of 1, or $53 \div 133 = .3984 +$. This added (the figures annexed) to 3275, gives 32753984 +. The characteristic, being 5, indicates that the number lies between the 5th and 6th powers of 10, and hence has 6 integral places. $\therefore$ 5.515264 = log. 327539.84 +. Q. E. D.

EXERCISES.

1. Find the number whose logarithm is 1.240050.
   Number, 17.38.
2. Find the number whose logarithm is 2.431203.
   Number, 269.9.
3. Find the number whose logarithm is 3.503780.
   Number, 3189.91.
4. Find the number whose logarithm is 0.138934.
   Number, 1.377.
5. Find the number whose logarithm is $\bar{1}$.368730.
   Number, .233738.
6. Find the number whose logarithm is $\bar{2}$.448375.
   Number, .028078.
7. Find the number whose logarithm is $\bar{3}$.538630.
   Number, .003456.
8. Find the number whose logarithm is .843970.
   Number, 6.98184.
9. Find the number whose logarithm is $\bar{1}$.867372.
   Number, .736837.
10. Find the number whose logarithm is .003985.
    Number, 1.00921.
11. Find the number whose logarithm is $\bar{3}$.723460.
    Number, .005290.

APPLICATIONS

1. Find, by means of logarithms, the product of 57.98 by 18.

SOLUTION. As the logarithm of the product equals the sum of the logarithms of the factors (9), we find the logarithms of 57.98, and 18 from the table, and

adding them together, find the number corresponding to the sum. The latter number is the product required. Thus,

$$\log 57.98 = 1.763278$$
$$\log 18 = \underline{1.225273}$$
$$3.018551 = \log 1043.64.$$

2. Multiply 23.14 by 5.062.	*Prod.*	117.1347.
3. Multiply 0.00563 by 17.	*Prod.*	0.09571.
4. Multiply 397.65 by 43.78.	*Prod.*	17409.117.
5. Multiply 0.3854 by 0.0576.	*Prod.*	0.022199.
6. Find the continued product of 3.902, 597.16, and 0.0314728.	*Prod.*	73.3354.
7. Multiply 832403 by 30243.	*Prod.*	25174363929.*
8. Multiply 9703407 by 90807.	*Prod.*	881137279449.
9. Multiply 3.47 by 9.83.	*Prod.*	34.1101.
10. Multiply 12.763 by 10.976.	*Prod.*	140.086688.*

[NOTE. The examples in division below will offer additional exercise, if necessary.]

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1. Divide 24163 by 4567.

SOLUTION. Since the logarithm of the quotient equals the logarithm of the dividend minus the logarithm of the divisor, we have the following operation:

$$\log 24163 = 4.383151$$
$$\log 4567 = \underline{3.659631}$$
$$0.723520 = \log 5.29078,$$

which number is the quotient.

2. Divide 56.4 by 0.00015.

OPERATION. $\log 56.4 = 1.751279$
$\log 0.00015 = \overline{4}.176091$
Difference of log's $= 5.575188$ $\therefore$ The quotient is 376000.

SUG. Observe that only the *characteristic* of the logarithm is negative, and that in subtracting we are to regard the nature of the logarithmic numbers as positive or negative.

3. Divide 461.02876 by 21.4.	*Ans.*	21.5434.
4. Divide 25.49052 by 2.46.	*Ans.*	10.362.
5. Divide 17610.8248 by 37.6.	*Ans.*	468.373.
6. Divide .00144 by 1.2.	*Ans.*	.0012.
7. Divide .0000025 by .005.	*Ans.*	.0005.

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* Ordinary 6 or 7 place logarithms will not give these products correct. When

8. Divide 43.2 by .24. *Ans.* 180
9. Divide 59.74514 by 1.36. *Ans.* 43.93025.
10. Divide .0001728 by 2.4. *Ans.* .000072.

[NOTE. The examples in multiplication given above will afford additional exercise, if necessary.]

17. SCH.—*Arithmetical Complement.*—The arithmetical complement of a number is simply the remainder after subtracting the number from some particular fixed number. Thus, the a. c. of 5 with reference to 9 is 4; of 3, 6; of 7, 2; etc. The a. c. of 7 with reference to 10 is 3; of 4, 6; of 2, 8; etc. When required to subtract one number from another, we may, if we choose, *add* its a. c. and then subtract the number with reference to which the a. c. is taken. This process will give the true difference. Thus, if we are to subtract 6 from 9, we may *add* to 9 what 6 lacks of being 10 ($10 - 6 = 4$, the a. c. of 6 with reference to 10) and then subtract 10. $9 - 6 = 9 + 4 - 10$. A few such questions as the following will render this simple process familiar. What number must I add to 576, in order that I may subtract 100 from the sum, and get the same remainder as if I had subtracted 58 in the first instance? Again, if I wish to take 37 from 160, what must I add to the latter, in order that I may subtract 40 from the result, and get the difference sought?

This principle is sometimes used in computing by means of logarithms. It is especially convenient when there are several multipliers and several divisors involved in the same computation. An example or two will make the process familiar. The complements of logarithms are usually taken with reference to 10. If the logarithm exceeds 10, 20 may be used, etc.

18. Required the result of the following combinations: $346 \times 27.8 \div 1156 \times 3426 \div 2.004 \times 27 \div 11.05$.

OPERATION.

	log 346 =	2.539076
	log 27.8 =	1.444045
a. c.	log 1156 =	6.937042
	log 3426 =	3.534787
a. c.	log 2.004 =	9.698102
	log 27 =	1.431364
a. c.	log 11.05 =	8.956638
		34.541054
Rejecting		30.000000 as three complements are
used.		4.541054 is the logarithm of the re-

quired result. ∴ As $4.541054 = \log 34757.92$, the latter is the result sought.

[NOTE. The preceding examples can be used to familiarize this principle if thought desirable.]

SUG. The a. c. of $\bar{2}.468216$ is 11.531784, since 2 is negative. An a. c. can be written directly from the table with nearly the same ease as the logarithm itself, by writing from left to right, and taking each figure from 9, except the

right-hand one, which is to be taken from 10. Thus, if the characteristic is 3, we write 6 · the next figure being 2, write 7; for 4, write 5, etc.

---

1. What is the cube of 32?

SOLUTION. Since the logarithm of the cube of a number is three times the logarithm of the number itself (***11***), we have

$$\log (32)^3 = 3 \log 32 = 4.515450 = \log 32767.97,$$

which number is the cube of 32, as accurately as the process gives it. $(32)^3$ by multiplication = 32768.

2. What is the cube root of 7896.34?

SUG. $\text{Log}\,(7896.34)^{\frac{1}{3}} = \frac{1}{3} \log 7896.34 = 1.299142 = \log 19.913.$ $\therefore (7896.34)^{\frac{1}{3}} = 19.913.$ (See ***12***.)

3. What is the 20th power of 1.06? *Ans.* 3.2071.
4. What is the 5th power of 2.846?
5. What is the 5th root of 31152784.1? *Ans.* 31.52+.
6. What is the cube root of 30? *Ans.* 3.107+.
7. What is the cube root of .03?

SUG. Log .03 = $\bar{2}.477121$. Now to divide this by 3, we have to bear in mind that the characteristic *alone* is negative; *i. e.*, $\bar{2}.477121 = -2 + .477121$, or -1.522879. This divided by 3 gives $-.507626$, or $0 - .507626 = \bar{1}.492374$. But a more convenient method of effecting this division is to write for the -2, $-3+1$, whence we have for $\bar{2}.477121$, $-3 + 1.477121$, which divided by 3 gives $\bar{1}.492374$, nearly.

8. Divide $\bar{3}.261453$ by 2, by 4, by 5. *Last quotient*, $\bar{1}.4522906$.
9. What is the 4th root of .00000081? *Ans.* .03.
10. What is the 7th root of 0.005846? *Ans.* .4797.

---

1. If 28.035 : 3.2781 :: 3114.27 : x, what logarithmic operations will find x?

SUG. The logarithm of the product of the means is the sum of their logarithms; and the logarithm of the quotient of this product divided by the first extreme, is the logarithm of said product minus the logarithm of the other extreme. $\therefore \log x = \log 3.2781 + \log 3114.27 - \log 28.035 = 0.515622 + 3.493356 - 1.447700 = 2.561278$. Hence, $x = 364.1478+$.

2. Given 72.34 : 2.519 :: 357.48 : x, to find x, by logarithms.
$x = 12.448.$
3. Given 6853 : 489 :: 38750 : x, to find x, by logarithms.
$x = 2765.$

SUG. The most elegant way to solve such propositions by logarithms is to take the sum of the logarithms of the means and the a. c. of the given extreme and reject 10. The result is log x.

4. Given 497 : 1891 :: 376 : x, to find x, using the a. c. log.

OPERATION.

$$\begin{aligned} \log\ 1891 &= 3.276692 \\ \log\ 376 &= 2.575188 \\ \text{a. c. log}\ 497 &= \underline{7.303644} \\ \text{Sum, less } 10 &= 3.155524 = \log 1430.62. \quad \therefore\ x = 1430.62. \end{aligned}$$

[NOTE. Solve the preceding in a similar manner, by using a. c. log.]

Let the student give the reasons for the following:

1. Given $(\frac{2}{3})^2 \div (\frac{5}{16})^{\frac{1}{4}} = x$, we have

$$\begin{aligned} 2 \log\ 2 &= 0.602060 \\ \tfrac{1}{4} \log\ 16 &= 0.903090 \\ \text{a. c. } 2 \log\ 3 &= 9.045758 \\ \text{a. c. } \tfrac{1}{4} \log\ 5 &= \underline{9.475773} \\ \text{Sum, less } 20 &= 0.026681. \quad \therefore\ x = 1.0633+. \end{aligned}$$

2. Given $\sqrt{\frac{2}{5}} : x :: (3\frac{1}{3})^2 : \sqrt[3]{6^2}$, to find x.

Log $x = 2 \log 3 + \frac{1}{2} \log 2 + \frac{2}{3} \log 6 +$ a. c. $2 \log 10 +$ a. c. $\frac{1}{2} \log 5 - 20 = .274039$. $\therefore\ x = 0.1879+$.

3. Given $\sqrt{115} \times \sqrt[3]{.016} : (0.0051)^2 :: x : \dfrac{(.32)^{\frac{1}{2}}}{(1146)^{\frac{2}{3}}}$.

Log $x = \frac{1}{2} \log 115 + \frac{1}{3} \log .016 + \frac{1}{2} \log .32 +$ a. c. $\frac{2}{3} \log 1146 +$ a. c. $2 \log .0051 - 20 = 2.729701$. $\therefore\ x = 536.66+$.

TABLE I.

CONTAINING

LOGARITHMS OF NUMBERS

From 1 to 11,000.

N.	Log.	N.	Log.	N.	Log.	N.	Log.
1	0·000000	26	1·414973	51	1·707570	76	1·880814
2	0·301030	27	1·431364	52	1·716003	77	1·886491
3	0·477121	28	1·447158	53	1·724276	78	1·892095
4	0·602060	29	1·462398	54	1·732394	79	1·897627
5	0·698970	30	1·477121	55	1·740363	80	1·903090
6	0·778151	31	1·491362	56	1·748188	81	1·908485
7	0·845098	32	1·505150	57	1·755875	82	1·913814
8	0·903090	33	1·518514	58	1·763428	83	1·919078
9	0·954243	34	1·531479	59	1·770852	84	1·924279
10	1·000000	35	1·544068	60	1·778151	85	1·929419
11	1·041393	36	1·556303	61	1·785330	86	1·934498
12	1·079181	37	1·568202	62	1·792392	87	1·939519
13	1·113943	38	1·579784	63	1·799341	88	1·944483
14	1·146128	39	1·591065	64	1·806180	89	1·949390
15	1·176091	40	1·602060	65	1·812913	90	1·954243
16	1·204120	41	1·612784	66	1·819544	91	1·959041
17	1·230449	42	1·623249	67	1·826075	92	1·963788
18	1·255273	43	1·633468	68	1·832509	93	1·968483
19	1·278754	44	1·643453	69	1·838849	94	1·973128
20	1·301030	45	1·653213	70	1·845098	95	1·977724
21	1·322219	46	1·662758	71	1·851258	96	1·982271
22	1·342423	47	1·672098	72	1·857333	97	1·986772
23	1·361728	48	1·681241	73	1·863323	98	1·991226
24	1·380211	49	1·690196	74	1·869232	99	1·995635
25	1·397940	50	1·698970	75	1·875061	100	2·000000

Remark.—In the following Table, the *first two figures*, in the first column of Logarithms, are to be prefixed to each of the numbers, in the same horizontal line, in the next nine columns; but when a point (•) occurs, a 0 is to be put in its place, and the *two initial figures* are to be taken from the next line below.

N.	0	1	2	3	4	5	6	7	8	9	D.
100	000000	0434	0868	1301	1734	2166	2598	3029	3461	3891	432
101	4321	4751	5181	5609	6038	6466	6894	7321	7748	8174	428
102	8600	9026	9451	9876	•300	•724	1147	1570	1993	2415	424
103	012837	3259	3680	4100	4521	4940	5360	5779	6197	6616	419
104	7033	7451	7868	8284	8700	9116	9532	9947	•361	•775	416
105	021189	1603	2016	2428	2841	3252	3664	4075	4486	4896	412
106	5306	5715	6125	6533	6942	7350	7757	8164	8571	8978	408
107	9384	9789	•195	•600	1004	1408	1812	2216	2619	3021	404
108	033424	3826	4227	4628	5029	5430	5830	6230	6629	7028	400
109	7426	7825	8223	8620	9017	9414	9811	•207	•602	•998	396
110	041393	1787	2182	2576	2969	3362	3755	4148	4540	4932	393
111	5323	5714	0105	6495	6885	7275	7664	8053	8442	8830	389
112	9218	9606	9993	•380	•766	1153	1538	1924	2309	2694	386
113	053078	3463	3846	4230	4613	4996	5378	5760	6142	6524	382
114	6905	7286	7666	8046	8426	8805	9185	9563	9942	•320	379
115	060698	1075	1452	1829	2206	2582	2958	3333	3709	4083	376
116	4458	4832	5206	5580	5953	6326	6699	7071	7443	7815	372
117	8186	8557	8928	9298	9668	••38	•407	•776	1145	1514	369
118	071882	2250	2617	2985	3352	3718	4085	4451	4816	5182	366
119	5547	5912	6276	6640	7004	7368	7731	8094	8457	8819	363
120	079181	9543	9904	•266	•626	•987	1347	1707	2067	2426	360
121	082785	3144	3503	3861	4219	4576	4934	5291	5647	6004	357
122	6360	6716	7071	7426	7781	8136	8490	8845	9198	9552	355
123	9905	•258	•611	•963	1315	1667	2018	2370	2721	3071	351
124	093422	3772	4122	4471	4820	5169	5518	5866	6215	6562	349
125	6910	7257	7604	7951	8298	8644	8990	9335	9681	••26	346
126	100371	0715	1059	1403	1747	2091	2434	2777	3119	3462	343
127	3804	4146	4487	4828	5169	5510	5851	6191	6531	6871	340
128	7210	7549	7888	8227	8565	8903	9241	9579	9916	•253	338
129	110590	0926	1263	1599	1934	2270	2605	2940	3275	3609	335
130	113943	4277	4611	4944	5278	5611	5943	6276	6608	6940	333
131	7271	7603	7934	8265	8595	8926	9256	9586	9915	•245	330
132	120574	0903	1231	1560	1888	2216	2544	2871	3198	3525	328
133	3852	4178	4504	4830	5156	5481	5806	6131	6456	6781	325
134	7105	7429	7753	8076	8399	8722	9045	9368	9690	••12	323
135	130334	0655	0977	1298	1619	1939	2260	2580	2900	3219	321
136	3539	3858	4177	4496	4814	5133	5451	5769	6086	6403	318
137	6721	7037	7354	7671	7987	8303	8618	8934	9249	9564	315
138	9879	•194	•508	•822	1136	1450	1763	2076	2389	2702	314
139	143015	3327	3639	3951	4263	4574	4885	5196	5507	5818	311
140	146128	6438	6748	7058	7367	7676	7985	8294	8603	8911	309
141	9219	9527	9835	•142	•449	•756	1063	1370	1676	1982	307
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366	3481	3600	3718	3837	3955	4074	4192	4311	4429	4548	119
367	4666	4784	4903	5021	5139	5257	5376	5494	5612	5730	118
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817	2222	2275	2328	2381	2435	2488	2541	2594	2647	2700	53
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1035	014940	4982	5024	5066	5108	5150	5192	5234	5276	5318	42
1036	5360	5402	5444	5485	5527	5569	5611	5653	5695	5737	42
1037	5779	5821	5863	5904	5946	5988	6030	6072	6114	6156	42
1038	6197	6239	6281	6323	6365	6407	6448	6490	6532	6574	42
1039	6616	6657	6699	6741	6783	6824	6866	6908	6950	6992	42
1040	017033	7075	7117	7159	7200	7242	7284	7326	7367	7409	42
1041	7451	7492	7534	7576	7618	7659	7701	7743	7784	7826	42
1042	7868	7909	7951	7993	8034	8076	8118	8159	8201	8243	42
1043	8284	8326	8368	8409	8451	8492	8534	8576	8617	8659	42
1044	8700	8742	8784	8825	8867	8908	8950	8992	9033	9075	42
1045	019116	9158	9199	9241	9282	9324	9366	9407	9449	9490	42
1046	9532	9573	9615	9656	9698	9739	9781	9822	9864	9905	42
1047	9947	9988	·0030	·0071	·0113	·0154	·0195	·0237	·0278	·0320	41
1048	020361	0403	0444	0486	0527	0568	0610	0651	0693	0734	41
1049	0775	0817	0858	0900	0941	0982	1024	1065	1107	1148	41
1050	021189	1231	1272	1313	1355	1396	1437	1479	1520	1561	41
1051	1603	1644	1685	1727	1768	1809	1851	1892	1933	1974	41
1052	2016	2057	2098	2140	2181	2222	2263	2305	2346	2387	41
1053	2428	2470	2511	2552	2593	2635	2676	2717	2758	2799	41
1054	2841	2882	2923	2964	3005	3047	3088	3129	3170	3211	41
1055	023252	3294	3335	3376	3417	3458	3499	3541	3582	3623	41
1056	3664	3705	3746	3787	3828	3870	3911	3952	3993	4034	41
1057	4075	4116	4157	4198	4239	4280	4321	4363	4404	4445	41
1058	4486	4527	4568	4609	4650	4691	4732	4773	4814	4855	41
1059	4896	4937	4978	5019	5060	5101	5142	5183	5224	5265	41
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
1060	025306	5347	5388	5429	5470	5511	5552	5593	5634	5674	41
1061	5715	5756	5797	5838	5879	5920	5961	6002	6043	6084	41
1062	6125	6165	6206	6247	6288	6329	6370	6411	6452	6492	41
1063	6533	6574	6615	6656	6697	6737	6778	6819	6860	6901	41
1064	6942	6982	7023	7064	7105	7146	7186	7227	7268	7309	41
1065	027350	7390	7431	7472	7513	7553	7594	7635	7676	7716	41
1066	7757	7798	7839	7879	7920	7961	8002	8042	8083	8124	41
1067	8164	8205	8246	8287	8327	8368	8409	8449	8490	8531	41
1068	8571	8612	8653	8693	8734	8775	8815	8856	8896	8937	41
1069	8978	9018	9059	9100	9140	9181	9221	9262	9303	9343	41
1070	029384	9424	9465	9506	9546	9587	9627	9668	9708	9749	41
1071	9789	9830	9871	9911	9952	9992	·0033	·0073	·0114	·0154	41
1072	030195	0235	0276	0316	0357	0397	0438	0478	0519	0559	40
1073	0600	0640	0681	0721	0762	0802	0843	0883	0923	0964	40
1074	1004	1045	1085	1126	1166	1206	1247	1287	1328	1368	40
1075	031408	1449	1489	1530	1570	1610	1651	1691	1732	1772	40
1076	1812	1853	1893	1933	1974	2014	2054	2095	2135	2175	40
1077	2216	2256	2296	2337	2377	2417	2458	2498	2538	2578	40
1078	2619	2659	2699	2740	2780	2820	2860	2901	2941	2981	40
1079	3021	3062	3102	3142	3182	3223	3263	3303	3343	3384	40
1080	033424	3464	3504	3544	3585	3625	3665	3705	3745	3786	40
1081	3826	3866	3906	3946	3986	4027	4067	4107	4147	4187	40
1082	4227	4267	4308	4348	4388	4428	4468	4508	4548	4588	40
1083	4628	4669	4709	4749	4789	4829	4869	4909	4949	4989	40
1084	5029	5069	5109	5149	5190	5230	5270	5310	5350	5390	40
1085	035430	5470	5510	5550	5590	5630	5670	5710	5750	5790	40
1086	5830	5870	5910	5950	5990	6030	6070	6110	6150	6190	40
1087	6230	6269	6309	6349	6389	6429	6469	6509	6549	6589	40
1088	6629	6669	6709	6749	6789	6828	6868	6908	6948	6988	40
1089	7028	7068	7108	7148	7187	7227	7267	7307	7347	7387	40
1090	037426	7466	7506	7546	7586	7626	7665	7705	7745	7785	40
1091	7825	7865	7904	7944	7984	8024	8064	8103	8143	8183	40
1092	8223	8262	8302	8342	8382	8421	8461	8501	8541	8580	40
1093	8620	8660	8700	8739	8779	8819	8859	8898	8938	8978	40
1094	9017	9057	9097	9136	9176	9216	9255	9295	9335	9374	40
1095	039414	9454	9493	9533	9573	9612	9652	9692	9731	9771	40
1096	9811	9850	9890	9929	9969	·0009	·0048	·0088	·0127	·0167	40
1097	040207	0246	0286	0325	0365	0405	0444	0484	0523	0563	40
1098	0602	0642	0681	0721	0761	0800	0840	0879	0919	0958	40
1099	0998	1037	1077	1116	1156	1195	1235	1274	1314	1353	39
N.	0	1	2	3	4	5	6	7	8	9	D.

TABLE II.

CONTAINING

NATURAL SINES AND COSINES,

AND

LOGARITHMIC SINES, COSINES, TANGENTS, AND COTANGENTS,

FOR

EVERY DEGREE AND MINUTE OF THE QUADRANT.

Nat. Functions.			Logarithmic Functions + 10.							
′	N. sine.	N. cos.	L. sine.	D. 1″	L. cos.	D.1″	L. tang.	D. 1″	L. cot.	
0	00000	Unit.	0·000000		10·000000		0·000000		Infinite.	60
1	00029	Unit.	6·463726	5017·17	000000	·00	6·463726	5017·17	13·536274	59
2	00058	Unit.	764756	2934·85	000000	·00	764756	2934·83	235244	58
3	00087	Unit.	940847	2082·31	000000	·00	940847	2082·31	059153	57
4	00116	Unit.	7·065786	1615·17	000000	·00	7·065786	1615·17	12·934214	56
5	00145	Unit.	162696	1319·68	000000	·00	162696	1319·69	837304	55
6	00175	Unit.	241877	1115·75	9·999999	·01	241878	1115·78	758122	54
7	00204	Unit.	308824	966·53	999999	·01	308825	996·53	691175	53
8	00233	Unit.	366816	852·54	999999	·01	366817	852·54	633183	52
9	00262	Unit.	417968	762·63	999999	·01	417970	762·63	58203o	51
10	00291	Unit.	463725	689·88	999998	·01	463727	689·88	536273	50
11	00320	99999	7·505118	629·81	9·999998	·01	7·505120	629·81	12·494880	49
12	00349	99999	542906	579·36	999997	·01	542909	579·33	457091	48
13	00378	99999	577668	536·41	999997	·01	577672	536·42	422328	47
14	00407	99999	609853	499·38	999996	·01	609857	499·39	390143	46
15	00436	99999	639816	467·14	999996	·01	639820	467·15	360180	45
16	00465	99999	667845	438·81	999995	·01	667849	438·82	332151	44
17	00495	99999	694173	413·72	999995	·01	694179	413·73	305821	43
18	00524	99999	718997	391·35	999994	·01	719003	391·36	280997	42
19	00553	99998	742477	371·27	999993	·01	742484	371·28	257516	41
20	00582	99998	764754	353·15	999993	·01	764761	351·36	235239	40
21	00611	99998	7·785943	336·72	9·999992	·01	7·785951	336·73	12·214049	39
22	00640	99998	806146	321·75	999991	·01	806155	321·76	193845	38
23	00669	99998	825451	308·05	999990	·01	825460	308·06	174540	37
24	00698	99998	843934	295·47	999989	·02	843944	295·49	156056	36
25	00727	99997	861662	283·88	999988	·02	861674	283·90	138326	35
26	00756	99997	878695	273·17	999988	·02	878708	273·18	121292	34
27	00785	99997	895085	263·23	999987	·02	895099	263·25	104901	33
28	00814	99997	910879	253·99	999986	·02	910894	254·01	089106	32
29	00844	99996	926119	245·38	999985	·02	926134	245·40	073866	31
30	00873	99996	940842	237·33	999983	·02	940858	237·35	059142	30
31	00902	99996	7·955082	229·80	9·999982	·02	7·955100	229·81	12·044900	29
32	00931	99996	968870	222·73	999981	·02	968889	222·75	031111	28
33	00960	99995	982233	216·08	999980	·02	982253	216·10	017747	27
34	00989	99995	995198	209·81	999979	·02	995219	209·83	004781	26
35	01018	99995	8·007787	203·90	999977	·02	8·007809	203·92	11·992191	25
36	01047	99995	020021	198·31	999976	·02	020043	198·33	979955	24
37	01076	99994	031919	193·02	999975	·02	031945	193·05	968055	23
38	01105	99994	043501	188·01	999973	·02	043527	188·03	956473	22
39	01134	99994	054781	183·25	999972	·02	054809	183·27	945191	21
40	01164	99993	065776	178·72	999971	·02	065806	178·74	934194	20
41	01193	99993	8·076500	174·41	9·999969	·02	8·076531	174·44	11·923469	19
42	01222	99993	086965	170·31	999968	·02	086997	170·34	913003	18
43	01251	99992	097183	166·39	999966	·02	097217	166·42	902783	17
44	01280	99992	107167	162·65	999964	·03	107202	162·68	892797	16
45	01309	99991	116926	159·08	999963	·03	116963	159·10	883037	15
46	01338	99991	126471	155·66	999961	·03	126510	155·68	873490	14
47	01367	99991	135810	152·38	999959	·03	135851	152·41	864149	13
48	01396	99990	144953	149·24	999958	·03	144996	149·27	855004	12
49	01425	99990	153907	146·22	999956	·03	153952	146·27	846048	11
50	01454	99989	162681	143·33	999954	·03	162727	143·36	837273	10
51	01483	99989	8·171280	140·54	9·999952	·03	8·171328	140·57	11·828672	9
52	01513	99989	179713	137·86	999950	·03	179763	137·90	820237	8
53	01542	99988	187985	135·29	999948	·03	188036	135·32	811964	7
54	01571	99988	196102	132·80	999946	·03	196156	132·84	803844	6
55	01600	99987	204070	130·41	999944	·03	204126	130·44	795874	5
56	01629	99987	211895	128·10	999942	·04	211953	128·14	788047	4
57	01658	99986	219581	125·87	999940	·04	219641	125·90	780359	3
58	01687	99986	227134	123·72	999938	·04	227195	123·76	772805	2
59	01716	99985	234557	121·64	999936	·04	234621	121·68	765379	1
60	01745	99985	241855	119·63	999934	·04	241921	119·67	758079	0
	N. cos.	N. sine.	L. cos.	D. 1″	L. sine.		L. cot.	D. 1″	L. tang.	′

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Nat. Functions.			Logarithmic Functions + 10.							
′	N. sine.	N. cos.	L. sine.	D. 1″	L. cos.	D.1″	L. tang.	D. 1″	L. cot.	
0	01745	99985	8·241855	119·63	9·999934	·04	8·241921	119·67	11·758079	60
1	01774	99984	249033	117·68	999932	·04	249102	117·72	750898	59
2	01803	99984	256094	115·80	999929	·04	256165	115·84	743835	58
3	01832	99983	263042	113·98	999927	·04	263115	114·02	736885	57
4	01862	99983	269881	112·21	999925	·04	269956	112·25	730044	56
5	01891	99982	276614	110·50	999922	·04	276691	110·54	723309	55
6	01920	99982	283243	108·83	999920	·04	283323	108·87	716677	54
7	01949	99981	289773	107·21	999918	·04	289856	107·26	710144	53
8	01978	99980	296207	105·65	999915	·04	296292	105·70	703708	52
9	02007	99980	302546	104·13	999913	·04	302634	104·18	697366	51
10	02036	99979	308794	102·66	999910	·04	308884	102·70	691116	50
11	02065	99979	8·314954	101·22	9·999907	·04	8·315046	101·26	11·684954	49
12	02094	99978	321027	99·82	999905	·04	321122	99·87	678878	48
13	02123	99977	327016	98·47	999902	·04	327114	98·51	672886	47
14	02152	99977	332924	97·14	999899	·05	333025	97·19	666975	46
15	02181	99976	338753	95·86	999897	·05	338856	95·90	661144	45
16	02211	99976	344504	94·60	999894	·05	344610	94·65	655390	44
17	02240	99975	350181	93·38	999891	·05	350289	93·43	649711	43
18	02269	99974	355783	92·19	999888	·05	355895	92·24	644105	42
19	02298	99974	361315	91·03	999885	·05	361430	91·08	638570	41
20	02327	99973	366777	89·90	999882	·05	366895	89·95	633105	40
21	02356	99972	8·372171	88·80	9·999879	·05	8·372292	88·85	11·627708	39
22	02385	99972	377499	87·72	999876	·05	377622	87·77	622378	38
23	02414	99971	382762	86·67	999873	·05	382889	86·72	617111	37
24	02443	99970	387962	85·64	999870	·05	388092	85·70	611908	36
25	02472	99969	393101	84·64	999867	·05	393234	84·70	606766	35
26	02501	99969	398179	83·66	999864	·05	398315	83·71	601685	34
27	02530	99968	403199	82·71	999861	·05	403338	82·76	596662	33
28	02560	99967	408161	81·77	999858	·05	408304	81·82	591696	32
29	02589	99966	413068	80·86	999854	·05	413213	80·91	586787	31
30	02618	99966	417919	79·96	999851	·06	418068	80·02	581932	30
31	02647	99965	8·422717	79·09	9·999848	·06	8·422869	79·14	11·577131	29
32	02676	99964	427462	78·23	999844	·06	427618	78·30	572382	28
33	02705	99963	432156	77·40	999841	·06	432315	77·45	567685	27
34	02734	99963	436800	76·57	999838	·06	436962	76·63	563038	26
35	02763	99962	441394	75·77	999834	·06	441560	75·83	558440	25
36	02792	99961	445941	74·99	999831	·06	446110	75·05	553890	24
37	02821	99960	450440	74·22	999827	·06	450613	74·28	549387	23
38	02850	99959	454893	73·46	999823	·06	455070	73·52	544930	22
39	02879	99959	459301	72·73	999820	·06	459481	72·79	540519	21
40	02908	99958	463665	72·00	999816	·06	463849	72·06	536151	20
41	02938	99957	8·467985	71·29	9·999812	·06	8·468172	71·35	11·531828	19
42	02967	99956	472263	70·60	999809	·06	472454	70·66	527546	18
43	02996	99955	476498	69·91	999805	·06	476693	69·98	523307	17
44	03025	99954	480693	69·24	999801	·06	480892	69·31	519108	16
45	03054	99953	484848	68·59	999797	·07	485050	68·65	514950	15
46	03083	99952	488963	67·94	999793	·07	489170	68·01	510830	14
47	03112	99952	493040	67·31	999790	·07	493250	67·38	506750	13
48	03141	99951	497078	66·69	999786	·07	497293	66·76	502707	12
49	03170	99950	501080	66·08	999782	·07	501298	66·15	498702	11
50	03199	99949	505045	65·48	999778	·07	505267	65·55	494733	10
51	03228	99948	8·508974	64·89	9·999774	·07	8·509200	64·96	11·490800	9
52	03257	99947	512867	64·31	999769	·07	513098	64·39	486902	8
53	03286	99946	516726	63·75	999765	·07	516961	63·82	483039	7
54	03316	99945	520551	63·19	999761	·07	520790	63·26	479210	6
55	03345	99944	524343	62·64	999757	·07	524586	62·72	475414	5
56	03374	99943	528102	62·11	999753	·07	528349	62·18	471651	4
57	03403	99942	531828	61·58	999748	·07	532080	61·65	467920	3
58	03432	99941	535523	61·06	999744	·07	535779	61·13	464221	2
59	03461	99940	539186	60·55	999740	·07	539447	60·62	460553	1
60	03490	99939	542819	60·04	999735	·07	543084	60·12	11·456916	0
	N. cos.	N. sine.	L. cos.	D. 1″	L. sine.		L. cot.	D. 1″	L. tang.	′

Nat. Functions.			Logarithmic Functions + 10.							
′	N. sine.	N. cos.	L. sine.	D. 1″	L. cos.	D. 1″	L. tang.	D. 1″	L. cot.	
0	03490	99939	8·542819	60·04	9·999735	·07	8·543084	60·12	11·456916	60
1	03519	99938	546422	59·55	999731	·07	546691	59·62	453309	59
2	03548	99937	549995	59·06	999726	·07	550268	59·14	449732	58
3	03577	99936	553539	58·58	999722	·08	553817	58·66	446183	57
4	03606	99935	557054	58·11	999717	·08	557336	58·19	442664	56
5	03635	99934	560540	57·65	999713	·08	560828	57·73	439172	55
6	03664	99933	563999	57·19	999708	·08	564291	57·27	435709	54
7	03693	99932	567431	56·74	999704	·08	567727	56·82	432273	53
8	03723	99931	570836	56·30	999699	·08	571137	56·38	428863	52
9	03752	99930	574214	55·87	999694	·08	574520	55·95	425480	51
10	03781	99929	577566	55·44	999689	·08	577877	55·52	422123	50
11	03810	99927	8·580892	55·02	9·999685	·08	8·581208	55·10	11·418792	49
12	03839	99926	584193	54·60	999680	·08	584514	54·68	415486	48
13	03868	99925	587469	54·19	999675	·08	587795	54·27	412205	47
14	03897	99924	590721	53·79	999670	·08	591051	53·87	408949	46
15	03926	99923	593948	53·39	999665	·08	594283	53·47	405717	45
16	03955	99922	597152	53·00	999660	·08	597492	53·08	402508	44
17	03984	99921	600332	52·61	999655	·08	600677	52·70	399323	43
18	04013	99919	603489	52·23	999650	·08	603839	52·32	396161	42
19	04042	99918	606623	51·86	999645	·09	606978	51·94	393022	41
20	04071	99917	609734	51·49	999640	·09	610094	51·58	389906	40
21	04100	99916	8·612823	51·12	9·999635	·09	8·613189	51·21	11·386811	39
22	04129	99915	615891	50·76	999629	·09	616262	50·85	383738	38
23	04159	99913	618937	50·41	999624	·09	619313	50·50	380687	37
24	04188	99912	621962	50·06	999619	·09	622343	50·15	377657	36
25	04217	99911	624965	49·72	999614	·09	625352	49·81	374648	35
26	04246	99910	627948	49·38	999608	·09	628340	49·47	371660	34
27	04275	99909	630911	49·04	999603	·09	631308	49·13	368692	33
28	04304	99907	633854	48·71	999597	·09	634256	48·80	365744	32
29	04333	99906	636776	48·39	999592	·09	637184	48·48	362816	31
30	04362	99905	639680	48·06	999586	·09	640093	48·16	359907	30
31	04391	99904	8·642563	47·75	9·999581	·09	8·642982	47·84	11·357018	29
32	04420	99902	645428	47·43	999575	·09	645853	47·53	354147	28
33	04449	99901	648274	47·12	999570	·09	648704	47·22	351296	27
34	04478	99900	651102	46·82	999564	·09	651537	46·91	348463	26
35	04507	99898	653911	46·52	999558	·10	654352	46·61	345648	25
36	04536	99897	656702	46·22	999553	·10	657149	46·31	342851	24
37	04565	99896	659475	45·92	999547	·10	659928	46·02	340072	23
38	04594	99894	662230	45·63	999541	·10	662689	45·73	337311	22
39	04623	99893	664968	45·35	999535	·10	665433	45·44	334567	21
40	04653	99892	667689	45·06	999529	·10	668160	45·26	331840	20
41	04682	99890	8·670393	44·79	9·999524	·10	8·670870	44·88	11·329130	19
42	04711	99889	673080	44·51	999518	·10	673563	44·61	326437	18
43	04740	99888	675751	44·24	999512	·10	676239	44·34	323761	17
44	04769	99886	678405	43·97	999506	·10	678900	44·17	321100	16
45	04798	99885	681043	43·70	999500	·10	681544	43·80	318456	15
46	04827	99883	683665	43·44	999493	·10	684172	43·54	315828	14
47	04856	99882	686272	43·18	999487	·10	686784	43·28	313216	13
48	04885	99881	688863	42·92	999481	·10	689381	43·03	310619	12
49	04914	99879	691438	42·67	999475	·10	691963	42·77	308037	11
50	04943	99878	693998	42·42	999469	·10	694529	42·52	305471	10
51	04972	99876	8·696543	42·17	9·999463	11	8·697081	42·28	11·302919	9
52	05001	99875	699073	41·92	999456	·11	699617	42·03	300383	8
53	05030	99873	701589	41·68	999450	·11	702139	41·79	297861	7
54	05059	99872	704090	41·44	999443	·11	704646	41·55	295354	6
55	05088	99870	706577	41·21	999437	·11	707140	41·32	292860	5
56	05117	99869	709049	40·97	999431	·11	709618	41·08	290382	4
57	05146	99867	711507	40·74	999424	·11	712083	40·85	287917	3
58	05175	99866	713952	40·51	999418	·11	714534	40·62	285465	2
59	05205	99864	716383	40·29	999411	·11	716972	40·40	283028	1
60	05234	99863	718800	40·06	999404	·11	719396	40·17	280604	0
	N. cos.	N. sine.	L. cos.	D. 1″	L. sine.		L. cot.	D. 1″	L. tang.	′

Nat. Functions.			Logarithmic Functions + 10.							
′	N. sine.	N. cos.	L. sine.	D. 1″	L. cos.	D.1″	L. tang.	D. 1″	L. cot.	
0	05234	99863	8·718800	40·06	9·999404	·11	8·719396	40·17	11·280604	60
1	05263	99861	721204	39·84	999398	·11	721806	39·95	278194	59
2	05292	99860	723595	39·62	999391	·11	724204	39·74	275796	58
3	05321	99858	725972	39·41	999384	·11	726588	39·52	273412	57
4	05350	99857	728337	39·19	999378	·11	728959	39·30	271041	56
5	05379	99855	730688	38·98	999371	·11	731317	39·09	268683	55
6	05408	99854	733027	38·77	999364	·12	733663	38·89	266337	54
7	05437	99852	735354	38·57	999357	·12	735996	38·68	264004	53
8	05466	99851	737667	38·36	999350	·12	738317	38·48	261683	52
9	05495	99849	739969	38·16	999343	·12	740626	38·27	259374	51
10	05524	99847	742259	37·96	999336	·12	742922	38·07	257078	50
11	05553	99846	8·744536	37·76	9·999329	·12	8·745207	37·87	11·254793	49
12	05582	99844	746802	37·56	999322	·12	747479	37·68	252521	48
13	05611	99842	749055	37·37	999315	·12	749740	37·49	250260	47
14	05640	99841	751297	37·17	999308	·12	751989	37·29	248011	46
15	05669	99839	753528	36·98	999301	·12	754227	37·10	245773	45
16	05698	99838	755747	36·79	999294	·12	756453	36·92	243547	44
17	05727	99836	757955	36·61	999286	·12	758668	36·73	241332	43
18	05756	99834	760151	36·42	999279	·12	760872	36·55	239128	42
19	05785	99833	762337	36·24	999272	·12	763065	36·36	236935	41
20	05814	99831	764511	36·06	999265	·12	765246	36·18	234754	40
21	05844	99829	8·766675	35·88	9·999257	·12	8·767417	36·00	11·232583	39
22	05873	99827	768828	35·70	999250	·13	769578	35·83	230422	38
23	05902	99826	770970	35·53	999242	·13	771727	35·65	228273	37
24	05931	99824	773101	35·35	999235	·13	773866	35·48	226134	36
25	05960	99822	775223	35·18	999227	·13	775995	35·31	224005	35
26	05989	99821	777333	35·01	999220	·13	778114	35·14	221886	34
27	06018	99819	779434	34·84	999212	·13	780222	34·97	219778	33
28	06047	99817	781524	34·67	999205	·13	782320	34·80	217680	32
29	06076	99815	783605	34·51	999197	·13	784408	34·64	215592	31
30	06105	99813	785675	34·31	999189	·13	786486	34·47	213514	30
31	06134	99812	8·787736	34·18	9·999181	·13	8·788554	34·31	11·211446	29
32	06163	99810	789787	34·02	999174	·13	790613	34·15	209387	28
33	06192	99808	791828	33·86	999166	·13	792662	33·99	207338	27
34	06221	99806	793859	33·70	999158	·13	794701	33·83	205299	26
35	06250	99804	795881	33·54	999150	·13	796731	33·68	203269	25
36	06279	99803	797894	33·39	999142	·13	798752	33·52	201248	24
37	06308	99801	799897	33·23	999134	·13	800763	33·37	199237	23
38	06337	99799	801892	33·08	999126	·13	802765	33·22	197235	22
39	06366	99797	803876	32·93	999118	·13	804758	33·07	195242	21
40	06395	99795	805852	32·78	999110	·13	806742	32·92	193258	20
41	06424	99793	8·807819	32·63	9·999102	·13	8·808717	32·78	11·191283	19
42	06453	99792	809777	32·49	999094	·14	810683	32·62	189317	18
43	06482	99790	811726	32·34	999086	·14	812641	32·48	187359	17
44	06511	99788	813667	32·19	999077	·14	814589	32·33	185411	16
45	06540	99786	815599	32·05	999069	·14	816529	32·19	183471	15
46	06569	99784	817522	31·91	999061	·14	818461	32·05	181539	14
47	06598	99782	819436	31·77	999053	·14	820384	31·91	179616	13
48	06627	99780	821343	31·63	999044	·14	822298	31·77	177702	12
49	06656	99778	823240	31·49	999036	·14	824205	31·63	175795	11
50	06685	99776	825130	31·35	999027	·14	826103	31·50	173897	10
51	06714	99774	8·827011	31·22	9·999019	·14	8·827992	31·36	11·172008	9
52	06743	99772	828884	31·08	999010	·14	829874	31·23	170126	8
53	06773	99770	830749	30·95	999002	·14	831748	31·10	168252	7
54	06802	99768	832607	30·82	998993	·14	833613	30·96	166387	6
55	06831	99766	834456	30·69	998984	·14	835471	30·83	164529	5
56	06860	99764	836297	30·56	998976	·14	837321	30·70	162679	4
57	06889	99762	838130	30·43	998967	·15	839163	30·57	160837	3
58	06918	99760	839956	30·30	998958	·15	840998	30·45	159002	2
59	06947	99758	841774	30·17	998950	·15	842825	30·32	157175	1
60	06976	99756	843585	30·00	998941	·15	844644	30·19	155356	0
	N. cos.	N. sine.	L. cos.	D. 1″	L. sine.		L. cot.	D. 1″	L. tang.	

Nat. Functions.			Logarithmic Functions + 10.						
′	N. sine.	N. cos.	L. sine.	D. 1″	L. cos.	D.1″	L. tang.	D 1.″	L. cot.	
0	06976	99756	8·843585	30·05	9·998941	·15	8·844644	30·19	11·155356	60
1	07005	99754	845387	29·92	998932	·15	846455	30·07	153545	59
2	07034	99752	847183	29·80	998923	·15	848260	29·95	151740	58
3	07063	99750	848971	29·67	998914	·15	850057	29·82	149943	57
4	07092	99748	850751	29·55	998905	·15	851846	29·70	148154	56
5	07121	99746	852525	29·43	998896	·15	853628	29·58	146372	55
6	07150	99744	854291	29·31	998887	·15	855403	29·46	144597	54
7	07179	99742	856049	29·19	998878	·15	857171	29·35	142829	53
8	07208	99740	857801	29·07	998869	·15	858932	29·23	141068	52
9	07237	99738	859546	28·96	998860	·15	860686	29·11	139314	51
10	07266	99736	861283	28·84	998851	·15	862433	29·00	137567	50
11	07295	99734	8·863014	28·73	9·998841	·15	8·864173	28·88	11·135827	49
12	07324	99731	864738	28·61	998832	·15	865906	28·77	134094	48
13	07353	99729	866455	28·50	998823	·16	867632	28·66	132368	47
14	07382	99727	868165	28·39	998813	·16	869351	28·54	130649	46
15	07411	99725	869868	28·28	998804	·16	871064	28·43	128936	45
16	07440	99723	871565	28·17	998795	·16	872770	28·32	127230	44
17	07469	99721	873255	28·06	998785	·16	874469	28·21	125531	43
18	07498	99719	874938	27·95	998776	·16	876162	28·11	123838	42
19	07527	99716	876615	27·86	998766	·16	877849	28·00	122151	41
20	07556	99714	878285	27·73	998757	·16	879529	27·89	120471	40
21	07585	99712	8·879949	27·63	9·998747	·16	8·881202	27·79	11·118798	39
22	07614	99710	881607	27·52	998738	·16	882869	27·68	117131	38
23	07643	99708	883258	27·42	998728	·16	884530	27·58	115470	37
24	07672	99705	884903	27·31	998718	·16	886185	27·47	113815	36
25	07701	99703	886542	27·21	998708	·16	887833	27·37	112167	35
26	07730	99701	888174	27·11	998699	·16	889476	27·27	110524	34
27	07759	99699	889801	27·00	998689	·16	891112	27·17	108888	33
28	07788	99696	891421	26·90	998679	·16	892742	27·07	107258	32
29	07817	99694	893035	26·80	998669	·17	894366	26·97	105634	31
30	07846	99692	894643	26·70	998659	·17	895984	26·87	104016	30
31	07875	99689	8·896246	26·60	9·998649	·17	8·897596	26·77	11·102404	29
32	07904	99687	897842	26·51	998639	·17	899203	26·67	100797	28
33	07933	99685	899432	26·41	998629	·17	900803	26·58	099197	27
34	07962	99683	901017	26·31	998619	·17	902398	26·48	097602	26
35	07991	99680	902596	26·22	998609	·17	903987	26·38	096013	25
36	08020	99678	904169	26·12	998599	·17	905570	26·29	094430	24
37	08049	99676	905736	26·03	998589	·17	907147	26·20	092853	23
38	08078	99673	907297	25·93	998578	·17	908719	26·10	091281	22
39	08107	99671	908853	25·84	998568	·17	910285	26·01	089715	21
40	08136	99668	910404	25·75	998558	·17	911846	25·92	088154	20
41	08165	99666	8·911949	25·66	9·998548	·17	8·913401	25·83	11·086599	19
42	08194	99664	913488	25·56	998537	·17	914951	25·74	085049	18
43	08223	99661	915022	25·47	998527	·17	916495	25·65	083505	17
44	08252	99659	916550	25·38	998516	·18	918034	25·56	081966	16
45	08281	99657	918073	25·29	998506	·18	919568	25·47	080432	15
46	08310	99654	919591	25·20	998495	·18	921096	25·38	078904	14
47	08339	99652	921103	25·12	998485	·18	922619	25·30	077381	13
48	08368	99649	922610	25·03	998474	·18	924136	25·21	075864	12
49	08397	99647	924112	24·94	998464	·18	925649	25·12	074351	11
50	08426	99644	925609	24·86	998453	·18	927156	25·03	072844	10
51	08455	99642	8·927100	24·77	9·998442	·18	8·928658	24·95	11·071342	9
52	08484	99639	928587	24·69	998431	·18	930155	24·86	069845	8
53	08513	99637	930068	24·60	998421	·18	931647	24·78	068353	7
54	08542	99635	931544	24·52	998410	·18	933134	24·70	066866	6
55	08571	99632	933015	24·43	998399	·18	934616	24·61	065384	5
56	08600	99630	934481	24·35	998388	·18	936093	24·53	063907	4
57	08629	99627	935942	24·27	998377	·18	937565	24·45	062435	3
58	08658	99625	937398	24·19	998366	·18	939032	24·37	060968	2
59	08687	99622	938850	24·11	998355	·18	940494	24·30	059506	1
60	08716	99619	940296	24·03	998344	·18	941952	24·21	058048	0
	N. cos.	N. sine.	L. cos.	D. 1″	L. sine.		L. cot.	D. 1″	L. tang.	′

Nat. Functions.			Logarithmic Functions + 10.							
′	N.sine.	N. cos.	L. sine.	D. 1″	L. cos.	D.1″	L. tang.	D. 1″	L. cot.	
0	08716	99619	8·940296	24·03	9·998344	·19	8·941952	24·21	11·058048	60
1	08745	99617	941738	23·94	998333	·19	943404	24·13	056596	59
2	08774	99614	943174	23·87	998322	·19	944852	24·05	055148	58
3	08803	99612	944606	23·79	998311	·19	946295	23·97	053705	57
4	08831	99609	946034	23·71	998300	·19	947734	23·90	052266	56
5	08860	99607	947456	23·63	998289	·19	949168	23·82	050832	55
6	08889	99604	948874	23·55	998277	·19	950597	23·74	049403	54
7	08918	99602	950287	23·48	998266	·19	952021	23·66	047979	53
8	08947	99599	951696	23·40	998255	·19	953441	23·60	046559	52
9	08976	99596	953100	23·32	998243	·19	954856	23·51	045144	51
10	09005	99594	954499	23·25	998232	·19	956267	23·44	043733	50
11	09034	99591	8·955894	23·17	9·998220	·19	8·957674	23·37	11·042326	49
12	09063	99588	957284	23·10	998209	·19	959075	23·29	040925	48
13	09092	99586	958670	23·02	998197	·19	960473	23·23	039527	47
14	09121	99583	960052	22·95	998186	·19	961866	23·14	038134	46
15	09150	99580	961429	22·88	998174	·19	963255	23·07	036745	45
16	09179	99578	962801	22·80	998163	·19	964639	23·00	035361	44
17	09208	99575	964170	22·73	998151	·19	966019	22·93	033981	43
18	09237	99572	965534	22·66	998139	·20	967394	22·86	032606	42
19	09266	99570	966893	22·59	998128	·20	968766	22·79	031234	41
20	09295	99567	968249	22·52	998116	·20	970133	22·71	029867	40
21	09324	99564	8·969600	22·44	9·998104	·20	8·971496	22·65	11·028504	39
22	09353	99562	970947	22·38	998092	·20	972855	22·57	027145	38
23	09382	99559	972289	22·31	998080	·20	974209	22·51	025791	37
24	09411	99556	973628	22·24	998068	·20	975560	22·44	024440	36
25	09440	99553	974962	22·17	998056	·20	976906	22·37	023094	35
26	09469	99551	976293	22·10	998044	·20	978248	22·30	021752	34
27	09498	99548	977619	22·03	998032	·20	979586	22·23	020414	33
28	09527	99545	978941	21·97	998020	·20	980921	22·17	019079	32
29	09556	99542	980259	21·90	998008	·20	982251	22·10	017749	31
30	09585	99540	981573	21·83	997996	·20	983577	22·04	016423	30
31	09614	99537	8·982883	21·77	9·997984	·20	8·984899	21·97	11·015101	29
32	09642	99534	984189	21·70	997972	·20	986217	21·91	013783	28
33	09671	99531	985491	21·63	997959	·20	987532	21·84	012468	27
34	09700	99528	986789	21·57	997947	·20	988842	21·78	011158	26
35	09729	99526	988083	21·50	997935	·21	990149	21·71	009851	25
36	09758	99523	989374	21·44	997922	·21	991451	21·65	008549	24
37	09787	99520	990660	21·38	997910	·21	992750	21·58	007250	23
38	09816	99517	991943	21·31	997897	·21	994045	21·52	005955	22
39	09845	99514	993222	21·25	997885	·21	995337	21·46	004663	21
40	09874	99511	994497	21·19	997872	·21	996624	21·40	003376	20
41	09903	99508	8·995768	21·12	9·997860	·21	8·997908	21·34	11·002092	19
42	09932	99506	997036	21·06	997847	·21	999188	21·27	000812	18
43	09961	99503	998299	21·00	997835	·21	9·000465	21·21	10·999535	17
44	09990	99500	999560	20·94	997822	·21	001738	21·15	998262	16
45	10019	99497	9·000816	20·87	997809	·21	003007	21·09	996993	15
46	10048	99494	002069	20·82	997797	·21	004272	21·03	995728	14
47	10077	99491	003318	20·76	997784	·21	005534	20·97	994466	13
48	10106	99488	004563	20·70	997771	·21	006792	20·91	993208	12
49	10135	99485	005805	20·64	997758	·21	008047	20·85	991953	11
50	10164	99482	007044	20·58	997745	·21	009298	20·80	990702	10
51	10192	99479	9·008278	20·52	9·997732	·21	9·010546	20·74	10·989454	9
52	10221	99476	009510	20·46	997719	·21	011790	20·68	988210	8
53	10250	99473	010737	20·40	997706	·21	013031	20·62	986969	7
54	10279	99470	011962	20·34	997693	·22	014268	20·56	985732	6
55	10308	99467	013182	20·29	997680	·22	015502	20·51	984498	5
56	10337	99464	014400	20·23	997667	·22	016732	20·45	983268	4
57	10366	99461	015613	20·17	997654	·22	017959	20·40	982041	3
58	10395	99458	016824	20·12	997641	·22	019183	20·33	980817	2
59	10424	99455	018031	20·06	997628	·22	020403	20·28	979597	1
60	10453	99452	019235	20·00	997614	·22	021620	20·23	978380	0
	N. cos.	N. sine.	L. cos.	D. 1″	L. sine.		L. cot.	D. 1″	L. tang.	′

Nat. Functions.			Logarithmic Functions + 10.							
′	N. sine.	N. cos.	L. sine.	D. 1″	L. cos.	D.1″	L. tang.	D. 1″	L. cot.	
0	10453	99452	9·019235	20·00	9·997614	·22	9·021620	20·23	10·978380	60
1	10482	99449	020435	19·95	997601	·22	022834	20·17	977166	59
2	10511	99446	021632	19·89	997588	·22	024044	20·11	975956	58
3	10540	99443	022825	19·84	997574	·22	025251	20·06	974749	57
4	10569	99440	024016	19·78	997561	·22	026455	20·00	973545	56
5	10597	99437	025203	19·73	997547	·22	027655	19·95	972345	55
6	10626	99434	026386	19·67	997534	·23	028852	19·90	971148	54
7	10655	99431	027567	19·62	997520	·23	030046	19·85	969954	53
8	10684	99428	028744	19·57	997507	·23	031237	19·79	968763	52
9	10713	99424	029918	19·51	997493	·23	032425	19·74	967575	51
10	10742	99421	031089	19·47	997480	·23	033609	19·69	966391	50
11	10771	99418	9·032257	19·41	9·997466	·23	9·034791	19·64	10·965209	49
12	10800	99415	033421	19·36	997452	·23	035969	19·58	964031	48
13	10829	99412	034582	19·30	997439	·23	037144	19·53	962856	47
14	10858	99409	035741	19·25	997425	·23	038316	19·48	961684	46
15	10887	99406	036896	19·20	997411	·23	039485	19·43	960515	45
16	10916	99402	038048	19·15	997397	·23	040651	19·38	959349	44
17	10945	99399	039197	19·10	997383	·23	041813	19·33	958187	43
18	10973	99396	040342	19·05	997369	·33	042973	19·28	957027	42
19	11002	99393	041485	18·99	997355	·23	044130	19·23	955870	41
20	11031	99390	042625	18·94	997341	·23	045284	19·18	954716	40
21	11060	99386	9·043762	18·89	9·997327	·24	9·046434	19·13	10·953566	39
22	11089	99383	044895	18·84	997313	·24	047582	19·08	952418	38
23	11118	99380	046026	18·79	997299	·24	048727	19·03	951273	37
24	11147	99377	047154	18·75	997285	·24	049869	18·98	950131	36
25	11176	99374	048279	18·70	997271	·24	051008	18·93	948992	35
26	11205	99370	049400	18·65	997257	·24	052144	18·89	947856	34
27	11234	99367	050519	18·60	997242	·24	053277	18·84	946723	33
28	11263	99364	051635	18·55	997228	·24	054407	18·79	945593	32
29	11291	99360	052749	18·50	997214	·24	055535	18·74	944465	31
30	11320	99357	053859	18·45	997199	·24	056659	18·70	943341	30
31	11349	99354	9·054966	18·41	9·997185	·24	9·057781	18·65	10·942219	29
32	11378	99351	056071	18·36	997170	·24	058900	18·69	941100	28
33	11407	99347	057172	18·31	997156	·24	060016	18·55	939984	27
34	11436	99344	058271	18·27	997141	·24	061130	18·51	938870	26
35	11465	99341	059367	18·22	997127	·24	062240	18·46	937760	25
36	11494	99337	060460	18·17	997112	·24	063348	18·42	936652	24
37	11523	99334	061551	18·13	997098	·24	064453	18·37	935547	23
38	11552	99331	062639	18·08	997083	·25	065556	18·33	934444	22
39	11580	99327	063724	18·04	997068	·25	066655	18·28	933345	21
40	11609	99324	064806	17·99	997053	·25	067752	18·24	932248	20
41	11638	99320	9·065885	17·94	9·997039	·25	9·068846	18·19	10·931154	19
42	11667	99317	066962	17·90	997024	·25	069938	18·15	930062	18
43	11696	99314	068036	17·86	997009	·25	071027	18·10	928973	17
44	11725	99310	069107	17·81	996994	·25	072113	18·06	927887	16
45	11754	99307	070176	17·77	996979	·25	073197	18·02	926803	15
46	11783	99303	071242	17·72	996964	·25	074278	17·97	925722	14
47	11812	99300	072306	17·68	996949	·25	075356	17·93	924644	13
48	11840	99297	073366	17·63	996934	·25	076432	17·89	923568	12
49	11869	99293	074424	17·59	996919	·25	077505	17·84	922495	11
50	11898	99290	075480	17·55	996904	·25	078576	17·80	921424	10
51	11927	99286	9·076533	17·50	9·996889	·25	9·079644	17·76	10·920356	9
52	11956	99283	077583	17·46	996874	·25	080710	17·72	919290	8
53	11985	99279	078631	17·42	996858	·25	081773	17·67	918227	7
54	12014	99276	079676	17·38	996843	·25	082833	17·63	917167	6
55	12043	99272	080719	17·33	996828	·25	083891	17·59	916109	5
56	12071	99269	081759	17·29	996812	·26	084947	17·55	915053	4
57	12100	99265	082797	17·25	996797	·26	086000	17·51	914000	3
58	12129	99262	083832	17·21	996782	·26	087050	17·47	912950	2
59	12158	99258	084864	17·17	996766	·26	088098	17·43	911902	1
60	12187	99255	085894	17·13	996751	·26	089144	17·38	910856	0
	N. cos.	N. sine.	L. cos.	D. 1″	L. sine.		L. cot.	D. 1″	L. tang.	′

Nat. Functions.			Logarithmic Functions + 10.							
	N. sine.	N. cos.	L. sine.	D. 1″	L. cos.	D.1″	L. tang.	D. 1″	L. cot.	
0	12187	99255	9·085894	17·13	9·996751	·26	9·089144	17·38	10·910856	60
1	12216	99251	086922	17·09	996735	·26	090187	17·34	909813	59
2	12245	99248	087947	17·04	996720	·26	091228	17·30	908772	58
3	12274	99244	088970	17·00	996704	·26	092266	17·27	907734	57
4	12302	99240	089990	16·96	996688	·26	093302	17·22	906698	56
5	12331	99237	091008	16·92	996673	·26	094336	17·19	905664	55
6	12360	99233	092024	16·88	996657	·26	095367	17·15	904633	54
7	12389	99230	093037	16·84	996641	·26	096395	17·11	903605	53
8	12418	99226	094047	16·80	996625	·26	097422	17·07	902578	52
9	12447	99222	095056	16·76	996610	·26	098446	17·03	901554	51
10	12476	99219	096062	16·73	996594	·26	099468	16·99	900532	50
11	12504	99215	9·097065	16·68	9·996578	·27	9·100487	16·95	10·899513	49
12	12533	99211	098066	16·65	996562	·27	101504	16·91	898496	48
13	12562	99208	099065	16·61	996546	·27	102519	16·87	897481	47
14	12591	99204	100062	16·57	996530	·27	103532	16·84	896468	46
15	12620	99200	101056	16·53	996514	·27	104542	16·80	895458	45
16	12649	99197	102048	16·49	996498	·27	105550	16·76	894450	44
17	12678	99193	103037	16·45	996482	·27	106556	16·72	893444	43
18	12706	99189	104025	16·41	996465	·27	107559	16·69	892441	42
19	12735	99186	105010	16·38	996449	·27	108560	16·65	891440	41
20	12764	99182	105992	16·34	996433	·27	109559	16·61	890441	40
21	12793	99178	9·106973	16·30	9·996417	·27	9·110556	16·58	10·889444	39
22	12822	99175	107951	16·27	996400	·27	111551	16·54	888449	38
23	12851	99171	108927	16·23	996384	·27	112543	16·50	887457	37
24	12880	99167	109901	16·19	996368	·27	113533	16·46	886467	36
25	12908	99163	110873	16·16	996351	·27	114521	16·43	885479	35
26	12937	99160	111842	16·12	996335	·27	115507	16·39	884493	34
27	12966	99156	112809	16·08	996318	·27	116491	16·36	883509	33
28	12995	99152	113774	16·05	996302	·28	117472	16·32	882528	32
29	13024	99148	114737	16·01	996285	·28	118452	16·29	881548	31
30	13053	99144	115698	15·97	996269	·28	119429	16·25	880571	30
31	13081	99141	9·116656	15·94	9·996252	·28	9·120404	16·22	10·879596	29
32	13110	99137	117613	15·90	996235	·28	121377	16·18	878623	28
33	13139	99133	118567	15·87	996219	·28	122348	16·15	877652	27
34	13168	99129	119519	15·83	996202	·28	123317	16·11	876683	26
35	13197	99125	120469	15·80	996185	·28	124284	16·07	875716	25
36	13226	99122	121417	15·76	996168	·28	125249	16·04	874751	24
37	13254	99118	122362	15·73	996151	·28	126211	16·01	873789	23
38	13283	99114	123306	15·69	996134	·28	127172	15·97	872828	22
39	13312	99110	124248	15·66	996117	·28	128130	15·94	871870	21
40	13341	99106	125187	15·62	996100	·28	129087	15·91	870913	20
41	13370	99102	9·126125	15·59	9·996083	·29	9·130041	15·87	10·869959	19
42	13399	99098	127060	15·56	996066	·29	130994	15·84	869006	18
43	13427	99094	127993	15·52	996049	·29	131944	15·81	868056	17
44	13456	99091	128925	15·49	996032	·29	132893	15·77	867107	16
45	13485	99087	129854	15·45	996015	·29	133839	15·74	866161	15
46	13514	99083	130781	15·42	995998	·29	134784	15·71	865216	14
47	13543	99079	131706	15·39	995980	·29	135726	15·67	864274	13
48	13572	99075	132630	15·35	995963	·29	136667	15·64	863333	12
49	13600	99071	133551	15·32	995946	·29	137605	15·61	862395	11
50	13629	99067	134470	15·29	995928	·29	138542	15·58	861458	10
51	13658	99063	9·135387	15·25	9·995911	·29	9·139476	15·55	10·860524	9
52	13687	99059	136303	15·22	995894	·29	140409	15·51	859591	8
53	13716	99055	137216	15·19	995876	·29	141340	15·48	858660	7
54	13744	99051	138128	15·16	995859	·29	142269	15·45	857731	6
55	13773	99047	139037	15·12	995841	·29	143196	15·42	856804	5
56	13802	99043	139944	15·09	995823	·29	144121	15·39	855879	4
57	13831	99039	140850	15·06	995806	·29	145044	15·35	854956	3
58	13860	99035	141754	15·03	995788	·29	145966	15·32	854034	2
59	13889	99031	142655	15·00	995771	·29	146885	15·29	853115	1
60	13917	99027	143555	14·96	995753	·29	147803	15·26	852197	0
	N. cos.	N. sine.	L. cos.	D. 1″	L. sine.		L. cot.	D. 1″	L. tang.	′

Nat. Functions.			Logarithmic Functions + 10.							
′	N.sine.	N. cos.	L. sine.	D. 1″	L. cos.	D.1″	L. tang.	D 1.″	L. cot.	
0	13917	99027	9·143555	14·96	9·995753	·30	9·147803	15·26	10·852197	60
1	13946	99023	144453	14·93	995735	·30	148718	15·23	851282	59
2	13975	99019	145349	14·90	995717	·30	149632	15·20	850368	58
3	14004	99015	146243	14·87	995699	·30	150544	15·17	849456	57
4	14033	99011	147136	14·84	995681	·30	151454	15·14	848546	56
5	14061	99006	148026	14·81	995664	·30	152363	15·11	847637	55
6	14090	99002	148915	14·78	995646	·30	153269	15·08	846731	54
7	14119	98998	149802	14·75	995628	·30	154174	15·05	845826	53
8	14148	98994	150686	14·72	995610	·30	155077	15·02	844923	52
9	14177	98990	151569	14·69	995591	·30	155978	14·99	844022	51
10	14205	98986	152451	14·66	995573	·30	156877	14·96	843123	50
11	14234	98982	9·153330	14·63	9·995555	·30	9·157775	14·93	10·842225	49
12	14263	98978	154208	14·60	995537	·30	158671	14·90	841329	48
13	14292	98973	155083	14·57	995519	·30	159565	14·87	840435	47
14	14320	98969	155957	14·54	995501	·31	160457	14·84	839543	46
15	14349	98965	156830	14·51	995482	·31	161347	14·81	838653	45
16	14378	98961	157700	14·48	995464	·31	162236	14·79	837764	44
17	14407	98957	158569	14·45	995446	·31	163123	14·76	836877	43
18	14436	98953	159435	14·42	995427	·31	164008	14·73	835992	42
19	14464	98948	160301	14·39	995409	·31	164892	14·70	835108	41
20	14493	98944	161164	14·36	995390	·31	165774	14·67	834226	40
21	14522	98940	9·162025	14·33	9·995372	·31	9·166654	14·64	10·833346	39
22	14551	98936	162885	14·30	995353	·31	167532	14·61	832468	38
23	14580	98931	163743	14·27	995334	·31	168409	14·58	831591	37
24	14608	98927	164600	14·24	995316	·31	169284	14·55	830716	36
25	14637	98923	165454	14·22	995297	·31	170157	14·53	829843	35
26	14666	98919	166307	14·19	995278	·31	171029	14·50	828971	34
27	14695	98914	167159	14·16	995260	·31	171899	14·47	828101	33
28	14723	98910	168008	14·13	995241	·32	172767	14·44	827233	32
29	14752	98906	168856	14·10	995222	·32	173634	14·42	826366	31
30	14781	98902	169702	14·07	995203	·32	174499	14·39	825501	30
31	14810	98897	9·170547	14·05	9·995184	·32	9·175362	14·36	10·824638	29
32	14838	98893	171389	14·02	995165	·32	176224	14·33	823776	28
33	14867	98889	172230	13·99	995146	·32	177084	14·31	822916	27
34	14896	98884	173070	13·96	995127	·32	177942	14·28	822058	26
35	14925	98880	173908	13·94	995108	·32	178799	14·25	821201	25
36	14954	98876	174744	13·91	995089	·32	179655	14·23	820345	24
37	14982	98871	175578	13·88	995070	·32	180508	14·20	819492	23
38	15011	98867	176411	13·86	995051	·32	181360	14·17	818640	22
39	15040	98863	177242	13·83	995032	·32	182211	14·15	817789	21
40	15069	98858	178072	13·80	995013	·32	183059	14·12	816941	20
41	15097	98854	9·178900	13·77	9·994993	·32	9·183907	14·09	10·816093	19
42	15126	98849	179726	13·74	994974	·32	184752	14·07	815248	18
43	15155	98845	180551	13·72	994955	·32	185597	14·04	814403	17
44	15184	98841	181374	13·69	994935	·32	186439	14·02	813561	16
45	15212	98836	182196	13·66	994916	·33	187280	13·99	812720	15
46	15241	98832	183016	13·64	994896	·33	188120	13·96	811880	14
47	15270	98827	183834	13·61	994877	·33	188958	13·93	811042	13
48	15299	98823	184651	13·59	994857	·33	189794	13·91	810206	12
49	15327	98818	185466	13·56	994838	·33	190629	13·89	809371	11
50	15356	98814	186280	13·53	994818	·33	191462	13·86	808538	10
51	15385	98809	9·187092	13·51	9·994798	·33	9·192294	13·84	10·807706	9
52	15414	98805	187903	13·48	994779	·33	193124	13·81	806876	8
53	15442	98800	188712	13·46	994759	·33	193953	13·79	806047	7
54	15471	98796	189519	13·43	994739	·33	194780	13·76	805220	6
55	15500	98791	190325	13·41	994719	·33	195606	13·74	804394	5
56	15529	98787	191130	13·38	994700	·33	196430	13·71	803570	4
57	15557	98782	191933	13·36	994680	·33	197253	13·69	802747	3
58	15586	98778	192734	13·33	994660	·33	198074	13·66	801926	2
59	15615	98773	193534	13·30	994640	·33	198894	13·64	801106	1
60	15643	98769	194332	13·28	994620	·33	199713	13·61	800287	0
	N. cos.	N. sine.	L. cos.	D. 1″	L. sine.		L. cot.	D. 1″	L. tang.	′

Nat. Functions.			Logarithmic Functions + 10.							
′	N. sine.	N. cos.	L. sine.	D. 1″	L. cos.	D.1″	L. tang.	D. 1″	L. cot.	
0	15643	98769	9·194332	13·28	9·994620	·33	9·199713	13·61	10·800287	60
1	15672	98764	195129	13·26	994600	·33	200529	13·59	799471	59
2	15701	98760	195925	13·23	994580	·33	201345	13·56	798655	58
3	15730	98755	196719	13·21	994560	·34	202159	13·54	797841	57
4	15758	98751	197511	13·18	994540	·34	202971	13·52	797029	56
5	15787	98746	198302	13·16	994519	·34	203782	13·49	796218	55
6	15816	98741	199091	13·13	994499	·34	204592	13·47	795408	54
7	15845	98737	199879	13·11	994479	·34	205400	13·45	794600	53
8	15873	98732	200666	13·08	994459	·34	206207	13·42	793793	52
9	15902	98728	201451	13·06	994438	·34	207013	13·40	792987	51
10	15931	98723	202234	13·04	994418	·34	207817	13·38	792183	50
11	15959	98718	9·203017	13·01	9·994397	·34	9·208619	13·35	10·791381	49
12	15988	98714	203797	12·99	994377	·34	209420	13·33	790580	48
13	16017	98709	204577	12·96	994357	·34	210220	13·31	789780	47
14	16046	98704	205354	12·94	994336	·34	211018	13·28	788982	46
15	16074	98700	206131	12·92	994316	·34	211815	13·26	788185	45
16	16103	98695	206906	12·89	994295	·34	212611	13·24	787389	44
17	16132	98690	207679	12·87	994274	·35	213405	13·21	786595	43
18	16160	98686	208452	12·85	994254	·35	214198	13·19	785802	42
19	16189	98681	209222	12·82	994233	·35	214989	13·17	785011	41
20	16218	98676	209992	12·80	994212	·35	215780	13·15	784220	40
21	16246	98671	9·210760	12·78	9·994191	·35	9·216568	13·12	10·783432	39
22	16275	98667	211526	12·75	994171	·35	217356	13·10	782644	38
23	16304	98662	212291	12·73	994150	·35	218142	13·08	781858	37
24	16333	98657	213055	12·71	994129	·35	218926	13·05	781074	36
25	16361	98652	213818	12·68	994108	·35	219710	13·03	780290	35
26	16390	98648	214579	12·66	994087	·35	220492	13·01	779508	34
27	16419	98643	215338	12·64	994066	·35	221272	12·99	778728	33
28	16447	98638	216097	12·61	994045	·35	222052	12·97	777948	32
29	16476	98633	216854	12·59	994024	·35	222830	12·94	777170	31
30	16505	98629	217609	12·57	994003	·35	223606	12·92	776394	30
31	16533	98624	9·218363	12·55	9·993981	·35	9·224382	12·90	10·775618	29
32	16562	98619	219116	12·53	993960	·35	225156	12·88	774844	28
33	16591	98614	219868	12·50	993939	·35	225929	12·86	774071	27
34	16620	98609	220618	12·48	993918	·35	226700	12·84	773300	26
35	16648	98604	221367	12·46	993896	·36	227471	12·81	772529	25
36	16677	98600	222115	12·44	993875	·36	228239	12·79	771761	24
37	16706	98595	222861	12·42	993854	·36	229007	12·77	770993	23
38	16734	98590	223606	12·39	993832	·36	229773	12·75	770227	22
39	16763	98585	224349	12·37	993811	·36	230539	12·73	769461	21
40	16792	98580	225092	12·35	993789	·36	231302	12·71	768698	20
41	16820	98575	9·225833	12·33	9·993768	·36	9·232065	12·69	10·767935	19
42	16849	98570	226573	12·31	993746	·36	232826	12·67	767174	18
43	16878	98565	227311	12·28	993725	·36	233586	12·65	766414	17
44	16906	98561	228048	12·26	993703	·36	234345	12·62	765655	16
45	16935	98556	228784	12·24	993681	·36	235103	12·60	764897	15
46	16964	98551	229518	12·22	993660	·36	235859	12·58	764141	14
47	16992	98546	230252	12·20	993638	·36	236614	12·56	763386	13
48	17021	98541	230984	12·18	993616	·36	237368	12·54	762632	12
49	17050	98536	231715	12·16	993594	·37	238120	12·52	761880	11
50	17078	98531	232444	12·14	993572	·37	238872	12·50	761128	10
51	17107	98526	9·233172	12·12	9·993550	·37	9·239622	12·48	10·760378	9
52	17136	98521	233899	12·09	993528	·37	240371	12·46	759629	8
53	17164	98516	234625	12·07	993506	·37	241118	12·44	758882	7
54	17193	98511	235349	12·05	993484	·37	241865	12·42	758135	6
55	17222	98506	236073	12·03	993462	·37	242610	12·40	757390	5
56	17250	98501	236795	12·01	993440	·37	243354	12·38	756646	4
57	17279	98496	237515	11·99	993418	·37	244097	12·36	755903	3
58	17308	98491	238235	11·97	993396	·37	244839	12·34	755161	2
59	17336	98486	238953	11·95	993374	·37	245579	12·32	754421	1
60	17365	98481	239670	11·93	993351	·37	246319	12·30	753681	0
	N. cos.	N. sine.	L. cos.	D. 1″	L. sine.		L. cot.	D. 1″	L. tang.	

Nat. Functions.			Logarithmic Functions + 10.							
′	N. sine.	N. cos.	L. sine.	D. 1″	L. cos.	D.1″	L. tang.	D. 1″	L. cot.	
0	17365	98481	9·239670	11·93	9·993351	·37	9·246319	12·30	10·753681	60
1	17393	98476	240386	11·91	993329	·37	247057	12·28	752943	59
2	17422	98471	241101	11·89	993307	·37	247794	12·26	752206	58
3	17451	98466	241814	11·87	993285	·37	248530	12·24	751470	57
4	17479	98461	242526	11·85	993262	·37	249264	12·22	750736	56
5	17508	98455	243237	11·83	993240	·37	249998	12·20	750002	55
6	17537	98450	243947	11·81	993217	·38	250730	12·18	749270	54
7	17565	98445	244656	11·79	993195	·38	251461	12·17	748539	53
8	17594	98440	245363	11·77	993172	·38	252191	12·15	747809	52
9	17623	98435	246069	11·75	993149	·38	252920	12·13	747080	51
10	17651	98430	246775	11·73	993127	·38	253648	12·11	746352	50
11	17680	98425	9·247478	11·71	9·993104	·38	9·254374	12·09	10·745626	49
12	17708	98420	248181	11·69	993081	·38	255100	12·07	744900	48
13	17737	98414	248883	11·67	993059	·38	255824	12·05	744176	47
14	17766	98409	249583	11·65	993036	·38	256547	12·03	743453	46
15	17794	98404	250282	11·63	993013	·38	257269	12·01	742731	45
16	17823	98399	250980	11·61	992990	·38	257990	12·00	742010	44
17	17852	98394	251677	11·59	992967	·38	258710	11·98	741290	43
18	17880	98389	252373	11·58	992944	·38	259429	11·96	740571	42
19	17909	98383	253067	11·56	992921	·38	260146	11·94	739854	41
20	17937	98378	253761	11·54	992898	·38	260863	11·92	739137	40
21	17966	98373	9·254453	11·52	9·992875	·38	9·261578	11·90	10·738422	39
22	17995	98368	255144	11·50	992852	·38	262292	11·89	737708	38
23	18023	98362	255834	11·48	992829	·39	263005	11·87	736995	37
24	18052	98357	256523	11·46	992806	·39	263717	11·85	736283	36
25	18081	98352	257211	11·44	992783	·39	264428	11·83	735572	35
26	18109	98347	257898	11·42	992759	·39	265138	11·81	734862	34
27	18138	98341	258583	11·41	992736	·39	265847	11·79	734153	33
28	18166	98336	259268	11·39	992713	·39	266555	11·78	733445	32
29	18195	98331	259951	11·37	992690	·39	267261	11·76	732739	31
30	18224	98325	260633	11·35	992666	·39	267967	11·74	732033	30
31	18252	98320	9·261314	11·33	9·992643	·39	9·268671	11·72	10·731329	29
32	18281	98315	261994	11·31	992619	·39	269375	11·70	730625	28
33	18309	98310	262673	11·30	992596	·39	270077	11·69	729923	27
34	18338	98304	263351	11·28	992572	·39	270779	11·67	729221	26
35	18367	98299	264027	11·26	992549	·39	271479	11·65	728521	25
36	18395	98294	264703	11·24	992525	·39	272178	11·64	727822	24
37	18424	98288	265377	11·22	992501	·39	272876	11·62	727124	23
38	18452	98283	266051	11·20	992478	·40	273573	11·60	726427	22
39	18481	98277	266723	11·19	992454	·40	274269	11·58	725731	21
40	18509	98272	267395	11·17	992430	·40	274964	11·57	725036	20
41	18538	98267	9·268065	11·15	9·992406	·40	9·275658	11·55	10·724342	19
42	18567	98261	268734	11·13	992382	·40	276351	11·53	723649	18
43	18595	98256	269402	11·11	992359	·40	277043	11·51	722957	17
44	18624	98250	270069	11·10	992335	·40	277734	11·50	722266	16
45	18652	98245	270735	11·08	992311	·40	278424	11·48	721576	15
46	18681	98240	271400	11·06	992287	·40	279113	11·47	720887	14
47	18710	98234	272064	11·05	992263	·40	279801	11·45	720199	13
48	18738	98229	272726	11·03	992239	·40	280488	11·43	719512	12
49	18767	98223	273388	11·01	992214	·40	281174	11·41	718826	11
50	18795	98218	274049	10·99	992190	·40	281858	11·40	718142	10
51	18824	98212	9·274708	10·98	9·992166	·40	9·282542	11·38	10·717458	9
52	18852	98207	275367	10·96	992142	·40	283225	11·36	716775	8
53	18881	98201	276024	10·94	992117	·41	283907	11·35	716093	7
54	18910	98196	276681	10·92	992093	·41	284588	11·33	715412	6
55	18938	98190	277337	10·91	992069	·41	285268	11·31	714732	5
56	18967	98185	277991	10·89	992044	·41	285947	11·30	714053	4
57	18995	98179	278644	10·87	992020	·41	286624	11·28	713376	3
58	19024	98174	279297	10·86	991996	·41	287301	11·26	712699	2
59	19052	98168	279948	10·84	991971	·41	287977	11·25	712023	1
60	19081	98163	280599	10·82	991947	·41	288652	11·23	711348	0
	N. cos.	N. sine.	L. cos.	D. 1″	L. sine.		L. cot.	D. 1″	L. tang.	′

Nat. Functions.			Logarithmic Functions + 10.						
′	N. sine.	N. cos.	L. sine.	D. 1″	L. cos.	D. 1″	L. tang.	D. 1″	L. cot.	
0	19081	98163	9·280599	10·82	9·991947	·41	9·288652	11·23	10·711348	60
1	19109	98157	281248	10·81	991922	·41	289326	11·22	710674	59
2	19138	98152	281897	10·79	991897	·41	289999	11·20	710001	58
3	19167	98146	282544	10·77	991873	·41	290671	11·18	709329	57
4	19195	98140	283190	10·76	991848	·41	291342	11·17	708658	56
5	19224	98135	283836	10·74	991823	·41	292013	11·15	707987	55
6	19252	98129	284480	10·72	991799	·41	292682	11·14	707318	54
7	19281	98124	285124	10·71	991774	·42	293350	11·12	706650	53
8	19309	98118	285766	10·69	991749	·42	294017	11·11	705983	52
9	19338	98112	286408	10·67	991724	·42	294684	11·09	705316	51
10	19366	98107	287048	10·66	991699	·42	295349	11·07	704651	50
11	19395	98101	9·287687	10·64	9·991674	·42	9·296013	11·06	10·703987	49
12	19423	98096	288326	10·63	991649	·42	296677	11·04	703323	48
13	19452	98090	288964	10·61	991624	·42	297339	11·03	702661	47
14	19481	98084	289600	10·59	991599	·42	298001	11·01	701999	46
15	19509	98079	290236	10·58	991574	·42	298662	11·00	701338	45
16	19538	98073	290870	10·56	991549	·42	299322	10·98	700678	44
17	19566	98067	291504	10·54	991524	·42	299980	10·96	700020	43
18	19595	98061	292137	10·53	991498	·42	300638	10·95	699362	42
19	19623	98056	292768	10·51	991473	·42	301295	10·93	698705	41
20	19652	98050	293399	10·50	991448	·42	301951	10·92	698049	40
21	19680	98044	9·294029	10·48	9·991422	·42	9·302607	10·90	10·697393	39
22	19709	98039	294658	10·46	991397	·42	303261	10·89	696739	38
23	19737	98033	295286	10·45	991372	·43	303914	10·87	696086	37
24	19766	98027	295913	10·43	991346	·43	304567	10·86	695433	36
25	19794	98021	296539	10·42	991321	·43	305218	10·84	694782	35
26	19823	98016	297164	10·40	991295	·43	305869	10·83	694131	34
27	19851	98010	297788	10·39	991270	·43	306519	10·81	693481	33
28	19880	98004	298412	10·37	991244	·43	307168	10·80	692832	32
29	19908	97998	299034	10·36	991218	·43	307815	10·78	692185	31
30	19937	97992	299655	10·34	991193	·43	308463	10·77	691537	30
31	19965	97987	9·300276	10·32	9·991167	·43	9·309109	10·75	10·690891	29
32	19994	97981	300895	10·31	991141	·43	309754	10·74	690246	28
33	20022	97975	301514	10·29	991115	·43	310398	19·73	689602	27
34	20051	97969	302132	10·28	991090	·43	311042	10·71	688958	26
35	20079	97963	302748	10·26	991064	·43	311685	10·70	688315	25
36	20108	97958	303364	10·25	991038	·43	312327	10·68	687673	24
37	20136	97952	303979	10·23	991012	·43	312967	10·67	687033	23
38	20165	97946	304593	10·22	990986	·43	313608	10·65	686392	22
39	20193	97940	305207	10·20	990960	·43	314247	10·64	685753	21
40	20222	97934	305819	10·19	990934	·44	314885	10·62	685115	20
41	20250	97928	9·306430	10·17	9·990908	·44	9·315523	10·61	10·684477	19
42	20279	97922	307041	10·16	990882	·44	316159	10·60	683841	18
43	20307	97916	307650	10·14	990855	·44	316793	10·58	683205	17
44	20336	97910	308259	10·13	990829	·44	317430	10·57	682570	16
45	20364	97905	308867	10·11	990803	·44	318064	10·55	681936	15
46	20393	97899	309474	10·10	990777	·44	318697	10·54	681303	14
47	20421	97893	310080	10·08	990750	·44	319329	10·53	680671	13
48	20450	97887	310685	10·07	990724	·44	319961	10·51	680039	12
49	20478	97881	311289	10·05	990697	·44	320592	10·50	679408	11
50	20507	97875	311893	10·04	990671	·44	321222	10·48	678778	10
51	20535	97869	9·312495	10·03	9·990644	·44	9·321851	10·47	10·678149	9
52	20563	97863	313097	10·01	990618	·44	322479	10·45	677521	8
53	20592	97857	313698	10·00	990591	·44	323106	10·44	676894	7
54	20620	97851	314297	9·98	990565	·44	323733	10·43	676267	6
55	20649	97845	314897	9·97	990538	·44	324358	10·41	675642	5
56	20677	97839	315495	9·96	990511	·45	324983	10·40	675017	4
57	20706	97833	316092	9·94	990485	·45	325607	10·39	674393	3
58	20734	97827	316689	9·93	990458	·45	326231	10·37	673769	2
59	20763	97821	317284	9·91	990431	·45	326853	10·36	673147	1
60	20791	97815	317879	9·90	990404	·45	327475	10·35	672525	0
	N. cos.	N. sine.	L. cos.	D. 1″	L. sine.		L. cot.	D. 1″	L. tang.	′

Nat. Functions.		Logarithmic Functions + 10.							
′	N.sine.	N. cos.	L. sine.	D. 1″	L. cos.	D.1″	L. tang.	D. 1″	L. cot.	
0	20791	97815	9·317879	9·90	9·990404	·45	9·327474	10·35	10·672526	60
1	20820	97809	318473	9·88	990378	·45	328095	10·33	671905	59
2	20848	97803	319066	9·87	990351	·45	328715	10·32	671285	58
3	20877	97797	319658	9·86	990324	·45	329334	10·30	670666	57
4	20905	97791	320249	9·84	990297	·45	329953	10·29	670047	56
5	20933	97784	320840	9·83	990270	·45	330570	10·28	669430	55
6	20962	97778	321430	9·82	990243	·45	331187	10·26	668813	54
7	20990	97772	322019	9·80	990215	·45	331803	10·25	668197	53
8	21019	97766	322607	9·79	990188	·45	332418	10·24	667582	52
9	21047	97760	323194	9·77	990161	·45	333033	10·23	666967	51
10	21076	97754	323780	9·76	990134	·45	333646	10·21	666354	50
11	21104	97748	9·324366	9·75	9·990107	·46	9·334259	10·20	10·665741	49
12	21132	97742	324950	9·73	990079	·46	334871	10·19	665129	48
13	21161	97735	325534	9·72	990052	·46	335482	10·17	664518	47
14	21189	97729	326117	9·70	990025	·46	336093	10·16	663907	46
15	21218	97723	326700	9·69	989997	·46	336702	10·15	663298	45
16	21246	97717	327281	9·68	989970	·46	337311	10·13	662689	44
17	21275	97711	327862	9·66	989942	·46	337919	10·12	662081	43
18	21303	97705	328442	9·65	989915	·46	338527	10·11	661473	42
19	21331	97698	329021	9·64	989887	·46	339133	10·10	660867	41
20	21360	97692	329599	9·62	989860	·46	339739	10·08	660261	40
21	21388	97686	9·330176	9·61	9·989832	·46	9·340344	10·07	10·659656	39
22	21417	97680	330753	9·60	989804	·46	340948	10·06	659052	38
23	21445	97673	331329	9·58	989777	·46	341552	10·04	658448	37
24	21474	97667	331903	9·57	989749	·47	342156	10·03	657845	36
25	21502	97661	332478	9·56	989721	·47	342757	10·02	657243	35
26	21530	97655	333051	9·54	989693	·47	343358	10·00	656642	34
27	21559	97648	333624	9·53	989665	·47	343958	9·99	656042	33
28	21587	97642	334195	9·52	989637	·47	344558	9·98	655442	32
29	21616	97636	334766	9·50	989609	·47	345157	9·97	654843	31
30	21644	97630	335337	9·49	989582	·47	345755	9·96	654245	30
31	21672	97623	9·335906	9·48	9·989553	·47	9·346353	9·94	10·653647	29
32	21701	97617	336475	9·46	989525	·47	346949	9·93	653051	28
33	21729	97611	337043	9·45	989497	·47	347545	9·92	652455	27
34	21758	97604	337610	9·44	989469	·47	348141	9·91	651859	26
35	21786	97598	338176	9·43	989441	·47	348735	9·90	661265	25
36	21814	97592	338742	9·41	989413	·47	349329	9·88	650671	24
37	21843	97585	339306	9·40	989384	·47	349922	9·87	650078	23
38	21871	97579	339871	9·39	989356	·47	350514	9·86	649486	22
39	21899	97573	340434	9·37	989328	·47	351106	9·85	648894	21
40	21928	97566	340996	9·36	989300	·47	351697	9·83	648303	20
41	21956	97560	9·341558	9·35	9·989271	·47	9·352287	9·82	10·647713	19
42	21985	97553	342119	9·34	989243	·47	352876	9·81	647124	18
43	22013	97547	342679	9·32	989214	·47	353465	9·80	646535	17
44	22041	97541	343239	9·31	989186	·47	354053	9·79	645947	16
45	22070	97534	343797	9·30	989157	·47	354640	9·77	645360	15
46	22098	97528	344355	9·29	989128	·48	355227	9·76	644773	14
47	22126	97521	344912	9·27	989100	·48	355813	9·75	644187	13
48	22155	97515	345469	9·26	989071	·48	356398	9·74	643602	12
49	22183	97508	346024	9·25	989042	·48	356982	9·73	643018	11
50	22212	97502	346579	9·24	989014	·48	357566	9·71	642434	10
51	22240	97496	9·347134	9·22	9·988985	·48	9·358149	9·70	10·641851	9
52	22268	97489	347687	9·21	988956	·48	358731	9·69	641269	8
53	22297	97483	348240	9·20	988927	·48	359313	9·68	640687	7
54	22325	97476	348792	9·19	988898	·48	359893	9·67	640107	6
55	22353	97470	349343	9·17	988869	·48	360474	9·66	639526	5
56	22382	97463	349893	9·16	988840	·48	361053	9·65	638947	4
57	22410	97457	350443	9·15	988811	·49	361632	9·63	638368	3
58	22438	97450	350992	9·14	988782	·49	362210	9·62	637790	2
59	22467	97444	351540	9·13	988753	·49	362787	9·61	637213	1
60	22495	97437	352088	9·11	988724	·49	363364	9·60	636636	0
	N. cos.	N. sine.	L. cos.	D. 1″	L. sine.		L. cot.	D. 1″	L. tang.	′

Nat. Functions.			Logarithmic Functions + 10.							
′	N. sine.	N. cos.	L. sine.	D. 1″	L. cos.	D.1″	L. tang.	D 1.″	L. cot.	
0	22495	97437	9·352088	9·11	9·988724	·49	9·363364	9·60	10·636636	60
1	22523	97430	352635	9·10	988695	·49	363940	9·59	636060	59
2	22552	97424	353181	9·09	988666	·49	364515	9·58	635485	58
3	22580	97417	353726	9·08	988636	·49	365090	9·57	634910	57
4	22608	97411	354271	9·07	988607	·49	365664	9·55	634336	56
5	22637	97404	354815	9·05	988578	·49	366237	9·54	633763	55
6	22665	97398	355358	9·04	988548	·49	366810	9·53	633190	54
7	22693	97391	355901	9·03	988519	·49	367382	9·52	632618	53
8	22722	97384	356443	9·02	988489	·49	367953	9·51	632047	52
9	22750	97378	356984	9·01	988460	·49	368524	9·50	631476	51
10	22778	97371	357524	8·99	988430	·49	369094	9·49	630906	50
11	22807	97365	9·358064	8·98	9·988401	·49	9·369663	9·48	10·630337	49
12	22835	97358	358603	8·97	988371	·49	370232	9·46	629768	48
13	22863	97351	359141	8·96	988342	·49	370799	9·45	629201	47
14	22892	97345	359678	8·95	988312	·50	371367	9·44	628633	46
15	22920	97338	360215	8·93	988282	·50	371933	9·43	628067	45
16	22948	97331	360752	8·92	988252	·50	372499	9·42	627501	44
17	22977	97325	361287	8·91	988223	·50	373064	9·41	626936	43
18	23005	97318	361822	8·90	988193	·50	373629	9·40	626371	42
19	23033	97311	362356	8·89	988163	·50	374193	9·39	625807	41
20	23062	97304	362889	8·88	988133	·50	374756	9·38	625244	40
21	23090	97298	9·363422	8·87	9·988103	·50	9·375319	9·37	10·624681	39
22	23118	97291	363954	8·85	988073	·50	375881	9·35	624119	38
23	23146	97284	364485	8·84	988043	·50	376442	9·34	623558	37
24	23175	97278	365016	8·83	988013	·50	377003	9·33	622997	36
25	23203	97271	365546	8·82	987983	·50	377563	9·32	622437	35
26	23231	97264	366075	8·81	987953	·50	378122	9·31	621878	34
27	23260	97257	366604	8·80	987922	·50	378681	9·30	621319	33
28	23288	97251	367131	8·79	987892	·50	379239	9·29	620761	32
29	23316	97244	367659	8·77	987862	·50	379797	9·28	620203	31
30	23345	97237	368185	8·76	987832	·51	380354	9·27	619646	30
31	23373	97230	9·368711	8·75	9·987801	·51	9·380910	9·26	10·619090	29
32	23401	97223	369236	8·74	987771	·51	381466	9·25	618534	28
33	23429	97217	369761	8·73	987740	·51	382020	9·24	617980	27
34	23458	97210	370285	8·72	987710	·51	382575	9·23	617425	26
35	23486	97203	370808	8·71	987679	·51	383129	9·22	616871	25
36	23514	97196	371330	8·70	987649	·51	383682	9·21	616318	24
37	23542	97189	371852	8·69	987618	·51	384234	9·20	615766	23
38	23571	97182	372373	8·67	987588	·51	384786	9·19	615214	22
39	23599	97176	372894	8·66	987557	·51	385337	9·18	614663	21
40	23627	97169	373414	8·65	987526	·51	385888	9·17	614112	20
41	23656	97162	9·373933	8·64	9·987496	·51	9·386438	9·15	10·613562	19
42	23684	97155	374452	8·63	987465	·51	386987	9·14	613013	18
43	23712	97148	374970	8·62	987434	·51	387536	9·13	612464	17
44	23740	97141	375487	8·61	987403	·52	388084	9·12	611916	16
45	23769	97134	376003	8·60	987372	·52	388631	9·11	611369	15
46	23797	97127	376519	8·59	987341	·52	389178	9·10	610822	14
47	23825	97120	377035	8·58	987310	·52	389724	9·09	610276	13
48	23853	97113	377549	8·57	987279	·52	390270	9·08	609730	12
49	23882	97106	378063	8·56	987248	·52	390815	9·07	609185	11
50	23910	97100	378577	8·54	987217	·52	391360	9·06	608640	10
51	23938	97093	9·379089	8·53	9·987186	·52	9·391903	9·05	10·608097	9
52	23966	97086	379601	8·52	987155	·52	392447	9·04	607553	8
53	23995	97079	380113	8·51	987124	·52	392989	9·03	607011	7
54	24023	97072	380624	8·50	987092	·52	393531	9·02	606469	6
55	24051	97065	381134	8·49	987061	·52	394073	9·01	605927	5
56	24079	97058	381643	8·48	987030	·52	394614	9·00	605386	4
57	24108	97051	382152	8·47	986998	·52	395154	8·99	604846	3
58	24136	97044	382661	8·46	986967	·52	395694	8·98	604306	2
59	24164	97037	383168	8·45	986936	·52	396233	8·97	603767	1
60	24192	97030	383675	8·44	986904	·52	396771	8·96	603229	0
	N. cos.	N. sine.	L. cos.	D. 1″	L. sine.		L. cot.	D. 1″	L. tang.	′

Nat. Functions.			Logarithmic Functions + 10.							
′	N.sine.	N. cos.	L. sine.	D. 1″	L. cos.	D.1″	L. tang.	D. 1″	L. cot.	
0	24192	97030	9·383675	8·44	9·986904	·52	9·396771	8·96	10·603229	60
1	24220	97023	384182	8·43	986873	·53	397309	8·96	602691	59
2	24249	97015	384687	8·42	986841	·53	397846	8·95	602154	58
3	24277	97008	385192	8·41	986809	·53	398383	8·94	601617	57
4	24305	97001	385697	8·40	986778	·53	398919	8·93	601081	56
5	24333	96994	386201	8·39	986746	·53	399455	8·92	600545	55
6	24362	96987	386704	8·38	986714	·53	399990	8·91	600010	54
7	24390	96980	387207	8·37	986683	·53	400524	8·90	599476	53
8	24418	96973	387709	8·36	986651	·53	401058	8·89	598942	52
9	24446	96966	388210	8·35	986619	·53	401591	8·88	598409	51
10	24474	96959	388711	8·34	986587	·53	402124	8·87	597876	50
11	24503	96952	9·389211	8·33	9·986555	·53	9·402656	8·86	10·597344	49
12	24531	96945	389711	8·32	986523	·53	403187	8·85	596813	48
13	24559	96937	390210	8·31	986491	·53	403718	8·84	596282	47
14	24587	96930	390708	8·30	986459	·53	404249	8·83	595751	46
15	24615	96923	391206	8·28	986427	·53	404778	8·82	595222	45
16	24644	96916	391703	8·27	986395	·53	405308	8·81	594692	44
17	24672	96909	392199	8·26	986363	·54	405836	8·80	594164	43
18	24700	96902	392695	8·25	986331	·54	406364	8·79	593636	42
19	24728	96894	393191	8·24	986299	·54	406892	8·78	593108	41
20	24756	96887	393685	8·23	986266	·54	407419	8·77	592581	40
21	24784	96880	9·394179	8·22	9·986234	·54	9·407945	8·76	10·592055	39
22	24813	96873	394673	8·21	986202	·54	408471	8·75	591529	38
23	24841	96866	395166	8·20	986169	·54	408997	8·74	591003	37
24	24869	96858	395658	8·19	986137	·54	409521	8·74	590479	36
25	24897	96851	396150	8·18	986104	·54	410045	8·73	589955	35
26	24925	96844	396641	8·17	986072	·54	410569	8·72	589431	34
27	24954	96837	397132	8·17	986039	·54	411092	8·71	588908	33
28	24982	96829	397621	8·16	986007	·54	411615	8·70	588385	32
29	25010	96822	398111	8·15	985974	·54	412137	8·69	587864	31
30	25038	96815	398600	8·14	985942	·54	412658	8·68	587342	30
31	25066	96807	9·399088	8·13	9·985909	·55	9·413179	8·67	10·586821	29
32	25094	96800	399575	8·12	985876	·55	413699	8·66	586301	28
33	25122	96793	400062	8·11	985843	·55	414219	8·65	585781	27
34	25151	96786	400549	8·10	985811	·55	414738	8·64	585262	26
35	25179	96778	401035	8·09	985778	·55	415257	8·64	584743	25
36	25207	96771	401520	8·08	985745	·55	415775	8·63	584225	24
37	25235	96764	402005	8·07	985712	·55	416293	8·62	583707	23
38	25263	96756	402489	8·06	985679	·55	416810	8·61	583190	22
39	25291	96749	402972	8·05	985646	·55	417326	8·60	582674	21
40	25320	96742	403455	8·04	985613	·55	417842	8·59	582158	20
41	25348	96734	9·403938	8·03	9·985580	·55	9·418358	8·58	10·581642	19
42	25376	96727	404420	8·02	985547	·55	418873	8·57	581127	18
43	25404	96719	404901	8·01	985514	·55	419387	8·56	580613	17
44	25432	96712	405382	8·00	985480	·55	419901	8·55	580099	16
45	25460	96705	405862	7 99	985447	·55	420415	8·55	579585	15
46	25488	96697	406341	7·98	985414	·56	420927	8·54	579073	14
47	25516	96690	406820	7·97	985380	·56	421440	8·53	578560	13
48	25545	96682	407299	7·96	985347	·56	421952	8·52	578048	12
49	25573	96675	407777	7·95	985314	·56	422463	8·51	577537	11
50	25601	96667	408254	7·94	985280	·56	422974	8·50	577026	10
51	25629	96660	9·408731	7·94	9·985247	·56	9·423484	8·49	10·576516	9
52	25657	96653	409207	7·93	985213	·56	423993	8·48	576007	8
53	25685	96645	409682	7·92	985180	·56	424503	8·48	575497	7
54	25713	96638	410157	7·91	985146	·56	425011	8·47	574989	6
55	25741	96630	410632	7·90	985113	·56	425519	8·46	574481	5
56	25769	96623	411106	7·89	985079	·56	426027	8·45	573973	4
57	25798	96615	411579	7·88	985045	·56	426534	8·44	573466	3
58	25826	96608	412052	7·87	985011	·56	427041	8·43	572959	2
59	25854	96600	412524	7·86	984978	·56	427547	8·43	572453	1
60	25882	96593	412996	7·85	984944	·56	428052	8·42	571948	0
	N. cos.	N. sine.	L. cos.	D. 1″	L. sine.		L. cot.	D. 1″	L. tang.	′

Nat. Functions.			Logarithmic Functions + 10.						
′	N. sine.	N. cos.	L. sine.	D. 1″	L. cos.	D. 1″	L. tang.	D. 1″	L. cot.	
0	25882	96593	9·412996	7·85	9·984944	·57	9·428052	8·42	10·571948	60
1	25910	96585	413467	7·84	984910	·57	428557	8·41	571443	59
2	25938	96578	413938	7·83	984876	·57	429062	8·40	570938	58
3	25966	96570	414408	7·83	984842	·57	429566	8·39	570434	57
4	25994	96562	414878	7·82	984808	·57	430070	8·38	569930	56
5	26022	96555	415347	7·81	984774	·57	430573	8·38	569427	55
6	26050	96547	415815	7·80	984740	·57	431075	8·37	568925	54
7	26079	96540	416283	7·79	984706	·57	431577	8·36	568423	53
8	26107	96532	416751	7·78	984672	·57	432079	8·35	567921	52
9	26135	96524	417217	7·77	984637	·57	432580	8·34	567420	51
10	26163	96517	417684	7·76	984603	·57	433080	8·33	566920	50
11	26191	96509	9·418150	7·75	9·984569	·57	9·433580	8·32	10·566420	49
12	26219	96502	418615	7·74	984535	·57	434080	8·32	565920	48
13	26247	96494	419079	7·73	984500	·57	434579	8·31	565421	47
14	26275	96486	419544	7·73	984466	·57	435078	8·30	564922	46
15	26303	96479	420007	7·72	984432	·58	435576	8·29	564424	45
16	26331	96471	420470	7·71	984397	·58	436073	8·28	563927	44
17	26359	96463	420933	7·70	984363	·58	436570	8·28	563430	43
18	26387	96456	421395	7·69	984328	·58	437067	8·27	562933	42
19	26415	96448	421857	7·68	984294	·58	437563	8·26	562437	41
20	26443	96440	422318	7·67	984259	·58	438059	8·25	561941	40
21	26471	96433	9·422778	7·67	9·984224	·58	9·438554	8·24	10·561446	39
22	26500	96425	423238	7·66	984190	·58	439048	8·23	560952	38
23	26528	96417	423697	7·65	984155	·58	439543	8·23	560457	37
24	26556	96410	424156	7·64	984120	·58	440036	8·22	559964	36
25	26584	96402	424615	7·63	984085	·58	440529	8·21	559471	35
26	26612	96394	425073	7·62	984050	·58	441022	8·20	558978	34
27	26640	96386	425530	7·61	984015	·58	441514	8·19	558486	33
28	26668	96379	425987	7·60	983981	·58	442006	8·19	557994	32
29	26696	96371	426443	7·60	983946	·58	442497	8·18	557503	31
30	26724	96363	426899	7·59	983911	·58	442988	8·17	557012	30
31	26752	96355	9·427354	7·58	9·983875	·58	9·443479	8·16	10·556521	29
32	26780	96347	427809	7·57	983840	·59	443968	8·16	556032	28
33	26808	96340	428263	7·56	983805	·59	444458	8·15	555542	27
34	26836	96332	428717	7·55	983770	·59	444947	8·14	555053	26
35	26864	96324	429170	7·54	983735	·59	445435	8·13	554565	25
36	26892	96316	429623	7·53	983700	·59	445923	8·12	554077	24
37	26920	96308	430075	7·52	983664	·59	446411	8·12	553589	23
38	26948	96301	430527	7·52	983629	·59	446898	8·11	553102	22
39	26976	96293	430978	7·51	983594	·59	447384	8·10	552616	21
40	27004	96285	431429	7·50	983558	·59	447870	8·09	552130	20
41	27032	96277	9·431879	7·49	9·983523	·59	9·448356	8·09	10·551644	19
42	27060	96269	432329	7·49	983487	·59	448841	8·08	551159	18
43	27088	96261	432778	7·48	983452	·59	449326	8·07	550674	17
44	27116	96253	433226	7·47	983416	·59	449810	8·06	550190	16
45	27144	96246	433675	7·46	983381	·59	450294	8·06	549706	15
46	27172	96238	434122	7·45	983345	·59	450777	8·05	549223	14
47	27200	96230	434569	7·44	983309	·59	451260	8·04	548740	13
48	27228	96222	435016	7·44	983273	·60	451743	8·03	548257	12
49	27256	96214	435462	7·43	983238	·60	452225	8·02	547775	11
50	27284	96206	435908	7·42	983202	·60	452706	8·02	547294	10
51	27312	96198	9·436353	7·41	9·983166	·60	9·453187	8·01	10·546813	9
52	27340	96190	436798	7·40	983130	·60	453668	8·00	546332	8
53	27368	96182	437242	7·40	983094	·60	454148	7·99	545852	7
54	27396	96174	437686	7·39	983058	·60	454628	7·99	545372	6
55	27424	96166	438129	7·38	983022	·60	455107	7·98	544893	5
56	27452	96158	438572	7·37	982986	·60	455586	7·97	544414	4
57	27480	96150	439014	7·36	982950	·60	456064	7·96	543936	3
58	27508	96142	439456	7·36	982914	·60	456542	7·96	543458	2
59	27536	96134	439897	7·35	982878	·60	457019	7·95	542981	1
60	27564	96126	440338	7·34	982842	·60	457496	7·94	542504	0
	N. cos.	N. sine.	L. cos.	D. 1″	L. sine.		L. cot.	D. 1″	L. tang.	′

Nat. Functions.			Logarithmic Functions + 10.							
′	N.sine.	N. cos.	L. sine.	D. 1″	L. cos.	D.1″	L. tang.	D. 1″	L. cot.	
0	27564	96126	9·440338	7·34	9·982842	·60	9·457496	7·94	10·542504	60
1	27592	96118	440778	7·33	982805	·60	457973	7·93	542027	59
2	27620	96110	441218	7·32	982769	·61	458449	7·93	541551	58
3	27648	96102	441658	7·31	982733	·61	458925	7·92	541075	57
4	27676	96094	442096	7·31	982696	·61	459400	7·91	540600	56
5	27704	96086	442535	7·30	982660	·61	459875	7·90	540125	55
6	27731	96078	442973	7·29	982624	·61	460349	7·90	539651	54
7	27759	96070	443410	7·28	982587	·61	460823	7·89	539177	53
8	27787	96062	443847	7·27	982551	·61	461297	7·88	538703	52
9	27815	96054	444284	7·27	982514	·61	461770	7·88	538230	51
10	27843	96046	444720	7·26	982477	·61	462242	7·87	537758	50
11	27871	96037	9·445155	7·25	9·982441	·61	9·462714	7·86	10·537286	49
12	27899	96029	445590	7·24	982404	·61	463186	7·85	536814	48
13	27927	96021	446025	7·23	982367	·61	463658	7·85	536342	47
14	27955	96013	446459	7·23	982331	·61	464129	7·84	535871	46
15	27983	96005	446893	7·22	982294	·61	464599	7·83	535401	45
16	28011	95997	447326	7·21	982257	·61	465069	7·83	534931	44
17	28039	95989	447759	7·20	982220	·62	465539	7·82	534461	43
18	28067	95981	448191	7·20	982183	·62	466008	7·81	533992	42
19	28095	95972	448623	7·19	982146	·62	466476	7·80	533524	41
20	28123	95964	449054	7·18	982109	·62	466945	7·80	533055	40
21	28151	95956	9·449485	7·17	9·982072	·62	9·467413	7·79	10·532587	39
22	28178	95948	449915	7·16	982035	·62	467880	7·78	532120	38
23	28206	95940	450345	7·16	981998	·62	468347	7·78	531653	37
24	28234	95931	450775	7·15	981961	·62	468814	7·77	531186	36
25	28262	95923	451204	7·14	981924	·62	469280	7·76	530720	35
26	28290	95915	451632	7·13	981886	·62	469746	7·75	530254	34
27	28318	95907	452060	7·13	981849	·02	470211	7·75	529789	33
28	28346	95898	452488	7·12	981812	·62	470676	7·74	529324	32
29	28374	95890	452915	7·11	981774	·62	471141	7·73	528859	31
30	28402	95882	453342	7·10	981737	·62	471605	7·73	528395	30
31	28429	95874	9·453768	7·10	9·981699	·63	9·472068	7·72	10·527932	29
32	28457	95865	454194	7·09	981662	·63	472532	7·71	527468	28
33	28485	95857	454619	7·08	981625	·63	472995	7·71	527005	27
34	28513	95849	455044	7·07	981587	·63	473457	7·70	526543	26
35	28541	95841	455469	7·07	981549	·63	473919	7·69	526081	25
36	28569	95832	455893	7·06	981512	·63	474381	7·69	525619	24
37	28597	95824	456316	7·05	981474	·63	474842	7·68	525158	23
38	28625	95816	456739	7·04	981436	·63	475303	7·67	524697	22
39	28652	95807	457162	7·04	981399	·63	475763	7·67	524237	21
40	28680	95799	457584	7·03	981361	·63	476223	7·66	523777	20
41	28708	95791	9·458006	7·02	9·981323	·63	9·476683	7·65	10·523317	19
42	28736	95782	458427	7·01	981285	·63	477142	7·65	522858	18
43	28764	95774	458848	7·01	981247	·63	477601	7·64	522399	17
44	28792	95766	459268	7·00	981209	·63	478059	7·63	521941	16
45	28820	95757	459688	6·99	981171	·63	478517	7·63	521483	15
46	28847	95749	460108	6·98	981133	·64	478975	7·62	521025	14
47	28875	95740	460527	6·98	981095	·64	479432	7·61	520568	13
48	28903	95732	460946	6·97	981057	·64	479889	7·61	520111	12
49	28931	95724	461364	6·96	981019	·64	480345	7·60	519655	11
50	28959	95715	461782	6·95	980981	·64	480801	7·59	519199	10
51	28987	95707	9·462199	6·95	9·980942	·64	9·481257	7·59	10·518743	9
52	29015	95698	462616	6·94	980904	·64	481712	7·58	518288	8
53	29042	95690	463032	6·93	980866	·64	482167	7·57	517833	7
54	29070	95681	463448	6·93	980827	·64	482621	7·57	517379	6
55	29098	95673	463864	6·92	980789	·64	483075	7·56	516925	5
56	29126	95664	464279	6·91	980750	·64	483529	7·55	516471	4
57	29154	95656	464694	6·90	980712	·64	483982	7·55	516018	3
58	29182	95647	465108	6·90	980673	·64	484435	7·54	515565	2
59	29209	95639	465522	6·89	980635	·64	484887	7·53	515113	1
60	29237	95630	465935	6·88	980596	·64	485339	7·53	514661	0
	N. cos.	N. sine.	L. cos.	D. 1″	L. sine.		L. cot.	D. 1″	L. tang.	′

Nat. Functions.			Logarithmic Functions + 10.						
′	N.sine.	N. cos.	L. sine.	D. 1″	L. cos.	D.1″	L. tang.	D. 1″	L. cot.	
0	29237	95630	9·465935	6·88	9·980596	·64	9·485339	7·55	10·514661	60
1	29265	95622	466348	6·88	980558	·64	485791	7·52	514209	59
2	29293	95613	466761	6·87	980519	·65	486242	7·51	513758	58
3	29321	95605	467173	6·86	980480	·65	486693	7·51	513307	57
4	29348	95596	467585	6·85	980442	·65	487143	7·50	512857	56
5	29376	95588	467996	6·85	980403	·65	487593	7·49	512407	55
6	29404	95579	468407	6·84	980364	·65	488043	7·49	511957	54
7	29432	95571	468817	6·83	980325	·65	488492	7·48	511508	53
8	29460	95562	469227	6·83	980286	·65	488941	7·47	511059	52
9	29487	95554	469637	6·82	980247	·65	489390	7·47	510610	51
10	29515	95545	470046	6·81	980208	·65	489838	7·46	510162	50
11	29543	95536	9·470455	6·80	9·980169	·65	9·490286	7·46	10·509714	49
12	29571	95528	470863	6·80	980130	·65	490733	7·45	509267	48
13	29599	95519	471271	6·79	980091	·65	491180	7·44	508820	47
14	29626	95511	471679	6·78	980052	·65	491627	7·44	508373	46
15	29654	95502	472086	6·78	980012	·65	492073	7·43	507927	45
16	29682	95493	472492	6·77	979973	·65	492519	7·43	507481	44
17	29710	95485	472898	6·76	979934	·66	492965	7·42	507035	43
18	29737	95476	473304	6·76	979895	·66	493410	7·41	506590	42
19	29765	95467	473710	6·75	979855	·66	493854	7·40	506146	41
20	29793	95459	474115	6·74	979816	·66	494299	7·40	505701	40
21	29821	95450	9·474519	6·74	9·979776	·66	9·494743	7·40	10·505257	39
22	29849	95441	474923	6·73	979737	·66	495186	7·39	504814	38
23	29876	95433	475327	6·72	979697	·66	495630	7·38	504370	37
24	29904	95424	475730	6·72	979658	·66	496073	7·37	503927	36
25	29932	95415	476133	6·71	979618	·66	496515	7·37	503485	35
26	29960	95407	476536	6·70	979579	·66	496957	7·36	503043	34
27	29987	95398	476938	6·69	979539	·66	497399	7·36	502601	33
28	30015	95389	477340	6·69	979499	·66	497841	7·35	502159	32
29	30043	95380	477741	6·68	979459	·66	498282	7·34	501718	31
30	30071	95372	478142	6·67	979420	·66	498722	7·34	501278	30
31	30098	95363	9·478542	6·67	9·979380	·66	9·499163	7·33	10·500837	29
32	30126	95354	478942	6·66	979340	·66	499603	7·33	500397	28
33	30154	95345	479342	6·65	979300	·67	500042	7·32	499958	27
34	30182	95337	479741	6·65	979260	·67	500481	7·31	499519	26
35	30209	95328	480140	6·64	979220	·67	500920	7·31	499080	25
36	30237	95319	480539	6·63	979180	·67	501359	7·30	498641	24
37	30265	95310	480937	6·63	979140	·67	501797	7·30	498203	23
38	30292	95301	481334	6·62	979100	·67	502235	7·29	497765	22
39	30320	95293	481731	6·61	979059	·67	502672	7·28	497328	21
40	30348	95284	482128	6·61	979019	·67	503109	7·28	496891	20
41	30376	95275	9·482525	6·60	9·978979	·67	9·503546	7·27	10·496454	19
42	30403	95266	482921	6·59	978939	·67	503982	7·27	496018	18
43	30431	95257	483316	6·59	978898	·67	504418	7·26	495582	17
44	30459	95248	483712	6·58	978858	·67	504854	7·25	495146	16
45	30486	95240	484107	6·57	978817	·67	505289	7·25	494711	15
46	30514	95231	484501	6·57	978777	·67	505724	7·24	494276	14
47	30542	95222	484895	6·56	978736	·67	506159	7·24	493841	13
48	30570	95213	485289	6·55	978696	·68	506593	7·23	493407	12
49	30597	95204	485682	6·55	978655	·68	507027	7·22	492973	11
50	30625	95195	486075	6·54	978615	·68	507460	7·22	492540	10
51	30653	95186	9·486467	6·53	9·978574	·68	9·507893	7·21	10·492107	9
52	30680	95177	486860	6·53	978533	·68	508326	7·21	491674	8
53	30708	95168	487251	6·52	978493	·68	508759	7·20	491241	7
54	30736	95159	487643	6·51	978452	·68	509191	7·19	490809	6
55	30763	95150	488034	6·51	978411	·68	509622	7·19	490378	5
56	30791	95142	488424	6·50	978370	·68	510054	7·18	489946	4
57	30819	95133	488814	6·50	978329	·68	510485	7·18	489515	3
58	30846	95124	489204	6·49	978288	·68	510916	7·17	489084	2
59	30874	95115	489593	6·48	978247	·68	511346	7·16	488654	1
60	30902	95106	489982	6·48	978206	·68	511776	7·16	488824	0
	N.cos.	N. sine.	L. cos.	D. 1″	L. sine.		L. cot.	D. 1″	L. tang.	′

Nat. Functions.			Logarithmic Functions + 10.							
′	N.sine.	N. cos.	L. sine.	D. 1″	L. cos.	D.1″	L. tang.	D. 1″	L. cot.	
0	30902	95106	9·489982	6·48	9·978206	·68	9·511776	7·16	10·488224	60
1	30929	95097	490071	6·48	978165	·68	512206	7·16	487794	59
2	30957	95088	490159	6·47	978124	·68	512635	7·15	487365	58
3	30985	95079	491147	6·46	978083	·69	513064	7·14	486936	57
4	31012	95070	491535	6·46	978042	·69	513493	7·14	486507	56
5	31040	95061	491922	6·45	978001	·69	513921	7·13	486079	55
6	31068	95052	492308	6·44	977959	·69	514349	7·13	485651	54
7	31095	95043	492695	6·44	977918	·69	514777	7·12	485223	53
8	31123	95033	493081	6·43	977877	·69	515204	7·12	484796	52
9	31151	95024	493466	6·42	977835	·69	515631	7·11	484369	51
10	31178	95015	493851	6·42	977794	·69	516057	7·10	483943	50
11	31206	95006	9·494236	6·41	9·977752	·69	9·516484	7·10	10·483516	49
12	31233	94997	494621	6·41	977711	·69	516910	7·09	483090	48
13	31261	94988	495005	6·40	977669	·69	517335	7·09	482665	47
14	31289	94979	495388	6·39	977628	·69	517761	7·08	482239	46
15	31316	94970	495772	6·39	977586	·69	518185	7·08	481815	45
16	31344	94961	496154	6·38	977544	·70	518610	7·07	481390	44
17	31372	94952	496537	6·37	977503	·70	519034	7·06	480966	43
18	31399	94943	496919	6·37	977461	·70	519458	7·06	480542	42
19	31427	94933	497301	6·36	977419	·70	519882	7·05	480118	41
20	31454	94924	497682	6·36	977377	·70	520305	7·05	479695	40
21	31482	94915	9·498064	6·35	9·977335	·70	9·520728	7·04	10·479272	39
22	31510	94906	498444	6·34	977293	·70	521151	7·03	478849	38
23	31537	94897	498825	6·34	977251	·70	521573	7·03	478427	37
24	31565	94888	499204	6·33	977209	·70	521995	7·03	478005	36
25	31593	94878	499584	6·32	977167	·70	522417	7·02	477583	35
26	31620	94869	499963	6·32	977125	·70	522838	7·02	477162	34
27	31648	94860	500342	6·31	977083	·70	523259	7·01	476741	33
28	31675	94851	500721	6·31	977041	·70	523680	7·01	476320	32
29	31703	94842	501099	6·30	976999	·70	524100	7·00	475900	31
30	31730	94832	501476	6·29	976957	·70	524520	6·99	475480	30
31	31758	94823	9·501854	6·29	9·976914	·70	9·524939	6·99	10·475061	29
32	31786	94814	502231	6·28	976872	·71	525359	6·98	474641	28
33	31813	94805	502607	6·28	976830	·71	525778	6·98	474222	27
34	31841	94795	502984	6·27	976787	·71	526197	6·97	473803	26
35	31868	94786	503360	6·26	976745	·71	526615	6·97	473385	25
36	31896	94777	503735	6·26	976702	·71	527033	6·96	472967	24
37	31923	94768	504110	6·25	976660	·71	527451	6·96	472549	23
38	31951	94758	504485	6·25	976617	·71	527868	6·95	472132	22
39	31979	94749	504860	6·24	976574	·71	528285	6·95	471715	21
40	32006	94740	505234	6·23	976532	·71	528702	6·94	471298	20
41	32034	94730	9·505608	6·23	9·976489	·71	9·529119	6·93	10·470881	19
42	32061	94721	505981	6·22	976446	·71	529535	6·93	470465	18
43	32089	94712	506354	6·22	976404	·71	529950	6·93	470050	17
44	32116	94702	506727	6·21	976361	·71	530366	6·92	469634	16
45	32144	94693	507099	6·20	976318	·71	530781	6·91	469219	15
46	32171	94684	507471	6·20	976275	·71	531196	6·91	468804	14
47	32199	94674	507843	6·19	976232	·72	531611	6·90	468389	13
48	32227	94665	508214	6·19	976189	·72	532025	6·90	467975	12
49	32254	94656	508585	6·18	976146	·72	532439	6·89	467561	11
50	32282	94646	508956	6·18	976103	·72	532853	6·89	467147	10
51	32309	94637	9·509326	6·17	9·976060	·72	9·533266	6·88	10·466734	9
52	32337	94627	509696	6·16	976017	·72	533679	6·88	466321	8
53	32364	94618	510065	6·16	975974	·72	534092	6·87	465908	7
54	32392	94609	510434	6·15	975930	·72	534504	6·87	465496	6
55	32419	94599	510803	6·15	975887	·72	534916	6·86	465084	5
56	32447	94590	511172	6·14	975844	·72	535328	6·86	464672	4
57	32474	94580	511540	6·13	975800	·72	535739	6·85	464261	3
58	32502	94571	511907	6·13	975757	·72	536150	6·85	463850	2
59	32529	94561	512275	6·12	975714	·72	536561	6·84	463439	1
60	32557	94552	512642	6·12	975670	·72	536972	6·84	463028	0
	N. cos.	N. sine.	L. cos.	D. 1″	L. sine.		L. cot.	D. 1″	L. tang.	′

Nat. Functions.			Logarithmic Functions + 10.							
′	N.sine.	N. cos.	L. sine.	D. 1″	L. cos.	D.1″	L. tang.	D 1.″	L. cot.	
0	32557	94552	9·512642	6·12	9·975670	·73	9·536972	6·84	10·463028	60
1	32584	94542	513009	6·11	975627	·73	537382	6·83	462613	59
2	32612	94533	513375	6·11	975583	·73	537791	6·83	462208	58
3	32639	94523	513741	6·10	975539	·73	538201	6·82	461798	57
4	32667	94514	514107	6·09	975496	·73	538611	6·82	461389	56
5	32694	94504	514472	6·09	975452	·73	539020	6·81	460980	55
6	32722	94495	514837	6·08	975408	·73	539429	6·81	460571	54
7	32749	94485	515202	6·08	975365	·73	539837	6·80	460163	53
8	32777	94476	515566	6·07	975321	·73	540245	6·80	459755	52
9	32804	94466	515930	6·07	975277	·73	540653	6·79	459347	51
10	32832	94457	516294	6·06	975233	·73	541061	6·79	458939	50
11	32859	94447	9·516657	6·05	9·975189	·73	9·541468	6·78	10·458532	49
12	32887	94438	517020	6·05	975145	·73	541875	6·78	458125	48
13	32914	94428	517382	6·04	975101	·73	542281	6·77	457719	47
14	32942	94418	517745	6·04	975057	·73	542688	6·77	457312	46
15	32969	94409	518107	6·03	975013	·73	543094	6·76	456906	45
16	32997	94399	518468	6·02	974969	·74	543499	6·76	456501	44
17	33024	94390	518829	6·02	974925	·74	543905	6·75	456095	43
18	33051	94380	519190	6·01	974880	·74	544310	6·75	455690	42
19	33079	94370	519551	6·01	974836	·74	544715	6·74	455285	41
20	33106	94361	519911	6·00	974792	·74	545119	6·74	454881	40
21	33134	94351	9·520271	6·00	9·974748	·74	9·545524	6·73	10·454476	39
22	33161	94342	520631	5·99	974703	·74	545928	6·73	454072	38
23	33189	94332	520990	5·99	974659	·74	546331	6·72	453669	37
24	33216	94322	521349	5·98	974614	·74	546735	6·72	453265	36
25	33244	94313	521707	5·98	974570	·74	547138	6·71	452862	35
26	33271	94303	522066	5·97	974525	·74	547540	6·71	452460	34
27	33298	94293	522424	5·96	974481	·74	547943	6·70	452057	33
28	33326	94284	522781	5·96	974436	·74	548345	6·70	451655	32
29	33353	94274	523138	5·95	974391	·74	548747	6·69	451253	31
30	33381	94264	523495	5·95	974347	·75	549149	6·69	450851	30
31	33408	94254	9·523852	5·94	9·974302	·75	9·549550	6·68	10·450450	29
32	33436	94245	524208	5·94	974257	·75	549951	6·68	450049	28
33	33463	94235	524564	5·93	974212	·75	550352	6·67	449648	27
34	33490	94225	524920	5·93	974167	·75	550752	6·67	449248	26
35	33518	94215	525275	5·92	974122	·75	551152	6·66	448848	25
36	33545	94206	525630	5·91	974077	·75	551552	6·66	448448	24
37	33573	94196	525984	5·91	974032	·75	551952	6·65	448048	23
38	33600	94186	526339	5·90	973987	·75	552351	6·65	447649	22
39	33627	94176	526693	5·90	973942	·75	552750	6·65	447250	21
40	33655	94167	527046	5·89	973897	·75	553149	6·64	446851	20
41	33682	94157	9·527400	5·89	9·973852	·75	9·553548	6·64	10·446452	19
42	33710	94147	527753	5·88	973807	·75	553946	6·63	446054	18
43	33737	94137	528105	5·88	973761	·75	554344	6·63	445656	17
44	33764	94127	528458	5·87	973716	·76	554741	6·62	445259	16
45	33792	94118	528810	5·87	973671	·76	555139	6·62	444861	15
46	33819	94108	529161	5·86	973625	·76	555536	6·61	444464	14
47	33846	94098	529513	5·86	973580	·76	555933	6·61	444067	13
48	33874	94088	529864	5·85	973535	·76	556329	6·60	443671	12
49	33901	94078	530215	5·85	973489	·76	556725	6·60	443275	11
50	33929	94068	530565	5·84	973444	·76	557121	6·59	442879	10
51	33956	94058	9·530915	5·84	9·973398	·76	9·557517	6·59	10·442483	9
52	33983	94049	531265	5·83	973352	·76	557913	6·59	442087	8
53	34011	94039	531614	5·82	973307	·76	558308	6·58	441692	7
54	34038	94029	531963	5·82	973261	·76	558702	6·58	441298	6
55	34065	94019	532312	5·81	973215	·76	559097	6·57	440903	5
56	34093	94009	532661	5·81	973169	·76	559491	6·57	440509	4
57	34120	93999	533009	5·80	973124	·76	559885	6·56	440115	3
58	34147	93989	533357	5·80	973078	·76	560279	6·56	439721	2
59	34175	93979	533704	5·79	973032	·77	560673	6·55	439327	1
60	34202	93969	534052	5·78	972986	·77	561066	6·55	438934	0
	N. cos.	N. sine.	L. cos.	D. 1″	L. sine.		L. cot.	D. 1″	L. tang.	′

Nat. Functions.			Logarithmic Functions + 10.							
′	N. sine.	N. cos.	L. sine.	D. 1″	L. cos.	D. 1″	L. tang.	D. 1″	L. cot.	
0	34202	93969	9·534052	5·78	9·972986	·77	9·561066	6·55	10·438934	60
1	34229	93959	534399	5·77	972940	·77	561459	6·54	438541	59
2	34257	93949	534745	5·77	972894	·77	561851	6·54	438149	58
3	34284	93939	535092	5·77	972848	·77	562244	6·53	437756	57
4	34311	93929	535438	5·76	972802	·77	562636	6·53	437364	56
5	34339	93919	535783	5·76	972755	·77	563028	6·53	436972	55
6	34366	93909	536129	5·75	972709	·77	563419	6·52	436581	54
7	34393	93899	536474	5·74	972663	·77	563811	6·52	436189	53
8	34421	93889	536818	5·74	972617	·77	564202	6·51	435798	52
9	34448	93879	537163	5·73	972570	·77	564592	6·51	435408	51
10	34475	93869	537507	5·73	972524	·77	564983	6·50	435017	50
11	34503	93859	9·537851	5·72	9·972478	·77	9·565373	6·50	10·434627	49
12	34530	93849	538194	5·72	972431	·78	565763	6·49	434237	48
13	34557	93839	538538	5·71	972385	·78	566153	6·49	433847	47
14	34584	93829	538880	5·71	972338	·78	566542	6·49	433458	46
15	34612	93819	539223	5·70	972291	·78	566932	6·48	433068	45
16	34639	93809	539565	5·70	972245	·78	567320	6·48	432680	44
17	34666	93799	539907	5·69	972198	·78	567709	6·47	432291	43
18	34694	93789	540249	5·69	972151	·78	568098	6·47	431902	42
19	34721	93779	540590	5·68	972105	·78	568486	6·46	431514	41
20	34748	93769	540931	5·68	972058	·78	568873	6·46	431127	40
21	34775	93759	9·541272	5·67	9·972011	·78	9·569261	6·45	10·430739	39
22	34803	93748	541613	5·67	971964	·78	569648	6·45	430352	38
23	34830	93738	541953	5·66	971917	·78	570035	6·45	429965	37
24	34857	93728	542293	5·66	971870	·78	570422	6·44	429578	36
25	34884	93718	542632	5·65	971823	·78	570809	6·44	429191	35
26	34912	93708	542971	5·65	971776	·78	571195	6·43	428805	34
27	34939	93698	543310	5·64	971729	·79	571581	6·43	428419	33
28	34966	93688	543649	5·64	971682	·79	571967	6·42	428033	32
29	34993	93677	543987	5·63	971635	·79	572352	6·42	427648	31
30	35021	93667	544325	5·63	971588	·79	572738	6·42	427262	30
31	35048	93657	9·544663	5·62	9·971540	·79	9·573123	6·41	10·426877	29
32	35075	93647	545000	5·62	971493	·79	573507	6·41	426493	28
33	35102	93637	545338	5·61	971446	·79	573892	6·40	426108	27
34	35130	93626	545674	5·61	971398	·79	574276	6·40	425724	26
35	35157	93616	546011	5·60	971351	·79	574660	6·39	425340	25
36	35184	93606	546347	5·60	971303	·79	575044	6·39	424956	24
37	35211	93596	546683	5·59	971256	·79	575427	6·39	424573	23
38	35239	93585	547019	5·59	971208	·79	575810	6·38	424190	22
39	35266	93575	547354	5·58	971161	·79	576193	6·38	423807	21
40	35293	93565	547689	5·58	971113	·79	576576	6·37	423424	20
41	35320	93555	9·548024	5·57	9·971066	·80	9·576958	6·37	10·423041	19
42	35347	93544	548359	5·57	971018	·80	577341	6·36	422659	18
43	35375	93534	548693	5·56	970970	·80	577723	6·36	422277	17
44	35402	93524	549027	5·56	970922	·80	578104	6·36	421896	16
45	35429	93514	549360	5·55	970874	·80	578486	6·35	421514	15
46	35456	93503	549693	5·55	970827	·80	578867	6·35	421133	14
47	35484	93493	550026	5·54	970779	·80	579248	6·34	420752	13
48	35511	93483	550359	5·54	970731	·80	579629	6·34	420371	12
49	35538	93472	550692	5·53	970683	·80	580009	6·34	419991	11
50	35565	93462	551024	5·53	970635	·80	580389	6·33	419611	10
51	35592	93452	9·551356	5·52	9·970586	·80	9·580769	6·33	10·419231	9
52	35619	93441	551687	5·52	970538	·80	581149	6·32	418851	8
53	35647	93431	552018	5·52	970490	·80	581528	6·32	418472	7
54	35674	93420	552349	5·51	970442	·80	581907	6·32	418093	6
55	35701	93410	552680	5·51	970394	·80	582286	6·31	417714	5
56	35728	93400	553010	5·50	970345	·81	582665	6·31	417335	4
57	35755	93389	553341	5·50	970297	·81	583043	6·30	416957	3
58	35782	93379	553670	5·49	970249	·81	583422	6·30	416578	2
59	35810	93368	554000	5·49	970200	·81	583800	6·29	416200	1
60	35837	93858	554329	5·48	970152	·81	584177	6·29	415823	0
	N. cos.	N. sine.	L. cos.	D. 1″	L. sine.		L. cot.	D. 1″	L. tang.	′

Nat. Functions.			Logarithmic Functions + 10.							
′	N. sine.	N. cos.	L. sine.	D. 1″	L. cos.	D.1″	L. tang.	D. 1″	L. cot.	
0	35837	93358	9·554329	5·48	9·970152	·81	9·584177	6·29	10·415823	60
1	35864	93348	554658	5·48	970103	·81	584555	6·29	415445	59
2	35891	93337	554987	5·47	970055	·81	584932	6·28	415068	58
3	35918	93327	555315	5·47	970006	·81	585309	6·28	414691	57
4	35945	93316	555643	5·46	969957	·81	585686	6·27	414314	56
5	35973	93306	555971	5·46	969909	·81	586062	6·27	413938	55
6	36000	93295	556299	5·45	969860	·81	586439	6·27	413561	54
7	36027	93285	556626	5·45	969811	·81	586815	6·26	413185	53
8	36054	93274	556953	5·44	969762	·81	587190	6·26	412810	52
9	36081	93264	557280	5·44	969714	·81	587566	6·25	412434	51
10	36108	93253	557606	5·43	969665	·81	587941	6·25	412059	50
11	36135	93243	9·557932	5·43	9·969616	·82	9·588316	6·25	10·411684	49
12	36162	93232	558258	5·43	969567	·82	588691	6·24	411309	48
13	36190	93222	558583	5·42	969518	·82	589066	6·24	410934	47
14	36217	93211	558909	5·42	969469	·82	589440	6·23	410560	46
15	36244	93201	559234	5·41	969420	·82	589814	6·23	410186	45
16	36271	93190	559558	5·41	969370	·82	590188	6·23	409812	44
17	36298	93180	559883	5·40	969321	·82	590562	6·22	409438	43
18	36325	93169	560207	5·40	969272	·82	590935	6·22	409065	42
19	36352	93159	560531	5·39	969223	·82	591308	6·22	408692	41
20	36379	93148	560855	5·39	969173	·82	591681	6·21	408319	40
21	36406	93137	9·561178	5·38	9·969124	·82	9·592054	6·21	10·407946	39
22	36434	93127	561501	5·38	969075	·82	592426	6·20	407574	38
23	36461	93116	561824	5·37	969025	·82	592798	6·20	407202	37
24	36488	93106	562146	5·37	968976	·82	593171	6·19	406829	36
25	36515	93095	562468	5·36	968926	·83	593542	6·19	406458	35
26	36542	93084	562790	5·36	968877	·83	593914	6·18	406086	34
27	36569	93074	563112	5·36	968827	·83	594285	6·18	405715	33
28	36596	93063	563433	5·35	968777	·83	594656	6·18	405344	32
29	36623	93052	563755	5·35	968728	·83	595027	6·17	404973	31
30	36650	93042	564075	5·34	968678	·83	595398	6·17	404602	30
31	36677	93031	9·564396	5·34	9·968628	·83	9·595768	6·17	10·404232	29
32	36704	93020	564716	5·33	968578	·83	596138	6·16	403862	28
33	36731	93010	565036	5·33	968528	·83	596508	6·16	403492	27
34	36758	92999	565356	5·32	968479	·83	596878	6·16	403122	26
35	36785	92988	565676	5·32	968429	·83	597247	6·15	402753	25
36	36812	92978	565995	5·31	968379	·83	597616	6·15	402384	24
37	36839	92967	566314	5·31	968329	·83	597985	6·15	402015	23
38	36867	92956	566632	5·31	968278	·83	598354	6·14	401646	22
39	36894	92945	566951	5·30	968228	·84	598722	6·14	401278	21
40	36921	92935	567269	5·30	968178	·84	599091	6·13	400909	20
41	36948	92924	9·567587	5·29	9·968128	·84	9·599459	6·13	10·400541	19
42	36975	92913	567904	5·29	968078	·84	599827	6·13	400173	18
43	37002	92902	568222	5·28	968027	·84	600194	6·12	399806	17
44	37029	92892	568539	5·28	967977	·84	600562	6·12	399438	16
45	37056	92881	568856	5·28	967927	·84	600929	6·11	399071	15
46	37083	92870	569172	5·27	967876	·84	601296	6·11	398704	14
47	37110	92859	569488	5·27	967826	·84	601662	6·11	398338	13
48	37137	92849	569804	5·26	967775	·84	602029	6·10	397971	12
49	37164	92838	570120	5·26	967725	·84	602395	6·10	397605	11
50	37191	92827	570435	5·25	967674	·84	602761	6·10	397239	10
51	37218	92816	9·570751	5·25	9·967624	·84	9·603127	6·09	10·396873	7
52	37245	92805	571066	5·24	967573	·84	603493	6·09	396507	9
53	37272	92794	571380	5·24	967522	·85	603858	6·09	396142	8
54	37299	92784	571695	5·23	967471	·85	604223	6·08	395777	6
55	37326	92773	572009	5·23	967421	·85	604588	6·08	395412	5
56	37353	92762	572323	5·23	967370	·85	604953	6·07	395047	4
57	37380	92751	572636	5·22	967319	·85	605317	6·07	394683	3
58	37407	92740	572950	5·22	967268	·85	605682	6·07	394318	2
59	37434	92729	573263	5·21	967217	·85	606046	6·06	393954	1
60	37461	92718	573575	5·21	967166	·85	606410	6·06	393590	0
	N. cos.	N. sine.	L. cos.	D. 1″	L. sine.		L. cot.	D. 1″	L. tang.	′

Nat. Functions.			Logarithmic Functions + 10.							
′	N.sine.	N. cos.	L. sine.	D. 1″	L. cos.	D.1″	L. tang.	D. 1″	L. cot.	
0	37461	92718	9·573575	5·21	9·967166	·85	9·606410	6·06	10·393590	60
1	37488	92707	573888	5·20	967115	·85	606773	6·06	393227	59
2	37515	92697	574200	5·20	967064	·85	607137	6·05	392863	58
3	37542	92686	574512	5·19	967013	·85	607500	6·05	392500	57
4	37569	92675	574824	5·19	966961	·85	607863	6·04	392137	56
5	37595	92664	575136	5·19	966910	·85	608225	6·04	391775	55
6	37622	92653	575447	5·18	966859	·85	608588	6·04	391412	54
7	37649	92642	575758	5·18	966808	·85	608950	6·03	391050	53
8	37676	92631	576069	5·17	966756	·86	609312	6·03	390688	52
9	37703	92620	576379	5·17	966705	·86	609674	6·03	390326	51
10	37730	92609	576689	5·16	966653	·86	610036	6·02	389964	50
11	37757	92598	9·576999	5·16	9·966602	·86	9·610397	6·02	10·389603	49
12	37784	92587	577309	5·16	966550	·86	610759	6·02	389241	48
13	37811	92576	577618	5·15	966499	·86	611120	6·01	388880	47
14	37838	92565	577927	5·15	966447	·86	611480	6·01	388520	46
15	37865	92554	578236	5·14	966395	·86	611841	6·01	388159	45
16	37892	92543	578545	5·14	966344	·86	612201	6·00	387799	44
17	37919	92532	578853	5·13	966292	·86	612561	6·00	387439	43
18	37946	92521	579162	5·13	966240	·86	612921	6·00	387079	42
19	37973	92510	579470	5·13	966188	·86	613281	5·99	386719	41
20	37999	92499	579777	5·12	966136	·86	613641	5·99	386359	40
21	38026	92488	9·580085	5·12	9·966085	·87	9·614000	5·98	10·386000	39
22	38053	92477	580392	5·11	966033	·87	614359	5·98	385641	38
23	38080	92466	580699	5·11	965981	·87	614718	5·98	385282	37
24	38107	92455	581005	5·11	965928	·87	615077	5·97	384923	36
25	38134	92444	581312	5·10	965876	·87	615435	5·97	384565	35
26	38161	92432	581618	5·10	965824	·87	615793	5·97	384207	34
27	38188	92421	581924	5·09	965772	·87	616151	5·96	383849	33
28	38215	92410	582229	5·09	965720	·87	616509	5·96	383491	32
29	38241	92399	582535	5·09	965668	·87	616867	5·96	383133	31
30	38268	92388	582840	5·08	965615	·87	617224	5·95	382776	30
31	38295	92377	9·583145	5·08	9·965563	·87	9·617582	5·95	10·382418	29
32	38322	92366	583449	5·07	965511	·87	617939	5·95	382061	28
33	38349	92355	583754	5·07	965458	·87	618295	5·94	381705	27
34	38376	92343	584058	5·06	965406	·87	618652	5·94	381348	26
35	38403	92332	584361	5·06	965353	·88	619008	5·94	380992	25
36	38430	92321	584665	5·06	965301	·88	619364	5·93	380636	24
37	38456	92310	584968	5·05	965248	·88	619721	5·93	380279	23
38	38483	92299	585272	5·05	965195	·88	620076	5·93	379924	22
39	38510	92287	585574	5·04	965143	·88	620432	5·92	379568	21
40	38537	92276	585877	5·04	965090	·88	620787	5·92	379213	20
41	38564	92265	9·586179	5·03	9·965037	·88	9·621142	5·92	10·378858	19
42	38591	92254	586482	5·03	964984	·88	621497	5·91	378503	18
43	38617	92243	586783	5·03	964931	·88	621852	5·91	378148	17
44	38644	92231	587085	5·02	964879	·88	622207	5·90	377793	16
45	38671	92220	587386	5·02	964826	·88	622561	5·90	377439	15
46	38698	92209	587688	5·01	964773	·88	622915	5·90	377085	14
47	38725	92198	587989	5·01	964719	·88	623269	5·89	376731	13
48	38752	92186	588289	5·01	964666	·89	623623	5·89	376377	12
49	38778	92175	588590	5·00	964613	·89	623976	5·89	376024	11
50	38805	92164	588890	5·00	964560	·89	624330	5·88	375670	10
51	38832	92152	9·589190	4·99	9·964507	·89	9·624683	5·88	10·375317	9
52	38859	92141	589489	4·99	964454	·89	625036	5·88	374964	8
53	38886	92130	589789	4·99	964400	·89	625388	5·87	374612	7
54	38912	92119	590088	4·98	964347	·89	625741	5·87	374259	6
55	38939	92107	590387	4·98	964294	·89	626093	5·87	373907	5
56	38966	92096	590686	4·97	964240	·89	626445	5·86	373555	4
57	38993	92085	590984	4·97	964187	·89	626797	5·86	373203	3
58	39020	92073	591282	4·97	964133	·89	627149	5·86	372851	2
59	39046	92062	591580	4·96	964080	·89	627501	5·85	372499	1
60	39073	92050	591878	4·96	964026	·89	627852	5·85	372148	0
	N. cos.	N. sine.	L. cos.	D. 1″	L. sine.		L. cot.	D. 1″	L. tang.	′

Nat. Functions.			Logarithmic Functions + 10.							
′	N. sine.	N. cos.	L. sine.	D. 1″	L. cos.	D.1″	L. tang.	D. 1″	L. cot.	
0	39073	92050	9·591878	4·96	9·964026	·89	9·627852	5·85	10·372148	60
1	39100	92039	592176	4·95	963972	·89	628203	5·85	371797	59
2	39127	92028	592473	4·95	963919	·89	628554	5·85	371446	58
3	39153	92016	592770	4·95	963865	·90	628905	5·84	371095	57
4	39180	92005	593067	4·94	963811	·90	629255	5·84	370745	56
5	39207	91994	593363	4·94	963757	·90	629606	5·83	370394	55
6	39234	91982	593659	4·93	963704	·90	629956	5·83	370044	54
7	39260	91971	593955	4·93	963650	·90	630306	5·83	369694	53
8	39287	91959	594251	4·93	963596	·90	630656	5·83	369344	52
9	39314	91948	594547	4·92	963542	·90	631005	5·82	368995	51
10	39341	91936	594842	4·92	963488	·90	631355	5·82	368645	50
11	39367	91925	9·595137	4·91	9·963434	·90	9·631704	5·82	10·368296	49
12	39394	91914	595432	4·91	963379	·90	632053	5·81	367947	48
13	39421	91902	595727	4·91	963325	·90	632401	5·81	367599	47
14	39448	91891	596021	4·90	963271	·90	632750	5·81	367250	46
15	39474	91879	596315	4·90	963217	·90	633098	5·80	366902	45
16	39501	91868	596609	4·89	963163	·90	633447	5·80	366553	44
17	39528	91856	596903	4·89	963108	·91	633795	5·80	366205	43
18	39555	91845	597196	4·89	963054	·91	634143	5·79	365857	42
19	39581	91833	597490	4·88	962999	·91	634490	5·79	365510	41
20	39608	91822	597783	4·88	962945	·91	634838	5·79	365162	40
21	39635	91810	9·598075	4·87	9·962890	·91	9·635185	5·78	10·364815	39
22	39661	91799	598368	4·87	962836	·91	635532	5·78	364468	38
23	39688	91787	598660	4·87	962781	·91	635879	5·78	364121	37
24	39715	91775	598952	4·86	962727	·91	636226	5·77	363774	36
25	39741	91764	599244	4·86	962672	·91	636572	5·77	363428	35
26	39768	91752	599536	4·85	962617	·91	636919	5·77	363081	34
27	39795	91741	599827	4·85	962562	·91	637265	5·77	362735	33
28	39822	91729	600118	4·85	962508	·91	637611	5·76	362389	32
29	39848	91718	600409	4·84	962453	·91	637956	5·76	362044	31
30	39875	91706	600700	4·84	962398	·92	638302	5·76	361698	30
31	39902	91694	9·600990	4·84	9·962343	·92	9·638647	5·75	10·361353	29
32	39928	91683	601280	4·83	962288	·92	638992	5·75	361008	28
33	39955	91671	601570	4·83	962233	·92	639337	5·75	360663	27
34	39982	91660	601860	4·82	962178	·92	639682	5·74	360318	26
35	40008	91648	602150	4·82	962123	·92	640027	5·74	359973	25
36	40035	91636	602439	4·82	962067	·92	640371	5·74	359629	24
37	40062	91625	602728	4·81	962012	·92	640716	5·73	359284	23
38	40088	91613	603017	4·81	961957	·92	641060	5·73	358940	22
39	40115	91601	603305	4·81	961902	·92	641404	5·73	358596	21
40	40141	91590	603594	4·80	961846	·92	641747	5·72	358253	20
41	40168	91578	9·603882	4·80	9·961791	·92	9·642091	5·72	10·357909	19
42	40195	91566	604170	4·79	961735	·92	642434	5·72	357566	18
43	40221	91555	604457	4·79	961680	·92	642777	5·72	357223	17
44	40248	91543	604745	4·79	961624	·93	643120	5·71	356880	16
45	40275	91531	605032	4·78	961569	·93	643463	5·71	356537	15
46	40301	91519	605319	4·78	961513	·93	643806	5·71	356194	14
47	40328	91508	605606	4·78	961459	·93	644148	5·70	355852	13
48	40355	91496	605892	4·77	961402	·93	644490	5·70	355510	12
49	40381	91484	606179	4·77	961346	·93	644832	5·70	355168	11
50	40408	91472	606465	4·76	961290	·93	645174	5·69	354826	10
51	40434	91461	9·606751	4·76	9·961235	·93	9·645516	5·69	10·354484	9
52	40461	91449	607036	4·76	961179	·93	645857	5·69	354143	8
53	40488	91437	607322	4·75	961123	·93	646199	5·69	353801	7
54	40514	91425	607607	4·75	961067	·93	646540	5·68	353460	6
55	40541	91414	607892	4·74	961011	·93	646881	5·68	353119	5
56	40567	91402	608177	4·74	960955	·93	647222	5·68	352778	4
57	40594	91390	608461	4·74	960899	·93	647562	5·67	352438	3
58	40621	91378	608745	4·73	960843	·94	647903	5·67	352097	2
59	40647	91366	609029	4·73	960786	·94	648243	5·67	351757	1
60	40674	91355	609313	4·73	960730	·94	648583	5·66	351417	0
	N. cos.	N. sine.	L. cos.	D. 1″	L. sine.		L. cot.	D. 1″	L. tang.	′

Nat. Functions.			Logarithmic Functions + 10.							
′	N. sine.	N. cos.	L. sine.	D. 1″	L. cos.	D.1″	L. tang.	D 1.″	L. cot.	
0	40674	91355	9·609313	4·73	9·960730	·94	9·648583	5·66	10·351417	60
1	40700	91343	609397	4·72	960674	·94	648923	5·66	351077	59
2	40727	91331	609880	4·72	960618	·94	649263	5·66	350737	58
3	40753	91319	610164	4·72	960561	·94	649602	5·66	350398	57
4	40780	91307	610447	4·71	960505	·94	649942	5·65	350058	56
5	40806	91295	610729	4·71	960448	·94	650281	5·65	349719	55
6	40833	91283	611012	4·70	960392	·94	650620	5·65	349380	54
7	40860	91272	611294	4·70	960335	·94	650959	5·64	349041	53
8	40886	91260	611576	4·70	960279	·94	651297	5·64	348703	52
9	40913	91248	611858	4·69	960222	·94	651636	5·64	348364	51
10	40939	91236	612140	4·69	960165	·94	651974	5·63	348026	50
11	40966	91224	9·612421	4·69	9·960109	·95	9·652312	5·63	10·347688	49
12	40992	91212	612702	4·68	960052	·95	652650	5·63	347350	48
13	41019	91200	612983	4·68	959995	·95	652988	5·63	347012	47
14	41045	91188	613264	4·67	959938	·95	653326	5·62	346674	46
15	41072	91176	613545	4·67	959882	·95	653663	5·62	346337	45
16	41098	91164	613825	4·67	959825	·95	654000	5·62	346000	44
17	41125	91152	614105	4·66	959768	·95	654337	5·61	345663	43
18	41151	91140	614385	4·66	959711	·95	654674	5·61	345326	42
19	41178	91128	614665	4·66	959654	·95	655011	5·61	344989	41
20	41204	91116	614944	4·65	959596	·95	655348	5·61	344652	40
21	41231	91104	9·615223	4·65	9·959539	·95	9·655684	5·60	10·344316	39
22	41257	91092	615502	4·65	959482	·95	656020	5·60	343980	38
23	41284	91080	615781	4·64	959425	·95	656356	5·60	343644	37
24	41310	91068	616060	4·64	959368	·95	656692	5·59	343308	36
25	41337	91056	616338	4·64	959310	·96	657028	5·59	342972	35
26	41363	91044	616616	4·63	959253	·96	657364	5·59	342636	34
27	41390	91032	616894	4·63	959195	·96	657699	5·59	342301	33
28	41416	91020	617172	4·62	959138	·96	658034	5·58	341966	32
29	41443	91008	617450	4·62	959081	·96	658369	5·58	341631	31
30	41469	90996	617727	4·62	959023	·96	658704	5·58	341296	30
31	41496	90984	9·618004	4·61	9·958965	·96	9·659039	5·58	10·340961	29
32	41522	90972	618281	4·61	958908	·96	659373	5·57	340627	28
33	41549	90960	618558	4·61	958850	·96	659708	5·57	340292	27
34	41575	90948	618834	4·60	958792	·96	660042	5·57	339958	26
35	41602	90936	619110	4·60	958734	·96	660376	5·57	339624	25
36	41628	90924	619386	4·60	958677	·96	660710	5·56	339290	24
37	41655	90911	619662	4·59	958619	·96	661043	5·56	338957	23
38	41681	90899	619938	4·59	958561	·96	661377	5·56	338623	22
39	41707	90887	620213	4·59	958503	·97	661710	5·55	338290	21
40	41734	90875	620488	4·58	958445	·97	662043	5·55	337957	20
41	41760	90863	9·620763	4·58	9·958387	·97	9·662376	5·55	10·337624	19
42	41787	90851	621038	4·57	958329	·97	662709	5·54	337291	18
43	41813	90839	621313	4·57	958271	·97	663042	5·54	336958	17
44	41840	90826	621587	4·57	958213	·97	663375	5·54	336625	16
45	41866	90814	621861	4·56	958154	·97	663707	5·54	336293	15
46	41892	90802	622135	4·56	958096	·97	664039	5·53	335961	14
47	41919	90790	622409	4·56	958038	·97	664371	5·53	335629	13
48	41945	90778	622682	4·55	957979	·97	664703	5·53	335297	12
49	41972	90766	622956	4·55	957921	·97	665035	5·53	334965	11
50	41998	90753	623229	4·55	957863	·97	665366	5·52	334634	10
51	42024	90741	9·623502	4·54	9·957804	·97	9·665697	5·52	10·334303	9
52	42051	90729	623774	4·54	957746	·98	666029	5·52	333971	8
53	42077	90717	624047	4·54	957687	·98	666360	5·51	333640	7
54	42104	90704	624319	4·53	957628	·98	666691	5·51	333309	6
55	42130	90692	624591	4·53	957570	·98	667021	5·51	332979	5
56	42156	90680	624863	4·53	957511	·98	667352	5·51	332648	4
57	42183	90668	625135	4·52	957452	·98	667682	5·50	332318	3
58	42209	90655	625406	4·52	957393	·98	668013	5·50	331987	2
59	42235	90643	625677	4·52	957335	·98	668343	5·50	331657	1
60	42262	90631	625948	4·51	957276	·98	668672	5·50	331328	0
	N. cos.	N. sine.	L. cos.	D. 1″	L. sine.		L. cot.	D. 1″	L. tang.	′

Nat. Functions.			Logarithmic Functions + 10.							
′	N. sine.	N. cos.	L. sine.	D. 1″	L. cos.	D.1″	L. tang.	D 1″	L. cot.	
0	42262	90631	9·625948	4·51	9·957276	·98	9·668673	5·50	10·331327	60
1	42288	90618	626219	4·51	957217	·98	669002	5·49	330998	59
2	42315	90606	626490	4·51	957158	·98	669332	5·49	330668	58
3	42341	90594	626760	4·50	957099	·98	669661	5·49	330339	57
4	42367	90582	627030	4·50	957040	·98	669991	5·48	330009	56
5	42394	90569	627300	4·50	956981	·98	670320	5·48	329680	55
6	42420	90557	627570	4·49	956921	·99	670649	5·48	329351	54
7	42446	90545	627840	4·49	956862	·99	670977	5·48	329023	53
8	42473	90532	628109	4·49	956803	·99	671306	5·47	328694	52
9	42499	90520	628378	4·48	956744	·99	671634	5·47	328366	51
10	42525	90507	628647	4·48	956684	·99	671963	5·47	328037	50
11	42552	90495	9·628916	4·47	9·956625	·99	9·672291	5·47	10·327709	49
12	42578	90483	629185	4·47	956566	·99	672619	5·46	327381	48
13	42604	90470	629453	4·47	956506	·99	672947	5·46	327053	47
14	42631	90458	629721	4·46	956447	·99	673274	5·46	326726	46
15	42657	90446	629989	4·46	956387	·99	673602	5·46	326398	45
16	42683	90433	630257	4·46	956327	·99	673929	5·45	326071	44
17	42709	90421	630524	4·46	956268	·99	674257	5·45	325743	43
18	42736	90408	630792	4·45	956208	1·00	674584	5·45	325416	42
19	42762	90396	631059	4·45	956148	1·00	674910	5·44	325090	41
20	42788	90383	631326	4·45	956089	1·00	675237	5·44	324763	40
21	42815	90371	9·631593	4·44	9·956029	1·00	9·675564	5·44	10·324436	39
22	42841	90358	631859	4·44	955969	1·00	675890	5·44	324110	38
23	42867	90346	632125	4·44	955909	1·00	676216	5·43	323784	37
24	42894	90334	632392	4·43	955849	1·00	676543	5·43	323457	36
25	42920	90321	632658	4·43	955789	1·00	676869	5·43	323131	35
26	42946	90309	632923	4·43	955729	1·00	677194	5·43	322806	34
27	42972	90296	633189	4·42	955669	1·00	677520	5·42	322480	33
28	42999	90284	633454	4·42	955609	1·00	677846	5·42	322154	32
29	43025	90271	633719	4·42	955548	1·00	678171	5·42	321829	31
30	43051	90259	633984	4·41	955488	1·00	678496	5·42	321504	30
31	43077	90246	9·634249	4·41	9·955428	1·01	9·678821	5·41	10·321179	29
32	43104	90233	634514	4·40	955368	1·01	679146	5·41	320854	28
33	43130	90221	634778	4·40	955307	1·01	679471	5·41	320529	27
34	43156	90208	635042	4·40	955247	1·01	679795	5·41	320205	26
35	43182	90196	635306	4·39	955186	1·01	680120	5·40	319880	25
36	43209	90183	635570	4·39	955126	1·01	680444	5·40	319556	24
37	43235	90171	635834	4·39	955065	1·01	680768	5·40	319232	23
38	43261	90158	636097	4·38	955005	1·01	681092	5·40	318908	22
39	43287	90146	636360	4·38	954944	1·01	681416	5·39	318584	21
40	43313	90133	636623	4·38	954883	1·01	681740	5·39	318260	20
41	43340	90120	9·636886	4·37	9·954823	1·01	9·682063	5·39	10·317937	19
42	43366	90108	637148	4·37	954762	1·01	682387	5·39	317613	18
43	43392	90095	637411	4·37	954701	1·01	682710	5·38	317290	17
44	43418	90082	637673	4·37	954640	1·01	683033	5·38	316967	16
45	43445	90070	637935	4·36	954579	1·01	683356	5·38	316644	15
46	43471	90057	638197	4·36	954518	1·02	683679	5·38	316321	14
47	43497	90045	638458	4·36	954457	1·02	684001	5·37	315999	13
48	43523	90032	638720	4·35	954396	1·02	684324	5·37	315676	12
49	43549	90019	638981	4·35	954335	1·02	684646	5·37	315354	11
50	43575	90007	639242	4·35	954274	1·02	684968	5·37	315032	10
51	43602	89994	9·639503	4·34	9·954213	1·02	9·685290	5·36	10·314710	9
52	43628	89981	639764	4·34	954152	1·02	685612	5·36	314388	8
53	43654	89968	640024	4·34	954090	1·02	685934	5·36	314066	7
54	43680	89956	640284	4·33	954029	1·02	686255	5·36	313745	6
55	43706	89943	640544	4·33	953968	1·02	686577	5·35	313423	5
56	43733	89930	640804	4·33	953906	1·02	686898	5·35	313102	4
57	43759	89918	641064	4·32	953845	1·02	687219	5·35	312781	3
58	43785	89905	641324	4·32	953783	1·02	687540	5·35	312460	2
59	43811	89892	641583	4·32	953722	1·03	687861	5·34	312139	1
60	43837	89879	641842	4·31	953660	1·03	688182	5·34	311818	0
	N. cos.	N. sine.	L. cos.	D. 1″	L. sine.		L. cot.	D. 1″	L. tang.	′

Nat. Functions.		Logarithmic Functions + 10.						
′	N. sine.	N. cos.	L. sine.	D. 1″	L. cos.	D. 1″	L. tang.	D. 1″	L. cot.	
0	43837	89879	9·641842	4·31	9·953660	1·03	9·688182	5·34	10·311818	60
1	43863	89867	642101	4·31	953599	1·03	688502	5·34	311498	59
2	43889	89854	642360	4·31	953537	1·03	688823	5·34	311177	58
3	43916	89841	642618	4·30	953475	1·03	689143	5·33	310857	57
4	43942	89828	642877	4·30	953413	1·03	689463	5·33	310537	56
5	43968	89816	643135	4·30	953352	1·03	689783	5·33	310217	55
6	43994	89803	643393	4·30	953290	1·03	690103	5·33	309897	54
7	44020	89790	643650	4·29	953228	1·03	690423	5·33	309577	53
8	44046	89777	643908	4·29	953166	1·03	690742	5·32	309258	52
9	44072	89764	644165	4·29	953104	1·03	691062	5·32	308938	51
10	44098	89752	644423	4·28	953042	1·03	691381	5·32	308619	50
11	44124	89739	9·644680	4·28	9·952980	1·04	9·691700	5·31	10·308300	49
12	44151	89726	644936	4·28	952918	1·04	692019	5·31	307981	48
13	44177	89713	645193	4·27	952855	1·04	692338	5·31	307662	47
14	44203	89700	645450	4·27	952793	1·04	692656	5·31	307344	46
15	44229	89687	645706	4·27	952731	1·04	692975	5·31	307025	45
16	44255	89674	645962	4·26	952669	1·04	693293	5·30	306707	44
17	44281	89662	646218	4·26	952606	1·04	693612	5·30	306388	43
18	44307	89649	646474	4·26	952544	1·04	693930	5·30	306070	42
19	44333	89636	646729	4·25	952481	1·04	694248	5·30	305752	41
20	44359	89623	646984	4·25	952419	1·04	694566	5·29	305434	40
21	44385	89610	9·647240	4·25	9·952356	1·04	9·694883	5·29	10·305117	39
22	44411	89597	647494	4·24	952294	1·04	695201	5·29	304799	38
23	44437	89584	647749	4·24	952231	1·04	695518	5·29	304482	37
24	44464	89571	648004	4·24	952168	1·05	695836	5·29	304164	36
25	44490	89558	648258	4·24	952106	1·05	696153	5·28	303847	35
26	44516	89545	648512	4·23	952043	1·05	696470	5·28	303530	34
27	44542	89532	648766	4·23	951980	1·05	696787	5·28	303213	33
28	44568	89519	649020	4·23	951917	1·05	697103	5·28	302897	32
29	44594	89506	649274	4·22	951854	1·05	697420	5·27	302580	31
30	44620	89493	649527	4·22	951791	1·05	697736	5·27	302264	30
31	44646	89480	9·649781	4·22	9·951728	1·05	9·698053	5·27	10·301947	29
32	44672	89467	650034	4·22	951665	1·05	698369	5·27	301631	28
33	44698	89454	650287	4·21	951602	1·05	698685	5·26	301315	27
34	44724	89441	650539	4·21	951539	1·05	699001	5·26	300999	26
35	44750	89428	650792	4·21	951476	1·05	699316	5·26	300684	25
36	44776	89415	651044	4·20	951412	1·05	699632	5·26	300368	24
37	44802	89402	651297	4·20	951349	1·06	699947	5·26	300053	23
38	44828	89389	651549	4·20	951286	1·06	700263	5·25	299737	22
39	44854	89376	651800	4·19	951222	1·06	700578	5·25	299422	21
40	44880	89363	652052	4·19	951159	1·06	700893	5·25	299107	20
41	44906	89350	9·652304	4·19	9·951096	1·06	9·701208	5·24	10·298792	19
42	44932	89337	652555	4·18	951032	1·06	701523	5·24	298477	18
43	44958	89324	652806	4·18	950968	1·06	701837	5·24	298163	17
44	44984	89311	653057	4·18	950905	1·06	702152	5·24	297848	16
45	45010	89298	653308	4·18	950841	1·06	702466	5·24	297534	15
46	45036	89285	653558	4·17	950778	1·06	702780	5·23	297220	14
47	45062	89272	653808	4·17	950714	1·06	703095	5·23	296905	13
48	45088	89259	654059	4·17	950650	1·06	703409	5·23	296591	12
49	45114	89245	654309	4·16	950586	1·06	703723	5·23	296277	11
50	45140	89232	654558	4·16	950522	1·07	704036	5·22	295964	10
51	45166	89219	9·654808	4·16	9·950458	1·07	9·704350	5·22	10·295650	9
52	45192	89206	655058	4·16	950394	1·07	704663	5·22	295337	8
53	45218	89193	655307	4·15	950330	1·07	704977	5·22	295023	7
54	45243	89180	655556	4·15	950266	1·07	705290	5·22	294710	6
55	45269	89167	655805	4·15	950202	1·07	705603	5·21	294397	5
56	45295	89153	656054	4·14	950138	1·07	705916	5·21	294084	4
57	45321	89140	656302	4·14	950074	1·07	706228	5·21	293772	3
58	45347	89127	656551	4·14	950010	1·07	706541	5·21	293459	2
59	45373	89114	656799	4·13	949945	1·07	706854	5·21	293146	1
60	45399	89101	657047	4·13	949881	1·07	707166	5·20	292834	0
	N. cos.	N. sine.	L. cos.	D. 1″	L. sine.		L. cot.	D. 1″	L. tang.	′

Nat. Functions.		Logarithmic Functions + 10.							
′	N. sine.	N. cos.	L. sine.	D. 1″	L. cos.	D. 1″	L. tang.	D. 1″	L. cot.	
0	45399	89101	9·657047	4·13	9·949881	1·07	9·707166	5·20	10·292834	60
1	45425	89087	657295	4·13	949816	1·07	707478	5·20	292522	59
2	45451	89074	657542	4·12	949752	1·07	707790	5·20	292210	58
3	45477	89061	657790	4·12	949688	1·08	708102	5·20	291898	57
4	45503	89048	658037	4·12	949623	1·08	708414	5·19	291586	56
5	45529	89035	658284	4·12	949558	1·08	708726	5·19	291274	55
6	45554	89021	658531	4·11	949494	1·08	709037	5·19	290963	54
7	45580	89008	658778	4·11	949429	1·08	709349	5·19	290651	53
8	45606	88995	659025	4·11	949364	1·08	709660	5·19	290340	52
9	45632	88981	659271	4·10	949300	1·08	709971	5·18	290029	51
10	45658	88968	659517	4·10	949235	1·08	710282	5·18	289718	50
11	45684	88955	9·659763	4·10	9·949170	1·08	9·710593	5·18	10·289407	49
12	45710	88942	660009	4·09	949105	1·08	710904	5·18	289096	48
13	45736	88928	660255	4·09	949040	1·08	711215	5·18	288785	47
14	45762	88915	660501	4·09	948975	1·08	711525	5·17	288475	46
15	45787	88902	660746	4·09	948910	1·08	711836	5·17	288164	45
16	45813	88888	660991	4·08	948845	1·08	712146	5·17	287854	44
17	45839	88875	661236	4·08	948780	1·09	712456	5·17	287544	43
18	45865	88862	661481	4·08	948715	1·09	712766	5·16	287234	42
19	45891	88848	661726	4·07	948650	1·09	713076	5·16	286924	41
20	45917	88835	661970	4·07	948584	1·09	713386	5·16	286614	40
21	45942	88822	9·662214	4·07	9·948519	1·09	9·713696	5·16	10·286304	39
22	45968	88808	662459	4·07	948454	1·09	714005	5·16	285995	38
23	45994	88795	662703	4·06	948388	1·09	714314	5·15	285686	37
24	46020	88782	662946	4·06	948323	1·09	714624	5·15	285376	36
25	46046	88768	663190	4·06	948257	1·09	714933	5·15	285067	35
26	46072	88755	663433	4·05	948192	1·09	715242	5·15	284758	34
27	46097	88741	663677	4·05	948126	1·09	715551	5·14	284449	33
28	46123	88728	663920	4·05	948060	1·09	715860	5·14	284140	32
29	46149	88715	664163	4·05	947995	1·10	716168	5·14	283832	31
30	46175	88701	664406	4·04	947929	1·10	716477	5·14	283523	30
31	46201	88688	9·664648	4·04	9·947863	1·10	9·716785	5·14	10·283215	29
32	46226	88674	664891	4·04	947797	1·10	717093	5·13	282907	28
33	46252	88661	665133	4·03	947731	1·10	717401	5·13	282599	27
34	46278	88647	665375	4·03	947665	1·10	717709	5·13	282291	26
35	46304	88634	665617	4·03	947600	1·10	718017	5·13	281983	25
36	46330	88620	665859	4·02	947533	1·10	718325	5·13	281675	24
37	46355	88607	666100	4·02	947467	1·10	718633	5·12	281367	23
38	46381	88593	666342	4·02	947401	1·10	718940	5·12	281060	22
39	46407	88580	666583	4·02	947335	1·10	719248	5·12	280752	21
40	46433	88566	666824	4·01	947269	1·10	719555	5·12	280445	20
41	46458	88553	9·667065	4·01	9·947203	1·10	9·719862	5·12	10·280138	19
42	46484	88539	667305	4·01	947136	1·11	720169	5·11	279831	18
43	46510	88526	667546	4·01	947070	1·11	720476	5·11	279524	17
44	46536	88512	667786	4·00	947004	1·11	720783	5·11	279217	16
45	46561	88499	668027	4·00	946937	1·11	721089	5·11	278911	15
46	46587	88485	668267	4·00	946871	1·11	721396	5·11	278604	14
47	46613	88472	668506	3·99	946804	1·11	721702	5·10	278298	13
48	46639	88458	668746	3·99	946738	1·11	722009	5·10	277991	12
49	46664	88445	668986	3·99	946671	1·11	722315	5·10	277685	11
50	46690	88431	669225	3·99	946604	1·11	722621	5·10	277379	10
51	46716	88417	9·669464	3·98	9·946538	1·11	9·722927	5·10	10·277073	9
52	46742	88404	669703	3·98	946471	1·11	723232	5·09	276768	8
53	46767	88390	669942	3·98	946404	1·11	723538	5·09	276462	7
54	46793	88377	670181	3·97	946337	1·11	723844	5·09	276156	6
55	46819	88363	670419	3·97	946270	1·12	724149	5·09	275851	5
56	46844	88349	670658	3·97	946203	1·12	724454	5·09	275546	4
57	46870	88336	670896	3·97	946136	1·12	724759	5·08	275241	3
58	46896	88322	671134	3·96	946069	1·12	725065	5·08	274935	2
59	46921	88308	671372	3·96	946002	1·12	725369	5·08	274631	1
60	46947	88295	671609	3·96	945935	1·12	725674	5·08	274326	0
	N. cos.	N. sine.	L. cos.	D. 1″	L. sine.		L. cot.	D. 1″	L. tang.	′

Nat. Functions.			Logarithmic Functions + 10.							
′	N. sine.	N. cos.	L. sine.	D. 1″	L. cos.	D. 1″	L. tang.	D. 1″	L. cot.	
0	46947	88295	9·671609	3·96	9·945935	1·12	9·725674	5·08	10·274326	60
1	46973	88281	671847	3·95	945568	1·12	725979	5·08	274021	59
2	46999	88267	672084	3·95	945800	1·12	726284	5·07	273716	58
3	47024	88254	672321	3·95	945733	1·12	726588	5·07	273412	57
4	47050	88240	672558	3·95	945666	1·12	726892	5·07	273108	56
5	47076	88226	672795	3·94	945598	1·12	727197	5·07	272803	55
6	47101	88213	673032	3·94	945531	1·12	727501	5·07	272499	54
7	47127	88199	673268	3·94	945464	1·13	727805	5·06	272195	53
8	47153	88185	673505	3·94	945396	1·13	728109	5·06	271891	52
9	47178	88172	673741	3·93	945328	1·13	728412	5·06	271588	51
10	47204	88158	673977	3·93	945261	1·13	728716	5·06	271284	50
11	47229	88144	9·674213	3·93	9·945193	1·13	9·729020	5·06	10·270980	49
12	47255	88130	674448	3·92	945125	1·13	729323	5·05	270677	48
13	47281	88117	674684	3·92	945058	1·13	729626	5·05	270374	47
14	47306	88103	674919	3·92	944990	1·13	729929	5·05	270071	46
15	47332	88089	675155	3·92	944922	1·13	730233	5·05	269767	45
16	47358	88075	675390	3·91	944854	1·13	730535	5·05	269465	44
17	47383	88062	675624	3·91	944786	1·13	730838	5·04	269162	43
18	47409	88048	675859	3·91	944718	1·13	731141	5·04	268859	42
19	47434	88034	676094	3·91	944650	1·13	731444	5·04	268556	41
20	47460	88020	676328	3·90	944582	1·14	731746	5·04	268254	40
21	47486	88006	9·676562	3·90	9·944514	1·14	9·732048	5·04	10·267952	39
22	47511	87993	676796	3·90	944446	1·14	732351	5·03	267649	38
23	47537	87979	677030	3·90	944377	1·14	732653	5·03	267347	37
24	47562	87965	677264	3·89	944309	1·14	732955	5·03	267045	36
25	47588	87951	677498	3·89	944241	1·14	733257	5·03	266743	35
26	47614	87937	677731	3·89	944172	1·14	733558	5·03	266442	34
27	47639	87923	677964	3·88	944104	1·14	733860	5·02	266140	33
28	47665	87909	678197	3·88	944036	1·14	734162	5·02	265838	32
29	47690	87896	678430	3·88	943967	1·14	734463	5·02	265537	31
30	47716	87882	678663	3·88	943899	1·14	734764	5·02	265236	30
31	47741	87868	9·678895	3·87	9·943830	1·14	9·735066	5·02	10·264934	29
32	47767	87854	679128	3·87	943761	1·14	735367	5·02	264633	28
33	47793	87840	679360	3·87	943693	1·15	735668	5·01	264332	27
34	47818	87826	679592	3·87	943624	1·15	735969	5·01	264031	26
35	47844	87812	679824	3·86	943555	1·15	736269	5·01	263731	25
36	47869	87798	680056	3·86	943486	1·15	736570	5·01	263430	24
37	47895	87784	680288	3·86	943417	1·15	736871	5·01	263129	23
38	47920	87770	680519	3·85	943348	1·15	737171	5·00	262829	22
39	47946	87756	680750	3·85	943279	1·15	737471	5·00	262529	21
40	47971	87743	680982	3·85	943210	1·15	737771	5·00	262229	20
41	47997	87729	9·681213	3·85	9·943141	1·15	9·738071	5·00	10·261929	19
42	48022	87715	681443	3·84	943072	1·15	738371	5·00	261629	18
43	48048	87701	681674	3·84	943003	1·15	738671	4·99	261329	17
44	48073	87687	681905	3·84	942934	1·15	738971	4·99	261029	16
45	48099	87673	682135	3·84	942864	1·15	739271	4·99	260729	15
46	48124	87659	682365	3·83	942795	1·16	739570	4·99	260430	14
47	48150	87645	682595	3·83	942726	1·16	739870	4·99	260130	13
48	48175	87631	682825	3·83	942656	1·16	740169	4·99	259831	12
49	48201	87617	683055	3·83	942587	1·16	740468	4·98	259532	11
50	48226	87603	683284	3·82	942517	1·16	740767	4·98	259233	10
51	48252	87589	9·683514	3·82	9·942448	1·16	9·741066	4·98	10·258934	9
52	48277	87575	683743	3·82	942378	1·16	741365	4·98	258635	8
53	48303	87561	683972	3·82	942308	1·16	741664	4·98	258336	7
54	48328	87546	684201	3·81	942239	1·16	741962	4·97	258038	6
55	48354	87532	684430	3·81	942169	1·16	742261	4·97	257739	5
56	48379	87518	684658	3·81	942099	1·16	742559	4·97	257441	4
57	48405	87504	684887	3·80	942029	1·16	742858	4·97	257142	3
58	48430	87490	685115	3·80	941959	1·16	743156	4·97	256844	2
59	48456	87476	685343	3·80	941889	1·17	743454	4·97	256546	1
60	48481	87462	685571	3·80	941819	1·17	743752	4·96	256248	0
	N. cos.	N. sine.	L. cos.	D. 1″	L. sine.		L. cot.	D. 1″	L. tang.	′

Nat. Functions.			Logarithmic Functions + 10.							
′	N. sine.	N. cos.	L. sine.	D. 1″	L. cos.	D.1″	L. tang.	D. 1″	L. cot.	
0	48481	87462	9·685571	3·80	9·941819	1·17	9·743752	4·96	10·256248	60
1	48506	87448	685799	3·79	941749	1·17	744050	4·96	255950	59
2	48532	87434	686027	3·79	941679	1·17	744348	4·96	255652	58
3	48557	87420	686254	3·79	941609	1·17	744645	4·96	255355	57
4	48583	87406	686482	3·79	941539	1·17	744943	4·96	255057	56
5	48608	87391	686709	3·78	941469	1·17	745240	4·96	254760	55
6	48634	87377	686936	3·78	941398	1·17	745538	4·95	254462	54
7	48659	87363	687163	3·78	941328	1·17	745835	4·95	254165	53
8	48684	87349	687389	3·78	941258	1·17	746132	4·95	253868	52
9	48710	87335	687616	3·77	941187	1·17	746429	4·95	253571	51
10	48735	87321	687843	3·77	941117	1·17	746726	4·95	253274	50
11	48761	87306	9·688069	3·77	9·941046	1·18	9·747023	4·94	10·252977	49
12	48786	87292	688295	3·77	940975	1·18	747319	4·94	252681	48
13	48811	87278	688521	3·76	940905	1·18	747616	4·94	252384	47
14	48837	87264	688747	3·76	940834	1·18	747913	4·94	252087	46
15	48862	87250	688972	3·76	940763	1·18	748209	4·94	251791	45
16	48888	87235	689198	3·76	940693	1·18	748505	4·93	251495	44
17	48913	87221	689423	3·75	940622	1·18	748801	4·93	251199	43
18	48938	87207	689648	3·75	940551	1·18	749097	4·93	250903	42
19	48964	87193	689873	3·75	940480	1·18	749393	4·93	250607	41
20	48989	87178	690098	3·75	940409	1·18	749689	4·93	250311	40
21	49014	87164	9·690323	3·74	9·940338	1·18	9·749985	4·93	10·250015	39
22	49040	87150	690548	3·74	940267	1·18	750281	4·92	249719	38
23	49065	87136	690772	3·74	940196	1·18	750576	4·92	249424	37
24	49090	87121	690996	3·74	940125	1·19	750872	4·92	249128	36
25	49116	87107	691220	3·73	940054	1·19	751167	4·92	248833	35
26	49141	87093	691444	3·73	939982	1·19	751462	4·92	248538	34
27	49166	87079	691668	3·73	939911	1·19	751757	4·92	248243	33
28	49192	87064	691892	3·72	939840	1·19	752052	4·91	247948	32
29	49217	87050	692115	3·72	939768	1·19	752347	4·91	247653	31
30	49242	87036	692339	3·72	939697	1·19	752642	4·91	247358	30
31	49268	87021	9·692562	3·72	9·939625	1·19	9·752937	4·91	10·247063	29
32	49293	87007	692785	3·71	939554	1·19	753231	4·91	246769	28
33	49318	86993	693008	3·71	939482	1·19	753526	4·91	246474	27
34	49344	86978	693231	3·71	939410	1·19	753820	4·90	246180	26
35	49369	86964	693453	3·71	939339	1·19	754115	4·90	245885	25
36	49394	86949	693676	3·70	939267	1·20	754409	4·90	245591	24
37	49419	86935	693898	3·70	939195	1·20	754703	4·90	245297	23
38	49445	86921	694120	3·70	939123	1·20	754997	4·90	245003	22
39	49470	86906	694342	3·70	939052	1·20	755291	4·90	244709	21
40	49495	86892	694564	3·69	938980	1·20	755585	4·89	244415	20
41	49521	86878	9·694786	3·69	9·938908	1·20	9·755878	4·89	10·244122	19
42	49546	86863	695007	3·69	938836	1·20	756172	4·89	243828	18
43	49571	86849	695229	3·69	938763	1·20	756465	4·89	243535	17
44	49596	86834	695450	3·68	938691	1·20	756759	4·89	243241	16
45	49622	86820	695671	3·68	938619	1·20	757052	4·89	242948	15
46	49647	86805	695892	3·68	938547	1·20	757345	4·88	242655	14
47	49672	86791	696113	3·68	938475	1·20	757638	4·88	242362	13
48	49697	86777	696334	3·67	938402	1·21	757931	4·88	242069	12
49	49723	86762	696554	3·67	938330	1·21	758224	4·88	241776	11
50	49748	86748	696775	3·67	938258	1·21	758517	4·88	241483	10
51	49773	86733	9·696995	3·67	9·938185	1·21	9·758810	4·88	10·241190	9
52	49798	86719	697215	3·66	938113	1·21	759102	4·87	240898	8
53	49824	86704	697435	3·66	938040	1·21	759395	4·87	240605	7
54	49849	86690	697654	3·66	937967	1·21	759687	4·87	240313	6
55	49874	86675	697874	3·66	937895	1·21	759979	4·87	240021	5
56	49899	86661	698094	3·65	937822	1·21	760272	4·87	239728	4
57	49924	86646	698313	3·65	937749	1·21	760564	4·87	239436	3
58	49950	86632	698532	3·65	937676	1·21	760856	4·86	239144	2
59	49975	86617	698751	3·65	937604	1·21	761148	4·86	238852	1
60	50000	86603	698970	3·64	937531	1·21	761439	4·86	238561	0
	N. cos.	N. sine.	L. cos.	D. 1″	L. sine.		L. cot.	D. 1″	L. tang.	′

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Nat. Functions.			Logarithmic Functions + 10.							
′	N. sine.	N. cos.	L. sine.	D. 1″	L. cos.	D. 1″	L. tang.	D 1.″	L. cot.	
0	50000	86603	9·698970	3·64	9·937531	1·21	9·761439	4·86	10·238561	60
1	50025	86588	699189	3·64	937458	1·22	761731	4·86	238269	59
2	50050	86573	699407	3·64	937385	1·22	762023	4·86	237977	58
3	50076	86559	699626	3·64	937312	1·22	762314	4·86	237686	57
4	50101	86544	699844	3·63	937238	1·22	762606	4·85	237394	56
5	50126	86530	700062	3·63	937165	1·22	762897	4·85	237103	55
6	50151	86515	700280	3·63	937092	1·22	763188	4·85	236812	54
7	50176	86501	700498	3·63	937019	1·22	763479	4·85	236521	53
8	50201	86486	700716	3·63	936946	1·22	763770	4·85	236230	52
9	50227	86471	700933	3·62	936872	1·22	764061	4·85	235939	51
10	50252	86457	701151	3·62	936799	1·22	764352	4·84	235648	50
11	50277	86442	9·701368	3·62	9·936725	1·22	9·764643	4·84	10·235357	49
12	50302	86427	701585	3·62	936652	1·23	764933	4·84	235067	48
13	50327	86413	701802	3·61	936578	1·23	765224	4·84	234776	47
14	50352	86398	702019	3·61	936505	1·23	765514	4·84	234486	46
15	50377	86384	702236	3·61	936431	1·23	765805	4·84	234195	45
16	50403	86369	702452	3·61	936357	1·23	766095	4·84	233905	44
17	50428	86354	702669	3·60	936284	1·23	766385	4·83	233615	43
18	50453	86340	702885	3·60	936210	1·23	766675	4·83	233325	42
19	50478	86325	703101	3·60	936136	1·23	766965	4·83	233035	41
20	50503	86310	703317	3·60	936062	1·23	767255	4·83	232745	40
21	50528	86295	9·703533	3·59	9·935988	1·23	9·767545	4·83	10·232455	39
22	50553	86281	703749	3·59	935914	1·23	767834	4·83	232166	38
23	50578	86266	703964	3·59	935840	1·23	768124	4·82	231876	37
24	50603	86251	704179	3·59	935766	1·24	768413	4·82	231587	36
25	50628	86237	704395	3·59	935692	1·24	768703	4·82	231297	35
26	50654	86222	704610	3·58	935618	1·24	768992	4·82	231008	34
27	50679	86207	704825	3·58	935543	1·24	769281	4·82	230719	33
28	50704	86192	705040	3·58	935469	1·24	769570	4·82	230430	32
29	50729	86178	705254	3·58	935395	1·24	769860	4·81	230140	31
30	50754	86163	705469	3·57	935320	1·24	770148	4·81	229852	30
31	50779	86148	9·705683	3·57	9·935246	1·24	9·770437	4·81	10·229563	29
32	50804	86133	705898	3·57	935171	1·24	770726	4·81	229274	28
33	50829	86119	706112	3·57	935097	1·24	771015	4·81	228985	27
34	50854	86104	706326	3·56	935022	1·24	771303	4·81	228697	26
35	50879	86089	706539	3·56	934948	1·24	771592	4·81	228408	25
36	50904	86074	706753	3·56	934873	1·24	771880	4·80	228120	24
37	50929	86059	706967	3·56	934798	1·25	772168	4·80	227832	23
38	50954	86045	707180	3·55	934723	1·25	772457	4·80	227543	22
39	50979	86030	707393	3·55	934649	1·25	772745	4·80	227255	21
40	51004	86015	707606	3·55	934574	1·25	773033	4·80	226967	20
41	51029	86000	9·707819	3·55	9·934499	1·25	9·773321	4·80	10·226679	19
42	51054	85985	708032	3·54	934424	1·25	773608	4·79	226392	18
43	51079	85970	708245	3·54	934349	1·25	773896	4·79	226104	17
44	51104	85956	708458	3·54	934274	1·25	774184	4·79	225816	16
45	51129	85941	708670	3·54	934199	1·25	774471	4·79	225529	15
46	51154	85926	708882	3·53	934123	1·25	774759	4·79	225241	14
47	51179	85911	709094	3·53	934048	1·25	775046	4·79	224954	13
48	51204	85896	709306	3·53	933973	1·25	775333	4·79	224667	12
49	51229	85881	709518	3·53	933898	1·26	775621	4·78	224379	11
50	51254	85866	709730	3·53	933822	1·26	775908	4·78	224092	10
51	51279	85851	9·709941	3·52	9·933747	1·26	9·776195	4·78	10·223805	9
52	51304	85836	710153	3·52	933671	1·26	776482	4·78	223518	8
53	51329	85821	710364	3·52	933596	1·26	776769	4·78	223231	7
54	51354	85806	710575	3·52	933520	1·26	777055	4·78	222945	6
55	51379	85792	710786	3·51	933445	1·26	777342	4·78	222658	5
56	51404	85777	710997	3·51	933369	1·26	777628	4·77	222372	4
57	51429	85762	711208	3·51	933293	1·26	777915	4·77	222085	3
58	51454	85747	711419	3·51	933217	1·26	778201	4·77	221799	2
59	51479	85732	711629	3·50	933141	1·26	778487	4·77	221512	1
60	51504	85717	711839	3·50	933066	1·26	778774	4·77	221226	0
	N. cos.	N. sine.	L. cos.	D. 1″	L. sine.		L. cot.	D. 1″	L. tang.	′

Nat. Functions.		Logarithmic Functions + 10.							
′	N.sine.	N. cos.	L. sine.	D. 1″	L. cos.	D.1″	L. tang.	D. 1″	L. cot.	
0	51504	85717	9·711839	3·50	9·933066	1·26	9·778774	4·77	10·221226	60
1	51529	85702	712050	3·50	932990	1·27	779060	4·77	220940	59
2	51554	85687	712260	3·50	932914	1·27	779346	4·76	220654	58
3	51579	85672	712469	3·49	932838	1·27	779632	4·76	220368	57
4	51604	85657	712679	3·49	932762	1·27	779918	4·76	220082	56
5	51628	85642	712889	3·49	932685	1·27	780203	4·76	219797	55
6	51653	85627	713098	3·49	932609	1·27	780489	4·76	219511	54
7	51678	85612	713308	3·49	932533	1·27	780775	4·76	219225	53
8	51703	85597	713517	3·48	932457	1·27	781060	4·76	218940	52
9	51728	85582	713726	3·48	932380	1·27	781346	4·75	218654	51
10	51753	85567	713935	3·48	932304	1·27	781631	4·75	218369	50
11	51778	85551	9·714144	3·48	9·932228	1·27	9·781916	4·75	10·218084	49
12	51803	85536	714352	3·47	932151	1·27	782201	4·75	217799	48
13	51828	85521	714561	3·47	932075	1·28	782486	4·75	217514	47
14	51852	85506	714769	3·47	931998	1·28	782771	4·75	217229	46
15	51877	85491	714978	3·47	931921	1·28	783056	4·75	216944	45
16	51902	85476	715186	3·47	931845	1·28	783341	4·75	216659	44
17	51927	85461	715394	3·46	931768	1·28	783626	4·74	216374	43
18	51952	85446	715602	3·46	931691	1·28	783910	4·74	216090	42
19	51977	85431	715809	3·46	931614	1·28	784195	4·74	215805	41
20	52002	85416	716017	3·46	931537	1·28	784479	4·74	215521	40
21	52026	85401	9·716224	3·45	9·931460	1·28	9·784764	4·74	10·215236	39
22	52051	85385	716432	3·45	931383	1·28	785048	4·74	214952	38
23	52076	85370	716639	3·45	931306	1·28	785332	4·73	214668	37
24	52101	85355	716846	3·45	931229	1·29	785616	4·73	214384	36
25	52126	85340	717053	3·45	931152	1·29	785900	4·73	214100	35
26	52151	85325	717259	3·44	931075	1·29	786184	4·73	213816	34
27	52175	85310	717466	3·44	930998	1·29	786468	4·73	213532	33
28	52200	85294	717673	3·44	930921	1·29	786752	4·73	213248	32
29	52225	85279	717879	3·44	930843	1·29	787036	4·73	212964	31
30	52250	85264	718085	3·43	930766	1·29	787319	4·72	212681	30
31	52275	85249	9·718291	3·43	9·930688	1·29	9·787603	4·72	10·212397	29
32	52299	85234	718497	3·43	930611	1·29	787886	4·72	212114	28
33	52324	85218	718703	3·43	930533	1·29	788170	4·72	211830	27
34	52349	85203	718909	3·43	930456	1·29	788453	4·72	211547	26
35	52374	85188	719114	3·42	930378	1·29	788736	4·72	211264	25
36	52399	85173	719320	3·42	930300	1·30	789019	4·72	210981	24
37	52423	85157	719525	3·42	930223	1·30	789302	4·71	210698	23
38	52448	85142	719730	3·42	930145	1·30	789585	4·71	210415	22
39	52473	85127	719935	3·41	930067	1·30	789868	4·71	210132	21
40	52498	85112	720140	3·41	929989	1·30	790151	4·71	209849	20
41	52522	85096	9·720345	3·41	9·929911	1·30	9·790433	4·71	10·209567	19
42	52547	85081	720549	3·41	929833	1·30	790716	4·71	209284	18
43	52572	85066	720754	3·40	929755	1·30	790999	4·71	209001	17
44	52597	85051	720958	3·40	929677	1·30	791281	4·71	208719	16
45	52621	85035	721162	3·40	929599	1·30	791563	4·70	208437	15
46	52646	85020	721366	3·40	929521	1·30	791846	4·70	208154	14
47	52671	85005	721570	3·40	929442	1·30	792128	4·70	207872	13
48	52696	84989	721774	3·39	929364	1·31	792410	4·70	207590	12
49	52720	84974	721978	3·39	929286	1·31	792692	4·70	207308	11
50	52745	84959	722181	3·39	929207	1·31	792974	4·70	207026	10
51	52770	84943	9·722385	3·39	9·929129	1·31	9·793256	4·70	10·206744	9
52	52794	84928	722588	3·39	929050	1·31	793538	4·69	206462	8
53	52819	84913	722791	3·38	928972	1·31	793819	4·69	206181	7
54	52844	84897	722994	3·38	928893	1·31	794101	4·69	205899	6
55	52869	84882	723197	3·38	928815	1·31	794383	4·69	205617	5
56	52893	84866	723400	3·38	928736	1·31	794664	4·69	205336	4
57	52918	84851	723603	3·37	928657	1·31	794945	4·69	205055	3
58	52943	84836	723805	3·37	928578	1·31	795227	4·69	204773	2
59	52967	84820	724007	3·37	928499	1·31	795508	4·68	204492	1
60	52992	84805	724210	3·37	928420	1·31	795789	4·68	204211	0
	N. cos.	N. sine.	L. cos.	D. 1″	L. sine.		L. cot.	D. 1″	L. tang.	′

Nat. Functions.			Logarithmic Functions + 10.						
′	N. sine.	N. cos.	L. sine.	D. 1″	L. cos.	D.1″	L. tang.	D. 1″	L. cot.	
0	52992	84805	9·724210	3·37	9·928420	1·32	9·795789	4·68	10·204211	60
1	53017	84789	724412	3·37	928342	1·32	796070	4·68	203930	59
2	53041	84774	724614	3·36	928263	1·32	796351	4·68	203649	58
3	53066	84759	724816	3·36	928183	1·32	796632	4·68	203368	57
4	53091	84743	725017	3·36	928104	1·32	796913	4·68	203087	56
5	53115	84728	725219	3·36	928025	1·32	797194	4·68	202806	55
6	53140	84712	725420	3·35	927946	1·32	797475	4·68	202525	54
7	53164	84697	725622	3·35	927867	1·32	797755	4·68	202245	53
8	53189	84681	725823	3·35	927787	1·32	798036	4·67	201964	52
9	53214	84666	726024	3·35	927708	1·32	798316	4·67	201684	51
10	53238	84650	726225	3·35	927629	1·32	798596	4·67	201404	50
11	53263	84635	9·726426	3·34	9·927549	1·32	9·798877	4·67	10·201123	49
12	53288	84619	726626	3·34	927470	1·33	799157	4·67	200843	48
13	53312	84604	726827	3·34	927390	1·33	799437	4·67	200563	47
14	53337	84588	727027	3·34	927310	1·33	799717	4·67	200283	46
15	53361	84573	727228	3·34	927231	1·33	799997	4·66	200003	45
16	53386	84557	727428	3·33	927151	1·33	800277	4·66	199723	44
17	53411	84542	727628	3·33	927071	1·33	800557	4·66	199443	43
18	53435	84526	727828	3·33	926991	1·33	800836	4·66	199164	42
19	53460	84511	728027	3·33	926911	1·33	801116	4·66	198884	41
20	53484	84495	728227	3·33	926831	1·33	801396	4·66	198604	40
21	53509	84480	9·728427	3·32	9·926751	1·33	9·801675	4·66	10·198325	39
22	53534	84464	728626	3·32	926671	1·33	801955	4·66	198045	38
23	53558	84448	728825	3·32	926591	1·33	802234	4·65	197766	37
24	53583	84433	729024	3·32	926511	1·34	802513	4·65	197487	36
25	53607	84417	729223	3·31	926431	1·34	802792	4·65	197208	35
26	53632	84402	729422	3·31	926351	1·34	803072	4·65	196928	34
27	53656	84386	729621	3·31	926270	1·34	803351	4·65	196649	33
28	53681	84370	729820	3·31	926190	1·34	803630	4·65	196370	32
29	53705	84355	730018	3·30	926110	1·34	803908	4·65	196092	31
30	53730	84339	730216	3·30	926029	1·34	804187	4·65	195813	30
31	53754	84324	9·730415	3·30	9·925949	1·34	9·804466	4·64	10·195534	29
32	53779	84308	730613	3·30	925868	1·34	804745	4·64	195255	28
33	53804	84292	730811	3·30	925788	1·34	805023	4·64	194977	27
34	53828	84277	731009	3·29	925707	1·34	805302	4·64	194698	26
35	53853	84261	731206	3·29	925626	1·34	805580	4·64	194420	25
36	53877	84245	731404	3·29	925545	1·35	805859	4·64	194141	24
37	53902	84230	731602	3·29	925465	1·35	806137	4·64	193863	23
38	53926	84214	731799	3·29	925384	1·35	806415	4·63	193585	22
39	53951	84198	731996	3·28	925303	1·35	806693	4·63	193307	21
40	53975	84182	732193	3·28	925222	1·35	806971	4·63	193029	20
41	54000	84167	9·732390	3·28	9·925141	1·35	9·807249	4·63	10·192751	19
42	54024	84151	732587	3·28	925060	1·35	807527	4·63	192473	18
43	54049	84135	732784	3·28	924979	1·35	807805	4·63	192195	17
44	54073	84120	732980	3·27	924897	1·35	808083	4·63	191917	16
45	54097	84104	733177	3·27	924816	1·35	808361	4·63	191639	15
46	54122	84088	733373	3·27	924735	1·36	808638	4·62	191362	14
47	54146	84072	733569	3·27	924654	1·36	808916	4·62	191084	13
48	54171	84057	733765	3·27	924572	1·36	809193	4·62	190807	12
49	54195	84041	733961	3·26	924491	1·36	809471	4·62	190529	11
50	54220	84025	734157	3·26	924409	1·36	809748	4·62	190252	10
51	54244	84009	9·734353	3·26	9·924328	1·36	9·810025	4·62	10·189975	9
52	54269	83994	734549	3·26	924246	1·36	810302	4·62	189698	8
53	54293	83978	734744	3·25	924164	1·36	810580	4·62	189420	7
54	54317	83962	734939	3·25	924083	1·36	810857	4·62	189143	6
55	54342	83946	735135	3·25	924001	1·36	811134	4·61	188866	5
56	54366	83930	735330	3·25	923919	1·36	811410	4·61	188590	4
57	54391	83915	735525	3·25	923837	1·36	811687	4·61	188313	3
58	54415	83899	735719	3·24	923755	1·37	811964	4·61	188036	2
59	54440	83883	735914	3·24	923673	1·37	812241	4·61	187759	1
60	54464	83867	736109	3·24	923591	1·37	812517	4·61	187483	0
	N. cos.	N. sine.	L. cos.	D. 1″	L. sine.		L. cot.	D. 1″	L. tang.	′

Nat. Functions.			Logarithmic Functions + 10.							
′	N. sine.	N. cos.	L. sine.	D. 1″	L. cos.	D.1″	L. tang.	D. 1″	L. cot.	
0	54464	83867	9·736109	3·24	9·923591	1·37	9·812517	4·61	10·187483	60
1	54488	83851	736303	3·24	923509	1·37	812794	4·61	187206	59
2	54513	83835	736498	3·24	923427	1·37	813070	4·61	186930	58
3	54537	83819	736692	3·23	923345	1·37	813347	4·60	186653	57
4	54561	83804	736886	3·23	923263	1·37	813623	4·60	186377	56
5	54586	83788	737080	3·23	923181	1·37	813899	4·60	186101	55
6	54610	83772	737274	3·23	923098	1·37	814175	4·60	185825	54
7	54635	83756	737467	3·23	923016	1·37	814452	4·60	185548	53
8	54659	83740	737661	3·22	922933	1·37	814728	4·60	185272	52
9	54683	83724	737855	3·22	922851	1·37	815004	4·60	184996	51
10	54708	83708	738048	3·22	922768	1·38	815279	4·60	184721	50
11	54732	83692	9·738241	3·22	9·922686	1·38	9·815555	4·59	10·184445	49
12	54756	83676	738434	3·22	922603	1·38	815831	4·59	184169	48
13	54781	83660	738627	3·21	922520	1·38	816107	4·59	183893	47
14	54805	83645	738820	3·21	922438	1·38	816382	4·59	183618	46
15	54829	83629	739013	3·21	922355	1·38	816658	4·59	183342	45
16	54854	83613	739206	3·21	922272	1·38	816933	4·59	183067	44
17	54878	83597	739398	3·21	922189	1·38	817209	4·59	182791	43
18	54902	83581	739590	3·20	922106	1·38	817484	4·59	182516	42
19	54927	83565	739783	3·20	922023	1·38	817759	4·59	182241	41
20	54951	83549	739975	3·20	921940	1·38	818035	4·58	181965	40
21	54975	83533	9·740167	3·20	9·921857	1·39	9·818310	4·58	10·181690	39
22	54999	83517	740359	3·20	921774	1·39	818585	4·58	181415	38
23	55024	83501	740550	3·19	921691	1·39	818860	4·58	181140	37
24	55048	83485	740742	3·19	921607	1·39	819135	4·58	180865	36
25	55072	83469	740934	3·19	921524	1·39	819410	4·58	180590	35
26	55097	83453	741125	3·19	921441	1·39	819684	4·58	180316	34
27	55121	83437	741316	3·19	921357	1·39	819959	4·58	180041	33
28	55145	83421	741508	3·18	921274	1·39	820234	4·58	179766	32
29	55169	83405	741699	3·18	921190	1·39	820508	4·57	179492	31
30	55194	83389	741889	3·18	921107	1·39	820783	4·57	179217	30
31	55218	83373	9·742080	3·18	9·921023	1·39	9·821057	4·57	10·178943	29
32	55242	83356	742271	3·18	920939	1·40	821332	4·57	178668	28
33	55266	83340	742462	3·17	920856	1·40	821606	4·57	178394	27
34	55291	83324	742652	3·17	920772	1·40	821880	4·57	178120	26
35	55315	83308	742842	3·17	920688	1·40	822154	4·57	177846	25
36	55339	83292	743033	3·17	920604	1·40	822429	4·57	177571	24
37	55363	83276	743223	3·17	920520	1·40	822703	4·57	177297	23
38	55388	83260	743413	3·16	920436	1·40	822977	4·56	177023	22
39	55412	83244	743602	3·16	920352	1·40	823250	4·56	176750	21
40	55436	83228	743792	3·16	920268	1·40	823524	4·56	176476	20
41	55460	83212	9·743982	3·16	9·920184	1·40	9·823798	4·56	10·176202	19
42	55484	83195	744171	3·16	920099	1·40	824072	4·56	175928	18
43	55509	83179	744361	3·15	920015	1·40	824345	4·56	175655	17
44	55533	83163	744550	3·15	919931	1·41	824619	4·56	175381	16
45	55557	83147	744739	3·15	919846	1·41	824893	4·56	175107	15
46	55581	83131	744928	3·15	919762	1·41	825166	4·56	174834	14
47	55605	83115	745117	3·15	919677	1·41	825439	4·55	174561	13
48	55630	83098	745306	3·14	919593	1·41	825713	4·55	174287	12
49	55654	83082	745494	3·14	919508	1·41	825986	4·55	174014	11
50	55678	83066	745683	3·14	919424	1·41	826259	4·55	173741	10
51	55702	83050	9·745871	3·14	9·919339	1·41	9·826532	4·55	10·173468	9
52	55726	83034	746059	3·14	919254	1·41	826805	4·55	173195	8
53	55750	83017	746248	3·13	919169	1·41	827078	4·55	172922	7
54	55775	83001	746436	3·13	919085	1·41	827351	4·55	172649	6
55	55799	82985	746624	3·13	919000	1·41	827624	4·55	172376	5
56	55823	82969	746812	3·13	918915	1·42	827897	4·54	172103	4
57	55847	82953	746999	3·13	918830	1·42	828170	4·54	171830	3
58	55871	82936	747187	3·12	918745	1·42	828442	4·54	171558	2
59	55895	82920	747374	3·12	918659	1·42	828715	4·54	171285	1
60	55919	82904	747562	3·12	918574	1·42	828987	4·54	171013	0
	N. cos.	N. sine.	L. cos.	D. 1″	L. sine.		L. cot.	D. 1″	L. tang.	′

Nat. Functions.		Logarithmic Functions + 10.							
′	N.sine.	N. cos.	L. sine.	D. 1″	L. cos.	D.1″	L. tang.	D. 1″	L. cot.	
0	55919	82904	9·747562	3·12	9·918574	1·42	9·828987	4·54	10·171013	60
1	55943	82887	747749	3·12	918489	1·42	829260	4·54	170740	59
2	55968	82871	747936	3·12	918404	1·42	829532	4·54	170468	58
3	55992	82855	748123	3·11	918318	1·42	829805	4·54	170195	57
4	56016	82839	748310	3·11	918233	1·42	830077	4·54	169923	56
5	56040	82822	748497	3·11	918147	1·42	830349	4·53	169651	55
6	56064	82806	748683	3·11	918062	1·42	830621	4·53	169379	54
7	56088	82790	748870	3·11	917976	1·43	830893	4·53	169107	53
8	56112	82773	749056	3·10	917891	1·43	831165	4·53	168835	52
9	56136	82757	749243	3·10	917805	1·43	831437	4·53	168563	51
10	56160	82741	749429	3·10	917719	1·43	831709	4·53	168291	50
11	56184	82724	9·749615	3·10	9·917634	1·43	9·831981	4·53	10·168019	49
12	56208	82708	749801	3·10	917548	1·43	832253	4·53	167747	48
13	56232	82692	749987	3·09	917462	1·43	832525	4·53	167475	47
14	56256	82675	750172	3·09	917376	1·43	832796	4·53	167204	46
15	56280	82659	750358	3·09	917290	1·43	833068	4·52	166932	45
16	56305	82643	750543	3·09	917204	1·43	833339	4·52	166661	44
17	56329	82626	750729	3·09	917118	1·44	833611	4·52	166389	43
18	56353	82610	750914	3·08	917032	1·44	833882	4·52	166118	42
19	56377	82593	751099	3·08	916946	1·44	834154	4·52	165846	41
20	56401	82577	751284	3·08	916859	1·44	834425	4·52	165575	40
21	56425	82561	9·751469	3·08	9·916773	1·44	9·834696	4·52	10·165304	39
22	56449	82544	751654	3·08	916687	1·44	834967	4·52	165033	38
23	56473	82528	751839	3·08	916600	1·44	835238	4·52	164762	37
24	56497	82511	752023	3·07	916514	1·44	835509	4·52	164491	36
25	56521	82495	752208	3·07	916427	1·44	835780	4·51	164220	35
26	56545	82478	752392	3·07	916341	1·44	836051	4·51	163949	34
27	56569	82462	752576	3·07	916254	1·44	836322	4·51	163678	33
28	56593	82446	752760	3·07	916167	1·45	836593	4·51	163407	32
29	56617	82429	752944	3·06	916081	1·45	836864	4·51	163136	31
30	56641	82413	753128	3·06	915994	1·45	837134	4·51	162866	30
31	56665	82396	9·753312	3·06	9·915907	1·45	9·837405	4·51	10·162595	29
32	56689	82380	753495	3·06	915820	1·45	837675	4·51	162325	28
33	56713	82363	753679	3·06	915733	1·45	837946	4·51	162054	27
34	56736	82347	753862	3·05	915646	1·45	838216	4·51	161784	26
35	56760	82330	754046	3·05	915559	1·45	838487	4·50	161513	25
36	56784	82314	754229	3·05	915472	1·45	838757	4·50	161243	24
37	56808	82297	754412	3·05	915385	1·45	839027	4·50	160973	23
38	56832	82281	754595	3·05	915297	1·45	839297	4·50	160703	22
39	56856	82264	754778	3·04	915210	1·45	839568	4·50	160432	21
40	56880	82248	754960	3·04	915123	1·46	839838	4·50	160162	20
41	56904	82231	9·755143	3·04	9·915035	1·46	9·840108	4·50	10·159892	19
42	56928	82214	755326	3·04	914948	1·46	840378	4·50	159622	18
43	56952	82198	755508	3·04	914860	1·46	840647	4·50	159353	17
44	56976	82181	755690	3·04	914773	1·46	840917	4·49	159083	16
45	57000	82165	755872	3·03	914685	1·46	841187	4·49	158813	15
46	57024	82148	756054	3·03	914598	1·46	841457	4·49	158543	14
47	57047	82132	756236	3·03	914510	1·46	841726	4·49	158274	13
48	57071	82115	756418	3·03	914422	1·46	841996	4·49	158004	12
49	57095	82098	756600	3·03	914334	1·46	842266	4·49	157734	11
50	57119	82082	756782	3·02	914246	1·47	842535	4·49	157465	10
51	57143	82065	9·756963	3·02	9·914158	1·47	9·842805	4·49	10·157195	9
52	57167	82048	757144	3·02	914070	1·47	843074	4·49	156926	8
53	57191	82032	757326	3·02	913982	1·47	843343	4·49	156657	7
54	57215	82015	757507	3·02	913894	1·47	843612	4·49	156388	6
55	57238	81999	757688	3·01	913806	1·47	843882	4·48	156118	5
56	57262	81982	757869	3·01	913718	1·47	844151	4·48	155849	4
57	57286	81965	758050	3·01	913630	1·47	844420	4·48	155580	3
58	57310	81949	758230	3·01	913541	1·47	844689	4·48	155311	2
59	57334	81932	758411	3·01	913453	1·47	844958	4·48	155042	1
60	57358	81915	758591	3·01	913365	1·47	845227	4·48	154773	0
	N. cos.	N. sine.	L. cos.	D. 1″	L. sine.		L. cot.	D. 1″	L. tang.	′

Nat. Functions.			Logarithmic Functions + 10.							
′	N. sine.	N. cos.	L. sine.	D. 1″	L. cos.	D. 1″	L. tang.	D. 1″	L. cot.	
0	57358	81915	9·758591	3·01	9·913365	1·47	9·845227	4·48	10·154773	60
1	57381	81899	758772	3·00	913276	1·47	845496	4·48	154504	59
2	57405	81882	758952	3·00	913187	1·48	845764	4·48	154236	58
3	57429	81865	759132	3·00	913099	1·48	846033	4·48	153967	57
4	57453	81848	759312	3·00	913010	1·48	846302	4·48	153698	56
5	57477	81832	759492	3·00	912922	1·48	846570	4·47	153430	55
6	57501	81815	759672	2·99	912833	1·48	846839	4·47	153161	54
7	57524	81798	759852	2·99	912744	1·48	847107	4·47	152893	53
8	57548	81782	760031	2·99	912655	1·48	847376	4·47	152624	52
9	57572	81765	760211	2·99	912566	1·48	847644	4·47	152356	51
10	57596	81748	760390	2·99	912477	1·48	847913	4·47	152087	50
11	57619	81731	9·760569	2·98	9·912388	1·48	9·848181	4·47	10·151819	49
12	57643	81714	760748	2·98	912299	1·49	848449	4·47	151551	48
13	57667	81698	760927	2·98	912210	1·49	848717	4·47	151283	47
14	57691	81681	761106	2·98	912121	1·49	848986	4·47	151014	46
15	57715	81664	761285	2·98	912031	1·49	849254	4·47	150746	45
16	57738	81647	761464	2·98	911942	1·49	849522	4·47	150478	44
17	57762	81631	761642	2·97	911853	1·49	849790	4·46	150210	43
18	57786	81614	761821	2·97	911763	1·49	850058	4·46	149942	42
19	57810	81597	761999	2·97	911674	1·49	850325	4·46	149675	41
20	57833	81580	762177	2·97	911584	1·49	850593	4·46	149407	40
21	57857	81563	9·762356	2·97	9·911495	1·49	9·850861	4·46	10·149139	39
22	57881	81546	762534	2·96	911405	1·49	851129	4·46	148871	38
23	57904	81530	762712	2·96	911315	1·50	851396	4·46	148604	37
24	57928	81513	762889	2·96	911226	1·50	851664	4·46	148336	36
25	57952	81496	763067	2·96	911136	1·50	851931	4·46	148069	35
26	57976	81479	763245	2·96	911046	1·50	852199	4·46	147801	34
27	57999	81462	763422	2·96	910956	1·50	852466	4·46	147534	33
28	58023	81445	763600	2·95	910866	1·50	852733	4·45	147267	32
29	58047	81428	763777	2·95	910776	1·50	853001	4·45	146999	31
30	58070	81412	763954	2·95	910686	1·50	853268	4·45	146732	30
31	58094	81395	9·764131	2·95	9·910596	1·50	9·853535	4·45	10·146465	29
32	58118	81378	764308	2·95	910506	1·50	853802	4·45	146198	28
33	58141	81361	764485	2·94	910415	1·50	854069	4·45	145931	27
34	58165	81344	764662	2·94	910325	1·51	854336	4·45	145664	26
35	58189	81327	764838	2·94	910235	1·51	854603	4·45	145397	25
36	58212	81310	765015	2·94	910144	1·51	854870	4·45	145130	24
37	58236	81293	765191	2·94	910054	1·51	855137	4·45	144863	23
38	58260	81276	765367	2·94	909963	1·51	855404	4·45	144596	22
39	58283	81259	765544	2·93	909873	1·51	855671	4·44	144329	21
40	58307	81242	765720	2·93	909782	1·51	855938	4·44	144062	20
41	58330	81225	9·765896	2·93	9·909691	1·51	9·856204	4·44	10·143796	19
42	58354	81208	766072	2·93	909601	1·51	856471	4·44	143529	18
43	58378	81191	766247	2·93	909510	1·51	856737	4·44	143263	17
44	58401	81174	766423	2·93	909419	1·51	857004	4·44	142996	16
45	58425	81157	766598	2·92	909328	1·52	857270	4·44	142730	15
46	58449	81140	766774	2·92	909237	1·52	857537	4·44	142463	14
47	58472	81123	766949	2·92	909146	1·52	857803	4·44	142197	13
48	58496	81106	767124	2·92	909055	1·52	858069	4·44	141931	12
49	58519	81089	767300	2·92	908964	1·52	858336	4·44	141664	11
50	58543	81072	767475	2·91	908873	1·52	858602	4·43	141398	10
51	58567	81055	9·767649	2·91	9·908781	1·52	9·858868	4·43	10·141132	9
52	58590	81038	767824	2·91	908690	1·52	859134	4·43	140866	8
53	58614	81021	767999	2·91	908599	1·52	859400	4·43	140600	7
54	58637	81004	768173	2·91	908507	1·52	859666	4·43	140334	6
55	58661	80987	768348	2·90	908416	1·53	859932	4·43	140068	5
56	58684	80970	768522	2·90	908324	1·53	860198	4·43	139802	4
57	58708	80953	768697	2·90	908233	1·53	860464	4·43	139536	3
58	58731	80936	768871	2·90	908141	1·53	860730	4·43	139270	2
59	58755	80919	769045	2·90	908049	1·53	860995	4·43	139005	1
60	58779	80902	769219	2·90	907958	1·53	861261	4·43	138739	0
	N. cos.	N. sine.	L. cos.	D. 1″	L. sine.		L. cot.	D. 1″	L. tang.	′

Nat. Functions.			Logarithmic Functions + 10.							
′	N. sine.	N. cos.	L. sine.	D. 1″	L. cos.	D.1″	L. tang.	D 1.″	L. cot.	
0	58779	80902	9·769219	2·90	9·907058	1·53	9·861261	4·43	10·138739	60
1	58802	80885	769393	2·89	907866	1·53	861527	4·43	138473	59
2	58826	80867	769566	2·89	907774	1·53	861792	4·42	138208	58
3	58849	80850	769740	2·89	907682	1·53	862058	4·42	137942	57
4	58873	80833	769913	2·89	907590	1·53	862323	4·42	137677	56
5	58896	80816	770087	2·89	907498	1·53	862589	4·42	137411	55
6	58920	80799	770260	2·88	907406	1·53	862854	4·42	137146	54
7	58943	80782	770433	2·88	907314	1·54	863119	4·42	136881	53
8	58967	80765	770606	2·88	907222	1·54	863385	4·42	136615	52
9	58990	80748	770779	2·88	907129	1·54	863650	4·42	136350	51
10	59014	80730	770952	2·88	907037	1·54	863915	4·42	136085	50
11	59037	80713	9·771125	2·88	9·906945	1·54	9·864180	4·42	10·135820	49
12	59061	80696	771298	2·87	906852	1·54	864445	4·42	135555	48
13	59084	80679	771470	2·87	906760	1·54	864710	4·42	135290	47
14	59108	80662	771643	2·87	906667	1·54	864975	4·41	135025	46
15	59131	80644	771815	2·87	906575	1·54	865240	4·41	134760	45
16	59154	80627	771987	2·87	906482	1·54	865505	4·41	134495	44
17	59178	80610	772159	2·87	906389	1·55	865770	4·41	134230	43
18	59201	80593	772331	2·86	906296	1·55	866035	4·41	133965	42
19	59225	80576	772503	2·86	906204	1·55	866300	4·41	133700	41
20	59248	80558	772675	2·86	906111	1·55	866564	4·41	133436	40
21	59272	80541	9·772847	2·86	9·906018	1·55	9·866829	4·41	10·133171	39
22	59295	80525	773018	2·86	905925	1·55	867094	4·41	132906	38
23	59318	80507	773190	2·86	905832	1·55	867358	4·41	132642	37
24	59342	80489	773361	2·85	905739	1·55	867623	4·41	132377	36
25	59365	80472	773533	2·85	905645	1·55	867887	4·41	132113	35
26	59389	80455	773704	2·85	905552	1·55	868152	4·40	131848	34
27	59412	80438	773875	2·85	905459	1·55	868416	4·40	131584	33
28	59436	80420	774046	2·85	905366	1·56	868680	4·40	131320	32
29	59459	80403	774217	2·85	905272	1·56	868945	4·40	131055	31
30	59482	80386	774388	2·84	905179	1·56	869209	4·40	130791	30
31	59506	80368	9·774558	2·84	9·905085	1·56	9·869473	4·40	10·130527	29
32	59529	80351	774729	2·84	904992	1·56	869737	4·40	130263	28
33	59552	80334	774899	2·84	904898	1·56	870001	4·40	129999	27
34	59576	80316	775070	2·84	904804	1·56	870265	4·40	129735	26
35	59599	80299	775240	2·84	904711	1·56	870529	4·40	129471	25
36	59622	80282	775410	2·83	904617	1·56	870793	4·40	129207	24
37	59646	80264	775580	2·83	904523	1·56	871057	4·40	128943	23
38	59669	80247	775750	2·83	904429	1·57	871321	4·40	128679	22
39	59693	80230	775920	2·83	904335	1·57	871585	4·40	128415	21
40	59716	80212	776090	2·83	904241	1·57	871849	4·39	128151	20
41	59739	80195	9·776259	2·83	9·904147	1·57	9·872112	4·39	10·127888	19
42	59763	80178	776429	2·82	904053	1·57	872376	4·39	127624	18
43	59786	80160	776598	2·82	903959	1·57	872640	4·39	127360	17
44	59809	80143	776768	2·82	903864	1·57	872903	4·39	127097	16
45	59832	80125	776937	2·82	903770	1·57	873167	4·39	126833	15
46	59856	80108	777106	2·82	903676	1·57	873430	4·39	126570	14
47	59879	80091	777275	2·81	903581	1·57	873694	4·39	126306	13
48	59902	80073	777444	2·81	903487	1·57	873957	4·39	126043	12
49	59926	80056	777613	2·81	903392	1·58	874220	4·39	125780	11
50	59949	80038	777781	2·81	903298	1·58	874484	4·39	125516	10
51	59972	80021	9·777950	2·81	9·903203	1·58	9·874747	4·39	10·125253	9
52	59995	80003	778119	2·81	903108	1·58	875010	4·39	124990	8
53	60019	79986	778287	2·80	903014	1·58	875273	4·38	124727	7
54	60042	79968	778455	2·80	902919	1·58	875536	4·38	124464	6
55	60065	79951	778624	2·80	902824	1·58	875800	4·38	124200	5
56	60089	79934	778792	2·80	902729	1·58	876063	4·38	123937	4
57	60112	79916	778960	2·80	902634	1·58	876326	4·38	123674	3
58	60135	79899	779128	2·80	902539	1·59	876589	4·38	123411	2
59	60158	79881	779295	2·79	902444	1·59	876851	4·38	123149	1
60	60182	79864	779463	2·79	902349	1·59	877114	4·38	122886	0
	N. cos.	N. sine.	L. cos.	D. 1″	L. sine.		L. cot.	D. 1″	L. tang.	′

Nat. Functions.			Logarithmic Functions + 10.							
′	N. sine.	N. cos.	L. sine.	D. 1″	L. cos.	D. 1″	L. tang.	D. 1″	L. cot.	
0	60182	79864	9·779463	2·79	9·902349	1·59	9·877114	4·38	10·122886	60
1	60205	79846	779631	2·79	902253	1·59	877377	4·38	122623	59
2	60228	79829	779798	2·79	902158	1·59	877640	4·38	122360	58
3	60251	79811	779966	2·79	902063	1·59	877903	4·38	122097	57
4	60274	79793	780133	2·79	901967	1·59	878165	4·38	121835	56
5	60298	79776	780300	2·78	901872	1·59	878428	4·38	121572	55
6	60321	79758	780467	2·78	901776	1·59	878691	4·38	121309	54
7	60344	79741	780634	2·78	901681	1·59	878953	4·37	121047	53
8	60367	79723	780801	2·78	901585	1·59	879216	4·37	120784	52
9	60390	79706	780968	2·78	901490	1·59	879478	4·37	120522	51
10	60414	79688	781134	2·78	901394	1·60	879741	4·37	120259	50
11	60437	79671	9·781301	2·77	9·901298	1·60	9·880003	4·37	10·119997	49
12	60460	79653	781468	2·77	901202	1·60	880265	4·37	119735	48
13	60483	79635	781634	2·77	901106	1·60	880528	4·37	119472	47
14	60506	79618	781800	2·77	901010	1·60	880790	4·37	119210	46
15	60529	79600	781966	2·77	900914	1·60	881052	4·37	118948	45
16	60553	79583	782132	2·77	900818	1·60	881314	4·37	118686	44
17	60576	79565	782298	2·76	900722	1·60	881576	4·37	118424	43
18	60599	79547	782464	2·76	900626	1·60	881839	4·37	118161	42
19	60622	79530	782630	2·76	900529	1·60	882101	4·37	117899	41
20	60645	79512	782796	2·76	900433	1·61	882363	4·36	117637	40
21	60668	79494	9·782961	2·76	9·900337	1·61	9·882625	4·36	10·117375	39
22	60691	79477	783127	2·76	900240	1·61	882887	4·36	117113	38
23	60714	79459	783292	2·75	900144	1·61	883148	4·36	116852	37
24	60738	79441	783458	2·75	900047	1·61	883410	4·36	116590	36
25	60761	79424	783623	2·75	899951	1·61	883672	4·36	116328	35
26	60784	79406	783788	2·75	899854	1·61	883934	4·36	116066	34
27	60807	79388	783953	2·75	899757	1·61	884196	4·36	115804	33
28	60830	79371	784118	2·75	899660	1·61	884457	4·36	115543	32
29	60853	79353	784282	2·74	899564	1·61	884719	4·36	115281	31
30	60876	79335	784447	2·74	899467	1·62	884980	4·36	115020	30
31	60899	79318	9·784612	2·74	9·899370	1·62	9·885242	4·36	10·114758	29
32	60922	79300	784776	2·74	899273	1·62	885503	4·36	114497	28
33	60945	79282	784941	2·74	899176	1·62	885765	4·36	114235	27
34	60968	79264	785105	2·74	899078	1·62	886026	4·36	113974	26
35	60991	79247	785269	2·73	898981	1·62	886288	4·36	113712	25
36	61015	79229	785433	2·73	898884	1·62	886549	4·35	113451	24
37	61038	79211	785597	2·73	898787	1·62	886810	4·35	113190	23
38	61061	79193	785761	2·73	898689	1·62	887072	4·35	112928	22
39	61084	79176	785925	2·73	898592	1·62	887333	4·35	112667	21
40	61107	79158	786089	2·73	898494	1·63	887594	4·35	112406	20
41	61130	79140	9·786252	2·72	9·898397	1·63	9·887855	4·35	10·112145	19
42	61153	79122	786416	2·72	898299	1·63	888116	4·35	111884	18
43	61176	79105	786579	2·72	898202	1·63	888377	4·35	111623	17
44	61199	79087	786742	2·72	898104	1·63	888639	4·35	111361	16
45	61222	79069	786906	2·72	898006	1·63	888900	4·35	111100	15
46	61245	79051	787069	2·72	897908	1·63	889160	4·35	110840	14
47	61268	79033	787232	2·71	897810	1·63	889421	4·35	110579	13
48	61291	79016	787395	2·71	897712	1·63	889682	4·35	110318	12
49	61314	78998	787557	2·71	897614	1·63	889943	4·35	110057	11
50	61337	78980	787720	2·71	897516	1·63	890204	4·34	109796	10
51	61360	78962	9·787883	2·71	9·897418	1·64	9·890465	4·34	10·109535	9
52	61383	78944	788045	2·71	897320	1·64	890725	4·34	109275	8
53	61406	78926	788208	2·71	897222	1·64	890986	4·34	109014	7
54	61429	78908	788370	2·70	897123	1·64	891247	4·34	108753	6
55	61451	78891	788532	2·70	897025	1·64	891507	4·34	108493	5
56	61474	78873	788694	2·70	896926	1·64	891768	4·34	108232	4
57	61497	78855	788856	2·70	896828	1·64	892028	4·34	107972	3
58	61520	78837	789018	2·70	896729	1·64	892289	4·34	107711	2
59	61543	78819	789180	2·70	896631	1·64	892549	4·34	107451	1
60	61566	78801	789342	2·69	896532	1·64	892810	4·34	107190	0
	N. cos.	N. sine.	L. cos.	D. 1″	L. sine.		L. cot.	D. 1″	L. tang.	′

Nat. Functions.			Logarithmic Functions + 10.							
′	N.sine.	N. cos.	L. sine.	D. 1″	L. cos.	D.1″	L. tang.	D. 1″	L. cot.	
0	61566	78801	9·789342	2·69	9·896532	1·64	9·892810	4·34	10·107190	60
1	61589	78783	789504	2·69	896433	1·65	893070	4·34	106930	59
2	61612	78765	789665	2·69	896335	1·65	893331	4·34	106669	58
3	61635	78747	789827	2·69	896236	1·65	893591	4·34	106409	57
4	61658	78729	789988	2·69	896137	1·65	893851	4·34	106149	56
5	61681	78711	790149	2·69	896038	1·65	894111	4·34	105889	55
6	61704	78694	790310	2·68	895939	1·65	894371	4·34	105629	54
7	61726	78676	790471	2·68	895840	1·65	894632	4·33	105368	53
8	61749	78658	790632	2·68	895741	1·65	894892	4·33	105108	52
9	61772	78640	790793	2·68	895641	1·65	895152	4·33	104848	51
10	61795	78622	790954	2·68	895542	1·65	895412	4·33	104588	50
11	61818	78604	9·791115	2·68	9·895443	1·66	9·895672	4·33	10·104328	49
12	61841	78586	791275	2·67	895343	1·66	895932	4·33	104068	48
13	61864	78568	791436	2·67	895244	1·66	896192	4·33	103808	47
14	61887	78550	791596	2·67	895145	1·66	896452	4·33	103548	46
15	61909	78532	791757	2·67	895045	1·66	896712	4·33	103288	45
16	61932	78514	791917	2·67	894945	1·66	896971	4·33	103029	44
17	61955	78496	792077	2·67	894846	1·66	897231	4·33	102769	43
18	61978	78478	792237	2·66	894746	1·66	897491	4·33	102509	42
19	62001	78460	792397	2·66	894646	1·66	897751	4·33	102249	41
20	62024	78442	792557	2·66	894546	1·66	898010	4·33	101990	40
21	62046	78424	9·792716	2·66	9·894446	1·67	9·898270	4·33	10·101730	39
22	62069	78405	792876	2·66	894346	1·67	898530	4·33	101470	38
23	62092	78387	793035	2·66	894246	1·67	898789	4·33	101211	37
24	62115	78369	793195	2·65	894146	1·67	899049	4·32	100951	36
25	62138	78351	793354	2·65	894046	1·67	899308	4·32	100692	35
26	62160	78333	793514	2·65	893946	1·67	899568	4·32	100432	34
27	62183	78315	793673	2·65	893846	1·67	899827	4·32	100173	33
28	62206	78297	793832	2·65	893745	1·67	900086	4·32	099914	32
29	62229	78279	793991	2·65	893645	1·67	900346	4·32	099654	31
30	62251	78261	794150	2·64	893544	1·67	900605	4·32	099395	30
31	62274	78243	9·794308	2·64	9·893444	1·68	9·900864	4·32	10·099136	29
32	62297	78225	794467	2·64	893343	1·68	901124	4·32	098876	28
33	62320	78206	794626	2·64	893243	1·68	901383	4·32	098617	27
34	62342	78188	794784	2·64	893142	1·68	901642	4·32	098358	26
35	62365	78170	794942	2·64	893041	1·68	901901	4·32	098099	25
36	62388	78152	795101	2·64	892940	1·68	902160	4·32	097840	24
37	62411	78134	795259	2·63	892839	1·68	902419	4·32	097581	23
38	62433	78116	795417	2·63	892739	1·68	902679	4·32	097321	22
39	62456	78098	795575	2·63	892638	1·68	902938	4·32	097062	21
40	62479	78079	795733	2·63	892536	1·68	903197	4·31	096803	20
41	62502	78061	9·795891	2·63	9·892435	1·69	9·903455	4·31	10·096545	19
42	62524	78043	796049	2·63	892334	1·69	903714	4·31	096286	18
43	62547	78025	796206	2·63	892233	1·69	903973	4·31	096027	17
44	62570	78007	796364	2·62	892132	1·69	904232	4·31	095768	16
45	62592	77988	796521	2·62	892030	1·69	904491	4·31	095509	15
46	62615	77970	796679	2·62	891929	1·69	904750	4·31	095250	14
47	62638	77952	796836	2·62	891827	1·69	905008	4·31	094992	13
48	62660	77934	796993	2·62	891726	1·69	905267	4·31	094733	12
49	62683	77916	797150	2·61	891624	1·69	905526	4·31	094474	11
50	62706	77897	797307	2·61	891523	1·70	905784	4·31	094216	10
51	62728	77879	9·797464	2·61	9·891421	1·70	9·906043	4·31	10·093957	9
52	62751	77861	797621	2·61	891319	1·70	906302	4·31	093698	8
53	62774	77843	797777	2·61	891217	1·70	906560	4·31	093440	7
54	62796	77824	797934	2·61	891115	1·70	906819	4·31	093181	6
55	62819	77806	798091	2·61	891013	1·70	907077	4·31	092923	5
56	62842	77788	798247	2·61	890911	1·70	907336	4·31	092664	4
57	62864	77769	798403	2·60	890809	1·70	907594	4·31	092406	3
58	62887	77751	798560	2·60	890707	1·70	907852	4·31	092148	2
59	62909	77733	798716	2·60	890605	1·70	908111	4·30	091889	1
60	62932	77715	798872	2·60	890503	1·70	908369	4·30	091631	0
	N. cos.	N. sine.	L. cos.	D. 1″	L. sine.		L. cot.	D. 1″	L. tang.	′

Nat. Functions.			Logarithmic Functions + 10.							
′	N. sine.	N. cos.	L. sine.	D. 1″	L. cos.	D.1″	L. tang.	D 1.″	L. cot.	
0	62932	77715	9·798872	2·60	9·890503	1·70	9·908369	4·30	10·091631	60
1	62955	77696	799028	2·60	890400	1·71	908628	4·30	091372	59
2	62977	77678	799184	2·60	890298	1·71	908886	4·30	091114	58
3	63000	77660	799339	2·59	890195	1·71	909144	4·30	090856	57
4	63022	77641	799495	2·59	890093	1·71	909402	4·30	090598	56
5	63045	77623	799651	2·59	889990	1·71	909660	4·30	090340	55
6	63068	77605	799806	2·59	889888	1·71	909918	4·30	090082	54
7	63090	77586	799962	2·59	889785	1·71	910177	4·30	089823	53
8	63113	77568	800117	2·59	889682	1·71	910435	4·30	089565	52
9	63135	77550	800272	2·58	889579	1·71	910693	4·30	089307	51
10	63158	77531	800427	2·58	889477	1·71	910951	4·30	089049	50
11	63180	77513	9·800582	2·58	9·889374	1·72	9·911209	4·30	10·088791	49
12	63203	77494	800737	2·58	889271	1·72	911467	4·30	088533	48
13	63225	77476	800892	2·58	889168	1·72	911724	4·30	088276	47
14	63248	77458	801047	2·58	889064	1·72	911982	4·30	088018	46
15	63271	77439	801201	2·58	888961	1·72	912240	4·30	087760	45
16	63293	77421	801356	2·57	888858	1·72	912498	4·30	087502	44
17	63316	77402	801511	2·57	888755	1·72	912756	4·30	087244	43
18	63338	77384	801665	2·57	888651	1·72	913014	4·29	086986	42
19	63361	77366	801819	2·57	888548	1·72	913271	4·29	086729	41
20	63383	77347	801973	2·57	888444	1·73	913529	4·29	086471	40
21	63406	77329	9·802128	2·57	9·888341	1·73	9·913787	4·29	10·086213	39
22	63428	77310	802282	2·56	888237	1·73	914044	4·29	085956	38
23	63451	77292	802436	2·56	888134	1·73	914302	4·29	085698	37
24	63473	77273	802589	2·56	888030	1·73	914560	4·29	085440	36
25	63496	77255	802743	2·56	887926	1·73	914817	4·29	085183	35
26	63518	77236	802897	2·56	887822	1·73	915075	4·29	084925	34
27	63540	77218	803050	2·56	887718	1·73	915332	4·29	084668	33
28	63563	77199	803204	2·56	887614	1·73	915590	4·29	084410	32
29	63585	77181	803357	2·55	887510	1·73	915847	4·29	084153	31
30	63608	77162	803511	2·55	887406	1·74	916104	4·29	083896	30
31	63630	77144	9·803664	2·55	9·887302	1·74	9·916362	4·29	10·083638	29
32	63653	77125	803817	2·55	887198	1·74	916619	4·29	083381	28
33	63675	77107	803970	2·55	887093	1·74	916877	4·29	083123	27
34	63698	77088	804123	2·55	886989	1·74	917134	4·29	082866	26
35	63720	77070	804276	2·54	886885	1·74	917391	4·29	082609	25
36	63742	77051	804428	2·54	886780	1·74	917648	4·29	082352	24
37	63765	77033	804581	2·54	886676	1·74	917905	4·29	082095	23
38	63787	77014	804734	2·54	886571	1·74	918163	4·28	081837	22
39	63810	76996	804886	2·54	886466	1·74	918420	4·28	081580	21
40	63832	76977	805039	2·54	886362	1·75	918677	4·28	081323	20
41	63854	76959	9·805191	2·54	9·886257	1·75	9·918934	4·28	10·081066	19
42	63877	76940	805343	2·53	886152	1·75	919191	4·28	080809	18
43	63899	76921	805495	2·53	886047	1·75	919448	4·28	080552	17
44	63922	76903	805647	2·53	885942	1·75	919705	4·28	080295	16
45	63944	76884	805799	2·53	885837	1·75	919962	4·28	080038	15
46	63966	76866	805951	2·53	885732	1·75	920219	4·28	079781	14
47	63989	76847	806103	2·53	885627	1·75	920476	4·28	079524	13
48	64011	76828	806254	2·53	885522	1·75	920733	4·28	079267	12
49	64033	76810	806406	2·52	885416	1·75	920990	4·28	079010	11
50	64056	76791	806557	2·52	885311	1·76	921247	4·28	078753	10
51	64078	76772	9·806709	2·52	9·885205	1·76	9·921503	4·28	10·078497	9
52	64100	76754	806860	2·52	885100	1·76	921760	4·28	078240	8
53	64123	76735	807011	2·52	884994	1·76	922017	4·28	077983	7
54	64145	76717	807163	2·52	884889	1·76	922274	4·28	077726	6
55	64167	76698	807314	2·52	884783	1·76	922530	4·28	077470	5
56	64190	76679	807465	2·51	884677	1·76	922787	4·28	077213	4
57	64212	76661	807615	2·51	884572	1·76	923044	4·28	076956	3
58	64234	76642	807766	2·51	884466	1·76	923300	4·28	076700	2
59	64256	76623	807917	2·51	884360	1·76	923557	4·27	076443	1
60	64279	76604	808067	2·51	884254	1·77	923813	4·27	076187	0
	N. cos.	N. sine.	L. cos.	D. 1″	L. sine.		L. cot.	D. 1″	L. tang.	′

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Nat. Functions.			Logarithmic Functions + 10.							
′	N.sine.	N. cos.	L. sine.	D. 1″	L. cos.	D.1″	L. tang.	D. 1″	L. cot.	
0	64279	76604	9·808067	2·51	9·884254	1·77	9·923813	4·27	10·076187	60
1	64301	76586	808218	2·51	884148	1·77	924070	4·27	075930	59
2	64323	76567	808368	2·51	884042	1·77	924327	4·27	075673	58
3	64346	76548	808519	2·50	883936	1·77	924583	4·27	075417	57
4	64368	76530	808669	2·50	883829	1·77	924840	4·27	075160	56
5	64390	76511	808819	2·50	883723	1·77	925096	4·27	074904	55
6	64412	76492	808969	2·50	883617	1·77	925352	4·27	074648	54
7	64435	76473	809119	2·50	883510	1·77	925609	4·27	074391	53
8	64457	76455	809269	2·50	883404	1·77	925865	4·27	074135	52
9	64479	76436	809419	2·49	883297	1·78	926122	4·27	073878	51
10	64501	76417	809569	2·49	883191	1·78	926378	4·27	073622	50
11	64524	76398	9·809718	2·49	9·883084	1·78	9·926634	4·27	10·073366	49
12	64546	76380	809868	2·49	882977	1·78	926890	4·27	073110	48
13	64568	76361	810017	2·49	882871	1·78	927147	4·27	072853	47
14	64590	76342	810167	2·49	882764	1·78	927403	4·27	072597	46
15	64612	76323	810316	2·48	882657	1·78	927659	4·27	072341	45
16	64635	76304	810465	2·48	882550	1·78	927915	4·27	072085	44
17	64657	76286	810614	2·48	882443	1·78	928171	4·27	071829	43
18	64679	76267	810763	2·48	882336	1·79	928427	4·27	071573	42
19	64701	76248	810912	2·48	882229	1·79	928683	4·27	071317	41
20	64723	76229	811061	2·48	882121	1·79	928940	4·27	071060	40
21	64746	76210	9·811210	2·48	9·882014	1·79	9·929196	4·27	10·070804	39
22	64768	76192	811358	2·47	881907	1·79	929452	4·27	070548	38
23	64790	76173	811507	2·47	881799	1·79	929708	4·27	070292	37
24	64812	76154	811655	2·47	881692	1·79	929964	4·26	070036	36
25	64834	76135	811804	2·47	881584	1·79	930220	4·26	069780	35
26	64856	76116	811952	2·47	881477	1·79	930475	4·26	069525	34
27	64878	76097	812100	2·47	881369	1·79	930731	4·26	069269	33
28	64901	76078	812248	2·47	881261	1·80	930987	4·26	069013	32
29	64923	76059	812396	2·46	881153	1·80	931243	4·26	068757	31
30	64945	76041	812544	2·46	881046	1·80	931499	4·26	068501	30
31	64967	76022	9·812692	2·46	9·880938	1·80	9·931755	4·26	10·068245	29
32	64989	76003	812840	2·46	880830	1·80	932010	4·26	067990	28
33	65011	75984	812988	2·46	880722	1·80	932266	4·26	067734	27
34	65033	75965	813135	2·46	880613	1·80	932522	4·26	067478	26
35	65055	75946	813283	2·46	880505	1·80	932778	4·26	067222	25
36	65077	75927	813430	2·45	880397	1·80	933033	4·26	066967	24
37	65100	75908	813578	2·45	880289	1·81	933289	4·26	066711	23
38	65122	75889	813725	2·45	880180	1·81	933545	4·26	066455	22
39	65144	75870	813872	2·45	880072	1·81	933800	4·26	066200	21
40	65166	75851	814019	2·45	879963	1·81	934056	4·26	065944	20
41	65188	75832	9·814166	2·45	9·879855	1·81	9·934311	4·26	10·065689	19
42	65210	75813	814313	2·45	879746	1·81	934567	4·26	065433	18
43	65232	75794	814460	2·44	879637	1·81	934823	4·26	065177	17
44	65254	75775	814607	2·44	879529	1·81	935078	4·26	064922	16
45	65276	75756	814753	2·44	879420	1·81	935333	4·26	064667	15
46	65298	75738	814900	2·44	879311	1·81	935589	4·26	064411	14
47	65320	75719	815046	2·44	879202	1·82	935844	4·26	064156	13
48	65342	75700	815193	2·44	879093	1·82	936100	4·26	063900	12
49	65364	75680	815339	2·44	878984	1·82	936355	4·26	063645	11
50	65386	75661	815485	2·43	878875	1·82	936610	4·26	063390	10
51	65408	75642	9·815631	2·43	9·878766	1·82	9·936866	4·25	10·063134	9
52	65430	75623	815778	2·43	878656	1·82	937121	4·25	062879	8
53	65452	75604	815924	2·43	878547	1·82	937376	4·25	062624	7
54	65474	75585	816069	2·43	878438	1·82	937632	4·25	062368	6
55	65496	75566	816215	2·43	878328	1·82	937887	4·25	062113	5
56	65518	75547	816361	2·43	878219	1·83	938142	4·25	061858	4
57	65540	75528	816507	2·42	878109	1·83	938398	4·25	061602	3
58	65562	75509	816652	2·42	877999	1·83	938653	4·25	061347	2
59	65584	75490	816798	2·42	877890	1·83	938908	4·25	061092	1
60	65606	75471	816943	2·42	877780	1·83	939163	4·25	060837	0
	N. cos.	N. sine.	L. cos.	D. 1″	L. sine.		L. cot.	D. 1″	L. tang.	′

Nat. Functions.		Logarithmic Functions + 10.							
′	N. sine.	N. cos.	L. sine.	D. 1″	L. cos.	D. 1″	L. tang.	D. 1″	L. cot.	
0	65606	75471	9·816943	2·42	9·877780	1·83	9·939163	4·25	10·060837	60
1	65628	75452	817088	2·42	877670	1·83	939418	4·25	060582	59
2	65650	75433	817233	2·42	877560	1·83	939673	4·25	060327	58
3	65672	75414	817379	2·42	877450	1·83	939928	4·25	060072	57
4	65694	75395	817524	2·41	877340	1·83	940183	4·25	059817	56
5	65716	75375	817668	2·41	877230	1·84	940438	4·25	059562	55
6	65738	75356	817813	2·41	877120	1·84	940694	4·25	059306	54
7	65759	75337	817958	2·41	877010	1·84	940949	4·25	059051	53
8	65781	75318	818103	2·41	876899	1·84	941204	4·25	058796	52
9	65803	75299	818247	2·41	876789	1·84	941458	4·25	058542	51
10	65825	75280	818392	2·41	876678	1·84	941714	4·25	058286	50
11	65847	75261	9·818536	2·40	9·876568	1·84	9·941968	4·25	10·058032	49
12	65869	75241	818681	2·40	876457	1·84	942223	4·25	057777	48
13	65891	75222	818825	2·40	876347	1·84	942478	4·25	057522	47
14	65913	75203	818969	2·40	876236	1·85	942733	4·25	057267	46
15	65935	75184	819113	2·40	876125	1·85	942988	4·25	057012	45
16	65956	75165	819257	2·40	876014	1·85	943243	4·25	056757	44
17	65978	75146	819401	2·40	875904	1·85	943498	4·25	056502	43
18	66000	75126	819545	2·39	875793	1·85	943752	4·25	056248	42
19	66022	75107	819689	2·39	875682	1·85	944007	4·25	055993	41
20	66044	75088	819832	2·39	875571	1·85	944262	4·25	055738	40
21	66066	75069	9·819976	2·39	9·875459	1·85	9·944517	4·25	10·055483	39
22	66088	75050	820120	2·39	875348	1·85	944771	4·24	055229	38
23	66109	75030	820263	2·39	875237	1·85	945026	4·24	054974	37
24	66131	75011	820406	2·39	875126	1·86	945281	4·24	054719	36
25	66153	74992	820550	2·38	875014	1·86	945535	4·24	054465	35
26	66175	74973	820693	2·38	874903	1·86	945790	4·24	054210	34
27	66197	74953	820836	2·38	874791	1·86	946045	4·24	053955	33
28	66218	74934	820979	2·38	874680	1·86	946299	4·24	053701	32
29	66240	74915	821122	2·38	874568	1·86	946554	4·24	053446	31
30	66262	74896	821265	2·38	874456	1·86	946808	4·24	053192	30
31	66284	74876	9·821407	2·38	9·874344	1·86	9·947063	4·24	10·052937	29
32	66306	74857	821550	2·38	874232	1·87	947318	4·24	052682	28
33	66327	74838	821693	2·37	874121	1·87	947572	4·24	052428	27
34	66349	74818	821835	2·37	874009	1·87	947826	4·24	052174	26
35	66371	74799	821977	2·37	873896	1·87	948081	4·24	051919	25
36	66393	74780	822120	2·37	873784	1·87	948336	4·24	051664	24
37	66414	74760	822262	2·37	873672	1·87	948590	4·24	051410	23
38	66436	74741	822404	2·37	873560	1·87	948844	4·24	051156	22
39	66458	74722	822546	2·37	873448	1·87	949099	4·24	050901	21
40	66480	74703	822688	2·36	873335	1·87	949353	4·24	050647	20
41	66501	74683	9·822830	2·36	9·873223	1·87	9·949607	4·24	10·050393	19
42	66523	74664	822972	2·36	873110	1·88	949862	4·24	050138	18
43	66545	74644	823114	2·36	872998	1·88	950116	4·24	049884	17
44	66566	74625	823255	2·36	872885	1·88	950370	4·24	049630	16
45	66588	74606	823397	2·36	872772	1·88	950625	4·24	049375	15
46	66610	74586	823539	2·36	872659	1·88	950879	4·24	049121	14
47	66632	74567	823680	2·35	872547	1·88	951133	4·24	048867	13
48	66653	74548	823821	2·35	872434	1·88	951388	4·24	048612	12
49	66675	74528	823963	2·35	872321	1·88	951642	4·24	048358	11
50	66697	74509	824104	2·35	872208	1·88	951896	4·24	048104	10
51	66718	74489	9·824245	2·35	9·872095	1·89	9·952150	4·24	10·047850	9
52	66740	74470	824386	2·35	871981	1·89	952405	4·24	047595	8
53	66762	74451	824527	2·35	871868	1·89	952659	4·24	047341	7
54	66783	74431	824668	2·34	871755	1·89	952913	4·24	047087	6
55	66805	74412	824808	2·34	871641	1·89	953167	4·23	046833	5
56	66827	74392	824949	2·34	871528	1·89	953421	4·23	046579	4
57	66848	74373	825090	2·34	871414	1·89	953675	4·23	046325	3
58	66870	74353	825230	2·34	871301	1·89	953929	4·23	046071	2
59	66891	74334	825371	2·34	871187	1·89	954183	4·23	045817	1
60	66913	74314	825511	2·34	871073	1·90	954437	4·23	045563	0
	N. cos.	N. sine.	L. cos.	D. 1″	L. sine.		L. cot.	D. 1″	L. tang.	′

Nat. Functions.			Logarithmic Functions + 10.							
′	N. sine.	N. cos.	L. sine.	D. 1″	L. cos.	D.1″	L. tang.	D. 1″	L. cot.	
0	66913	74314	9·825511	2·34	9·871073	1·90	9·954437	4·23	10·045563	60
1	66935	74295	825651	2·33	870960	1·90	954691	4·23	045309	59
2	66956	74276	825791	2·33	870846	1·90	954945	4·23	045055	58
3	66978	74256	825931	2·33	870732	1·90	955200	4·23	044800	57
4	66999	74237	826071	2·33	870618	1·90	955454	4·23	044546	56
5	67021	74217	826211	2·33	870504	1·90	955707	4·23	044293	55
6	67043	74198	826351	2·33	870390	1·90	955961	4·23	044039	54
7	67064	74178	826491	2·33	870276	1·90	956215	4·23	043785	53
8	67086	74159	826631	2·33	870161	1·90	956469	4·23	043531	52
9	67107	74139	826770	2·32	870047	1·91	956723	4·23	043277	51
10	67129	74120	826910	2·32	869933	1·91	956977	4·23	043023	50
11	67151	74100	9·827049	2·32	9·869818	1·91	9·957231	4·23	10·042769	49
12	67172	74080	827189	2·32	869704	1·91	957485	4·23	042515	48
13	67194	74061	827328	2·32	869589	1·91	957739	4·23	042261	47
14	67215	74041	827467	2·32	869474	1·91	957993	4·23	042007	46
15	67237	74022	827606	2·32	869360	1·91	958246	4·23	041754	45
16	67258	74002	827745	2·32	869245	1·91	958500	4·23	041500	44
17	67280	73983	827884	2·31	869130	1·91	958754	4·23	041246	43
18	67301	73963	828023	2·31	869015	1·92	959008	4·23	040992	42
19	67323	73944	828162	2·31	868900	1·92	959262	4·23	040738	41
20	67344	73924	828301	2·31	868785	1·92	959516	4·23	040484	40
21	67366	73904	9·828439	2·31	9·868670	1·92	9·959769	4·23	10·040231	39
22	67387	73885	828578	2·31	868555	1·92	960023	4·23	039977	38
23	67409	73865	828716	2·31	868440	1·92	960277	4·23	039723	37
24	67430	73846	828855	2·30	868324	1·92	960531	4·23	039469	36
25	67452	73826	828993	2·30	868209	1·92	960784	4·23	039216	35
26	67473	73806	829131	2·30	868093	1·92	961038	4·23	038962	34
27	67495	73787	829269	2·30	867978	1·93	961291	4·23	038709	33
28	67516	73767	829407	2·30	867862	1·93	961545	4·23	038455	32
29	67538	73747	829545	2·30	867747	1·93	961799	4·23	038201	31
30	67559	73728	829683	2·30	867631	1·93	962052	4·23	037948	30
31	67580	73708	9·829821	2·29	9·867515	1·93	9·962306	4·23	10·037694	29
32	67602	73688	829959	2·29	867399	1·93	962560	4·23	037440	28
33	67623	73669	830097	2·29	867283	1·93	962813	4·23	037187	27
34	67645	73649	830234	2·29	867167	1·93	963067	4·23	036933	26
35	67666	73629	830372	2·29	867051	1·93	963320	4·23	036680	25
36	67688	73610	830509	2·29	866935	1·94	963574	4·23	036426	24
37	67709	73590	830646	2·29	866819	1·94	963827	4·23	036173	23
38	67730	73570	830784	2·29	866703	1·94	964081	4·23	035919	22
39	67752	73551	830921	2·28	866586	1·94	964335	4·23	035665	21
40	67773	73531	831058	2·28	866470	1·94	964588	4·22	035412	20
41	67795	73511	9·831195	2·28	9·866353	1·94	9·964842	4·22	10·035158	19
42	67816	73491	831332	2·28	866237	1·94	965095	4·22	034905	18
43	67837	73472	831469	2·28	866120	1·94	965349	4·22	034651	17
44	67859	73452	831606	2·28	866004	1·95	965602	4·22	034398	16
45	67880	73432	831742	2·28	865887	1·95	965855	4·22	034145	15
46	67901	73413	831879	2·28	865770	1·95	966109	4·22	033891	14
47	67923	73393	832015	2·27	865653	1·95	966362	4·22	033638	13
48	67944	73373	832152	2·27	865536	1·95	966616	4·22	033384	12
49	67965	73353	832288	2·27	865419	1·95	966869	4·22	033131	11
50	67987	73333	832425	2·27	865302	1·95	967123	4·22	032877	10
51	68008	73314	9·832561	2·27	9·865185	1·95	9·967376	4·22	10·032624	9
52	68029	73294	832697	2·27	865068	1·95	967629	4·22	032371	8
53	68051	73274	832833	2·27	864950	1·95	967883	4·22	032117	7
54	68072	73254	832969	2·26	864833	1·96	968136	4·22	031864	6
55	68093	73234	833105	2·26	864716	1·96	968389	4·22	031611	5
56	68115	73215	833241	2·26	864598	1·96	968643	4·22	031357	4
57	68136	73195	833377	2·26	864481	1·96	968896	4·22	031104	3
58	68157	73175	833512	2·26	864363	1·96	969149	4·22	030851	2
59	68179	73155	833648	2·26	864245	1·96	969403	4·22	030597	1
60	68200	73135	833783	2·26	864127	1·96	969656	4·22	030344	0
	N. cos.	N.sine.	L. cos.	D. 1″	L. sine.		L. cot.	D. 1″	L. tang.	′

Nat. Functions.			Logarithmic Functions + 10.							
′	N.sine.	N. cos.	L. sine.	D. 1″	L. cos.	D.1″	L. tang.	D. 1″	L. cot.	
0	68200	73135	9·833783	2·26	9·864127	1·96	9·969656	4·22	10·030344	60
1	68221	73116	833919	2·25	864010	1·96	969909	4·22	030091	59
2	68242	73096	834054	2·25	863892	1·97	970162	4·22	029838	58
3	68264	73076	834189	2·25	863774	1·97	970416	4·22	029584	57
4	68285	73056	834325	2·25	863656	1·97	970669	4·22	029331	56
5	68306	73036	834460	2·25	863538	1·97	970922	4·22	029078	55
6	68327	73016	834595	2·25	863419	1·97	971175	4·22	028825	54
7	68349	72996	834730	2·25	863301	1·97	971429	4·22	028571	53
8	68370	72976	834865	2·25	863183	1·97	971682	4·22	028318	52
9	68391	72957	834999	2·24	863064	1·97	971935	4·22	028065	51
10	68412	72937	835134	2·24	862946	1·98	972188	4·22	027812	50
11	68434	72917	9·835269	2·24	9·862827	1·98	9·972441	4·22	10·027559	49
12	68455	72897	835403	2·24	862709	1·98	972694	4·22	027306	48
13	68476	72877	835538	2·24	862590	1·98	972948	4·22	027052	47
14	68497	72857	835672	2·24	862471	1·98	973201	4·22	026799	46
15	68518	72837	835807	2·24	862353	1·98	973454	4·22	026546	45
16	68539	72817	835941	2·24	862234	1·98	973707	4·22	026293	44
17	68561	72797	836075	2·23	862115	1·98	973960	4·22	026040	43
18	68582	72777	836209	2·23	861996	1·98	974213	4·22	025787	42
19	68603	72757	836343	2·23	861877	1·98	974466	4·22	025534	41
20	68624	72737	836477	2·23	861758	1·99	974719	4·22	025281	40
21	68645	72717	9·836611	2·23	9·861638	1·99	9·974973	4·22	10·025027	39
22	68666	72697	836745	2·23	861519	1·99	975226	4·22	024774	38
23	68688	72677	836878	2·23	861400	1·99	975479	4·22	024521	37
24	68709	72657	837012	2·22	861280	1·99	975732	4·22	024268	36
25	68730	72637	837146	2·22	861161	1·99	975985	4·22	024015	35
26	68751	72617	837279	2·22	861041	1·99	976238	4·22	023762	34
27	68772	72597	837412	2·22	860922	1·99	976491	4·22	023509	33
28	68793	72577	837546	2·22	860802	1·99	976744	4·22	023256	32
29	68814	72557	837679	2·22	860682	2·00	976997	4·22	023003	31
30	68835	72537	837812	2·22	860562	2·00	977250	4·22	022750	30
31	68857	72517	9·837945	2·22	9·860442	2·00	9·977503	4·22	10·022497	29
32	68878	72497	838078	2·21	860322	2·00	977756	4·22	022244	28
33	68899	72477	838211	2·21	860202	2·00	978009	4·22	021991	27
34	68920	72457	838344	2·21	860082	2·00	978262	4·22	021738	26
35	68941	72437	838477	2·21	859962	2·00	978515	4·22	021485	25
36	68962	72417	838610	2·21	859842	2·00	978768	4·22	021232	24
37	68983	72397	838742	2·21	859721	2·01	979021	4·22	020979	23
38	69004	72377	838875	2·21	859601	2·01	979274	4·22	020726	22
39	69025	72357	839007	2·21	859480	2·01	979527	4·22	020473	21
40	69046	72337	839140	2·20	859360	2·01	979780	4·22	020220	20
41	69067	72317	9·839272	2·20	9·859239	2·01	9·980033	4·22	10·019967	19
42	69088	72297	839404	2·20	859119	2·01	980286	4·22	019714	18
43	69109	72277	839536	2·20	858998	2·01	980538	4·22	019462	17
44	69130	72257	839668	2·20	858877	2·01	980791	4·21	019209	16
45	69151	72236	839800	2·20	858756	2·02	981044	4·21	018956	15
46	69172	72216	839932	2·20	858635	2·02	981297	4·21	018703	14
47	69193	72196	840064	2·19	858514	2·02	981550	4·21	018450	13
48	69214	72176	840196	2·19	858393	2·02	981803	4·21	018197	12
49	69235	72156	840328	2·19	858272	2·02	982056	4·21	017944	11
50	69256	72136	840459	2·19	858151	2·02	982309	4·21	017691	10
51	69277	72116	9·840591	2·19	9·858029	2·02	9·982562	4·21	10·017438	9
52	69298	72095	840722	2·19	857908	2·02	982814	4·21	017186	8
53	69319	72075	840854	2·19	857786	2·02	983067	4·21	016933	7
54	69340	72055	840985	2·19	857665	2·03	983320	4·21	016680	6
55	69361	72035	841116	2·18	857543	2·03	983573	4·21	016427	5
56	69382	72015	841247	2·18	857422	2·03	983826	4·21	016174	4
57	69403	71995	841378	2·18	857300	2·03	984079	4·21	015921	3
58	69424	71974	841509	2·18	857178	2·03	984331	4·21	015669	2
59	69445	71954	841640	2·18	857056	2·03	984584	4·21	015416	1
60	69466	71934	841771	2·18	856934	2·03	984837	4·21	015163	0
	N. cos.	N. sine.	L. cos.	D. 1″	L. sine.		L. cot.	D. 1″	L. tang.	′

Nat. Functions.			Logarithmic Functions + 10.						
′	N. sine.	N. cos.	L. sine.	D. 1″	L. cos.	D. 1″	L. tang.	D 1.″	L. cot.	
0	69466	71934	9·841771	2·18	9·856934	2·03	9·984837	4·21	10·015163	60
1	69487	71914	841902	2·18	856812	2·03	985090	4·21	014910	59
2	69508	71894	842033	2·18	856690	2·04	985343	4·21	014657	58
3	69529	71873	842163	2·17	856568	2·04	985596	4·21	014404	57
4	69549	71853	842294	2·17	856446	2·04	985848	4·21	014152	56
5	69570	71833	842424	2·17	856323	2·04	986101	4·21	013899	55
6	69591	71813	842555	2·17	856201	2·04	986354	4·21	013646	54
7	69612	71792	842685	2·17	856078	2·04	986607	4·21	013393	53
8	69633	71772	842815	2·17	855956	2·04	986860	4·21	013140	52
9	69654	71752	842946	2·17	855833	2·04	987112	4·21	012888	51
10	69675	71732	843076	2·17	855711	2·05	987365	4·21	012635	50
11	69696	71711	9·843206	2·16	9·855588	2·05	9·987618	4·21	10·012382	49
12	69717	71691	843336	2·16	855465	2·05	987871	4·21	012129	48
13	69737	71671	843466	2·16	855342	2·05	988123	4·21	011877	47
14	69758	71650	843595	2·16	855219	2·05	988376	4·21	011624	46
15	69779	71630	843725	2·16	855096	2·05	988629	4·21	011371	45
16	69800	71610	843855	2·16	854973	2·05	988882	4·21	011118	44
17	69821	71590	843984	2·16	854850	2·05	989134	4·21	010866	43
18	69842	71569	844114	2·15	854727	2·06	989387	4·21	010613	42
19	69862	71549	844243	2·15	854603	2·06	989640	4·21	010360	41
20	69883	71529	844372	2·15	854480	2·06	989893	4·21	010107	40
21	69904	71508	9·844502	2·15	9·854356	2·06	9·990145	4·21	10·009855	39
22	69925	71488	844631	2·15	854233	2·06	990398	4·21	009602	38
23	69946	71468	844760	2·15	854109	2·06	990651	4·21	009349	37
24	69966	71447	844889	2·15	853986	2·06	990903	4·21	009097	36
25	69987	71427	845018	2·15	853862	2·06	991156	4·21	008844	35
26	70008	71407	845147	2·15	853738	2·06	991409	4·21	008591	34
27	70029	71386	845276	2·14	853614	2·07	991662	4·21	008338	33
28	70049	71366	845405	2·14	853490	2·07	991914	4·21	008086	32
29	70070	71345	845533	2·14	853366	2·07	992167	4·21	007833	31
30	70091	71325	845662	2·14	853242	2·07	992420	4·21	007580	30
31	70112	71305	9·845790	2·14	9·853118	2·07	9·992672	4·21	10·007328	29
32	70132	71284	845919	2·14	852994	2·07	992925	4·21	007075	28
33	70153	71264	846047	2·14	852869	2·07	993178	4·21	006822	27
34	70174	71243	846175	2·14	852745	2·07	993430	4·21	006570	26
35	70195	71223	846304	2·14	852620	2·07	993683	4·21	006317	25
36	70215	71203	846432	2·13	852496	2·08	993936	4·21	006064	24
37	70236	71182	846560	2·13	852371	2·08	994189	4·21	005811	23
38	70257	71162	846688	2·13	852247	2·08	994441	4·21	005559	22
39	70277	71141	846816	2·13	852122	2·08	994694	4·21	005306	21
40	70298	71121	846944	2·13	851997	2·08	994947	4·21	005053	20
41	70319	71100	9·847071	2·13	9·851872	2·08	9·995199	4·21	10·004801	19
42	70339	71080	847199	2·13	851747	2·08	995452	4·21	004548	18
43	70360	71059	847327	2·13	851622	2·08	995705	4·21	004295	17
44	70381	71039	847454	2·12	851497	2·09	995957	4·21	004043	16
45	70401	71019	847582	2·12	851372	2·09	996210	4·21	003790	15
46	70422	70998	847709	2·12	851246	2·09	996463	4·21	003537	14
47	70443	70978	847836	2·12	851121	2·09	996715	4·21	003285	13
48	70463	70957	847964	2·12	850996	2·09	996968	4·21	003032	12
49	70484	70937	848091	2·12	850870	2·09	997221	4·21	002779	11
50	70505	70916	848218	2·12	850745	2·09	997473	4·21	002527	10
51	70525	70896	9·848345	2·12	9·850619	2·09	9·997726	4·21	10·002274	9
52	70546	70875	848472	2·11	850493	2·10	997979	4·21	002021	8
53	70567	70855	848599	2·11	850368	2·10	998231	4·21	001769	7
54	70587	70834	848726	2·11	850242	2·10	998484	4·21	001516	6
55	70608	70813	848852	2·11	850116	2·10	998737	4·21	001263	5
56	70628	70793	848979	2·11	849990	2·10	998989	4·21	001011	4
57	70649	70772	849106	2·11	849864	2·10	999242	4·21	000758	3
58	70670	70752	849232	2·11	849738	2·10	999495	4·21	000505	2
59	70690	70731	849359	2·11	849611	2·10	999747	4·21	000253	1
60	70711	70711	849485	2·11	849485	2·10	10·000000	4·21	10·000000	0
	N. cos.	N. sine.	L. cos.	D. 1″	L. sine.		L. cot.	D. 1″	L. tang.	′

TABLE III.

FOR

PRECISE CALCULATION

OF

FUNCTIONS NEAR THEIR LIMITS.

log. sin. $x'' = 4.6855575 + \log. x - \text{diff.}$

FOR THE SINES OF SMALL ANGLES.

Angles.	Seconds.	Diff.	Angles.	Seconds	Diff.	Angles.	Seconds.	Diff.
0″	0	0	1° 29′ 50″	5390	50	2° 7′ 30″	7650	100
9′	540	1	30′ 50″	5450	51	8′ 10″	7690	101
15′ 50″	950	2	31′ 40″	5500	52	8′ 45″	7725	102
20′ 20″	1220	3	32′ 30″	5550	53	9′ 20″	7760	103
23′ 50″	1430	4	33′ 30″	5610	54	10′	7800	104
27′	1620	5	34′ 20″	5660	55	10′ 40″	7840	105
29′ 50″	1790	6	35′ 10″	5710	56	11′ 15″	7875	106
32′ 30″	1950	7	36′	5760	57	11′ 50″	7910	107
35′	2100	8	36′ 50″	5810	58	12′ 30″	7950	108
37′ 20″	2240	9	37′ 40″	5860	59	13′ 5″	7985	109
39′ 30′	2370	10	38′ 30″	5910	60	13′ 40′	8020	110
41′ 30″	2490	11	39′ 30″	5970	61	14′ 20″	8060	111
43′ 20″	2600	12	40′ 20″	6020	62	15′	8100	112
45′ 10″	2710	13	41′ 10″	6070	63	15′ 35″	8135	113
47′	2820	14	41′ 50″	6110	64	16′ 10″	8170	114
48′ 40″	2920	15	42′ 40″	6160	65	16′ 45″	8205	115
50′ 20″	3020	16	43′ 30″	6210	66	17′ 20″	8240	116
52′	3120	17	44′ 10″	6250	67	17′ 55″	8275	117
53′ 30″	3210	18	45′	6300	68	18′ 30″	8310	118
55′	3300	19	45′ 50″	6350	69	19′ 5″	8345	119
56′ 30′	3390	20	46′ 30″	6390	70	19′ 40″	8380	120
58′	3480	21	47′ 20″	6440	71	20′ 15″	8415	121
59′ 20″	3560	22	48′	6480	72	20′ 50″	8450	122
1° 00′ 40″	3640	23	48′ 50″	6530	73	21′ 25″	8485	123
2′	3720	24	49′ 30″	6570	74	22′	8520	124
3′ 20″	3800	25	50′ 20″	6620	75	22′ 35″	8555	125
4′ 40″	3880	26	51′	6660	76	23′ 10″	8590	126
5′ 50″	3950	27	51′ 50″	6710	77	23′ 45″	8625	127
7′	4020	28	52′ 30″	6750	78	24′ 20″	8660	128
8′ 10″	4090	29	53′ 10″	6790	79	24′ 55″	8695	129
9′ 20″	4160	30	54′	6840	80	25′ 30″	8730	130
10′ 30″	4230	31	54′ 40″	6880	81	26′	8760	131
11′ 40″	4300	32	55′ 20″	6920	82	26′ 35″	8795	132
12′ 50″	4370	33	56′ 10″	6970	83	27′ 5″	8825	133
14′	4440	34	56′ 50″	7010	84	27′ 40″	8860	134
15′	4500	35	57′ 30″	7050	85	28′ 10″	8890	135
16′ 10″	4570	36	58′ 10″	7090	86	28′ 45″	8925	136
17′ 10″	4630	37	58′ 50″	7130	87	29′ 15″	8955	137
18′ 10″	4690	38	59′ 30″	7170	88	29′ 50″	8990	138
19′ 20″	4760	39	20 00′ 10″	7210	89	30′ 20″	9020	139
20′ 20″	4820	40	50″	7250	90	30′ 55″	9055	140
21′ 20″	4880	41	1′ 40″	7300	91	31′ 25″	9085	141
22′ 20″	4940	42	2′ 20″	7340	92	32′	9120	142
23′ 20″	5000	43	3′	7380	93	32′ 30″	9150	143
24′ 20″	5060	44	3′ 35″	7415	94	33′ 5″	9185	144
25′ 10″	5110	45	4′ 10″	7450	95	33′ 35″	9215	145
26′ 10″	5170	46	4′ 50″	7490	96	34′ 5″	9245	146
27′ 10″	5230	47	5′ 30″	7530	97	34′ 40″	9280	147
28′ 10″	5290	48	6′ 10″	7570	98	35′ 10″	9310	148
29′	5340	49	6′ 50″	7610	99	35′ 40″	9340	149
29′ 50″	5390		7′ 30″	7650		36′ 15″	9375	

log. tan. $x'' = 4.685575 + \log. x + \text{diff.}$
log. cot. $x'' = 15.314425 - \log. x - \text{diff.}$

FOR TANGENTS AND COTANGENTS OF SMALL ANGLES.

Angles.	Seconds.	Diff.	Angles.	Seconds.	Diff.	Angles.	Seconds.	Diff.
0″	0	0	1° 3′ 3o″	3810	50	1° 3o′ 10″	5410	100
7′ 10″	430	1	4′ 10″	3850	51	3c′ 3o″	5430	101
11′ 10″	670	2	4′ 5o″	3890	52	31′	5460	102
14′ 10″	850	3	5′ 3o″	3930	53	31′ 3o″	5490	103
17′	1020	4	6′	3960	54	32′	5520	104
19′	1440	5	6′ 40″	4000	55	32′ 20″	5540	105
21′	1260	6	7′ 20″	4040	56	32′ 5o″	5570	106
23′	1380	7	7′ 5o″	4070	57	33′ 10″	5590	107
24′ 5o″	1490	8	8′ 3o″	4110	58	33′ 40″	5620	108
26′ 3o″	1590	9	9′	4140	59	34′ 10″	5650	109
27′ 5o″	1670	10	9′ 40″	4180	60	34′ 3o″	5670	110
29′ 20″	1760	11	10′ 20″	4220	61	35′	5700	111
3o′ 40″	1840	12	10′ 5o″	4250	62	35′ 20″	5720	112
32′	1920	13	11′ 3o″	4290	63	35′ 5o″	5750	113
33′ 10″	1990	14	12′	4320	64	36′ 10″	5770	114
34′ 20″	2060	15	12′ 3o″	4350	65	36′ 40″	5800	115
35′ 3o″	2130	16	13′ 10″	4390	66	37′ 10″	5830	116
36′ 40″	2200	17	13′ 40″	4420	67	37′ 3o″	5850	117
37′ 5o″	2270	18	14′ 10″	4450	68	38′	5880	118
38′ 5o″	2330	19	14′ 5o″	4490	69	38′ 20″	5900	119
39′ 5o″	2390	20	15′ 20″	4520	70	38′ 5o″	5930	120
40′ 5o″	2450	21	15′ 5o″	4550	71	39′ 10″	5950	121
41′ 5o″	2510	22	16′ 20″	4580	72	39′ 3o″	5970	122
42′ 5o″	2570	23	17′	4620	73	40′	6000	123
43′ 5o″	2630	24	17′ 3o″	4650	74	40′ 20″	6020	124
44′ 40″	2680	25	18′	4680	75	40′ 5o″	6050	125
45′ 40″	2740	26	18′ 3o″	4710	76	41′ 10″	6070	126
46′ 3o″	2790	27	19′	4740	77	41′ 40″	6100	127
47′ 20″	2840	28	19′ 3o″	4770	78	42′	6120	128
48′ 10″	2890	29	20′	4800	79	42′ 3o″	6150	129
49′	2940	30	20′ 3o″	4830	80	42′ 5o″	6170	130
49′ 5o″	2990	31	21′	4860	81	43′ 10″	6190	131
5o′ 40″	3040	32	21′ 3o″	4890	82	43′ 40″	6220	132
51′ 3o″	3090	33	22′	4920	83	44′	6240	133
52′ 20″	3140	34	22′ 3o″	4950	84	44′ 3o″	6270	134
53′	3180	35	23′	4980	85	44′ 5o″	6290	135
53′ 5o″	3230	36	23′ 3o″	5010	86	45′ 20″	6320	136
54′ 40″	3280	37	24′	5040	87	45′ 40″	6340	137
55′ 20″	3320	38	24′ 3o″	5070	88	46′	6360	138
56′	3360	39	25′	5100	89	46′ 20″	6380	139
56′ 5o″	3410	40	25′ 3o″	5130	90	46′ 40″	6400	140
57′ 3o″	3450	41	26′	5160	91	47′ 10″	6430	141
58′ 10″	3490	42	26′ 3o″	5190	92	47′ 3o″	6450	142
58′ 5o″	3530	43	26′ 5o″	5210	93	48′	6480	143
59′ 3o″	3570	44	27′ 20″	5240	94	48′ 20″	6500	144
1° 0′ 20″	3620	45	27′ 5o″	5270	95	48′ 40″	6520	145
1′	3660	46	28′ 20″	5300	96	49′	6540	146
1′ 40″	3700	47	28′ 40″	5320	97	49′ 20″	6560	147
2′ 10″	3730	48	29′ 10″	5350	98	49′ 40″	9580	148
2′ 5o″	3770	49	29′ 40″	5380	99	5o′ 10″	6610	148
3′ 3o″	3810		3o′ 10″	5410		5o′ 3o″	6630	149

log. tan. A″= 4.685575 + log. A + diff.
log. cot. A″= 15.314425 − log. A − diff.

FOR TANGENTS AND COTANGENTS OF SMALL ANGLES.

Angles.	Seconds.	Diff.	Angles.	Seconds.	Diff.	Angles.	Seconds.	Diff.
50′ 30″	6630	150	2° 7′ 40″	7660	200	2° 22′ 35″	8555	250
50′ 50″	6650	151	8′	7680	201	22′ 55″	8575	251
51′ 10″	6670	152	8′ 15″	7695	202	23′ 10″	8590	252
51′ 30″	6690	153	8′ 30″	7710	203	23′ 30″	8610	253
52′	6720	154	8′ 50″	7730	204	23′ 45″	8625	254
52′ 20″	6740	155	9′ 10″	7750	205	24′	8640	255
52′ 40″	6760	156	9′ 30″	7770	206	24′ 20″	8660	256
53′	6780	157	9′ 50″	7790	207	24′ 35″	8675	257
53′ 20″	6800	158	10′ 10″	7810	208	24′ 55″	8695	258
53′ 50″	6830	159	10′ 20″	7820	209	25′ 10″	8710	259
54′ 10″	6850	160	10′ 40″	7840	210	25′ 25″	8725	260
54′ 30″	6870	161	10′ 55″	7855	211	25′ 45″	8745	261
54′ 50″	6890	162	11′ 15″	7875	212	26′	8760	262
55′ 10″	6910	163	11′ 35″	7895	213	26′ 20″	8780	263
55′ 30″	6930	164	11′ 55″	7915	214	26′ 35″	8795	264
55′ 50″	6950	165	12′ 15″	7935	215	26′ 50″	8810	265
56′ 10″	6970	166	12′ 35″	7955	216	27′ 10″	8830	266
56′ 30″	6990	167	12′ 55″	7975	217	27′ 25″	8845	267
56′ 50″	7010	168	13′ 15″	7995	218	27′ 45″	8865	268
57′ 10″	7030	169	13′ 35″	8015	219	28′	8880	269
57′ 40″	7060	170	13′ 50″	8030	220	28′ 15″	8895	270
58′	7080	171	14′ 10″	8050	221	28′ 35″	8915	271
58′ 20″	7100	172	14′ 30″	8070	222	28′ 50″	8930	272
58′ 40″	7120	173	14′ 45″	8085	223	29′ 10″	8950	273
59′	7140	174	15′ 5″	8105	224	29′ 25″	8965	274
59′ 20″	7160	175	15′ 20″	8120	225	29′ 40″	8980	275
59′ 40″	7180	176	15′ 40″	8140	226	30′	9000	276
2° 00′ 00″	7200	177	15′ 55″	8155	227	30′ 15″	9015	277
20″	7220	178	16′ 15″	8175	228	30′ 30″	9030	278
40″	7240	179	16′ 30″	8190	229	30′ 50″	9050	279
1′	7260	180	16′ 50″	8210	230	31′ 5″	9065	280
1′ 20″	7280	181	17′ 5″	8225	231	31′ 20″	9080	281
1′ 40″	7300	182	17′ 25″	8245	232	31′ 35″	9095	282
2′	7320	183	17′ 40″	8260	233	31′ 55″	9115	283
2′ 20″	7340	184	18′	8280	234	32′ 10″	9130	284
2′ 40″	7360	185	18′ 15″	8295	235	32′ 25″	9145	285
3′	7380	186	18′ 35″	8315	236	32′ 40″	9160	286
3′ 20″	7400	187	18′ 55″	8335	237	32′ 55″	9175	287
3′ 40″	7420	188	19′ 15″	8355	238	33′ 15″	9195	288
4′	7440	189	19′ 30″	8370	239	33′ 30″	9210	289
4′ 20″	7460	190	19′ 45″	8385	240	33′ 45″	9225	290
4′ 40″	7480	191	20′ 5″	8405	241	34′	9240	291
5′	7500	192	20′ 20″	8420	242	34′ 15″	9255	292
5′ 20″	7520	193	20′ 40″	8440	243	34′ 30″	9270	293
5′ 40″	7540	194	20′ 55″	8455	244	34′ 45″	9285	294
6′	7560	195	21′ 15″	8475	245	35′	9300	295
6′ 20″	7580	196	21′ 30″	8490	246	35′ 20″	9320	296
6′ 40″	7600	197	21′ 45″	8505	247	35′ 35″	9335	297
7′	7620	198	22′ 5″	8525	248	35′ 50″	9350	298
7′ 20″	7640	199	22′ 20″	8540	249	36′ 5″	9365	299
7′ 40″	7660		22′ 35″	8555		36′ 20″	9380	

TABLE IV.

CONTAINING

THE NATURAL TANGENTS AND COTANGENTS

FOR

EVERY DEGREE AND MINUTE OF THE QUADRANT.

Min.	0°	1°	2°	3°	4°	5°	6°	7°	8°	9°	
0	000000	017455	034921	052408	069927	087489	105104	122785	140541	158384	60
1	291	746	5212	2699	070219	7782	5398	3080	0837	8683	59
2	582	018037	5503	2991	0511	8075	5692	3375	1134	8981	58
3	873	328	5795	3283	0804	8368	5987	3670	1431	9279	57
4	001164	619	6086	3575	1096	8661	6281	3966	1728	9577	56
5	454	910	6377	3866	1389	8954	6575	4261	2024	9876	55
6	745	019201	6668	4158	1681	9248	6869	4557	2321	160174	54
7	002036	492	6960	4450	1973	9541	7163	4852	2618	0472	53
8	327	783	7251	4742	2266	9832	7458	5147	2915	0771	52
9	618	020074	7542	5033	2558	090127	7752	5443	3212	1069	51
10	909	365	7834	5325	2851	0421	8046	5738	3508	1368	50
11	003200	656	8125	5617	3143	0714	8340	7034	3805	1666	49
12	491	947	8416	5909	3435	1007	8635	6329	4102	1965	48
13	782	021238	8707	6200	3728	1300	8929	6625	4399	2263	47
14	004072	529	8999	6492	4020	1594	9223	6920	4696	2562	46
15	363	820	9290	6784	4313	1887	9518	7216	4993	2860	45
16	654	022111	9581	7076	4605	2180	9812	7512	5290	3159	44
17	945	402	9873	7368	4898	2474	110107	7807	5587	3458	43
18	005236	693	040164	7660	5190	2767	0401	8103	5884	3756	42
19	527	984	0456	7951	5483	3061	0695	8399	6181	4055	41
20	818	023275	0747	8243	5775	3354	0990	8694	6478	4354	40
21	006109	566	1038	8535	6068	3647	1284	8990	6776	4652	39
22	400	857	1330	8827	6361	3941	1579	9286	7073	4951	38
23	690	024148	1621	9119	6653	4234	1873	9582	7370	5250	37
24	981	439	1912	9411	6946	4528	2168	9877	7667	5549	36
25	007272	730	2204	9703	7238	4821	2463	130173	7964	4848	35
26	563	025022	2495	9995	7531	5115	2757	0469	8262	6147	34
27	854	313	2787	060287	7824	5408	3052	0765	8559	6446	33
28	008145	604	3078	0579	8116	5702	3346	1061	8856	6745	32
29	436	895	3370	0871	8409	5995	3641	1357	9154	7044	31
30	727	026186	3661	1163	8702	6289	3936	1652	9451	7343	30
31	009018	477	3952	1455	8994	6583	4230	1948	9748	7622	29
32	309	768	4244	1747	9287	6876	4525	2244	150046	7941	28
33	600	027059	4535	2039	9580	7170	4820	2540	0343	8240	27
34	891	350	4827	2331	9873	7464	5114	2836	0641	8539	26
35	010181	641	5118	2623	080165	7757	5409	3132	0938	8838	25
36	472	933	5410	2915	0458	8051	5704	3428	1236	9137	24
37	763	028224	5701	3207	0751	8345	5999	3725	1533	9437	23
38	011054	515	5993	3499	1044	8638	6294	4021	1831	9736	22
39	345	806	6284	3791	1336	8932	6588	4317	2129	170035	21
40	636	029097	6576	4083	1629	9226	6883	4613	2426	0334	20
41	927	388	6867	4375	1922	9519	7178	4909	2724	0634	19
42	012218	679	7159	4667	2215	9813	7473	5205	3022	0933	18
43	509	970	7450	4959	2508	100107	7768	5502	3319	1233	17
44	800	030262	7742	5251	2801	0401	8063	5798	3617	1532	16
45	013091	553	8033	5543	3094	0695	8358	6094	3915	1831	15
46	382	844	8325	5836	3386	0989	8653	6390	4213	2131	14
47	673	031135	8617	6128	3679	1282	8948	6687	4510	2430	13
48	964	426	8908	6420	3972	1576	9243	6983	4808	2730	12
49	014254	717	9200	6712	4265	1870	9538	7279	5106	3030	11
50	545	032009	9491	7004	4558	2164	9833	7576	5404	3329	10
51	836	300	9783	7296	4851	2458	120128	7872	5702	3629	9
52	015127	591	030075	7589	5144	2752	0423	8169	6000	3929	8
53	418	882	0366	7881	5437	3046	0718	8465	6298	4228	7
54	709	033173	0658	8173	5730	3340	1013	8761	6596	4528	6
55	016000	465	0949	8465	6023	3634	1308	9058	6894	4828	5
56	291	756	1241	8758	6316	3928	1604	9354	7192	5127	4
57	582	034047	1533	9050	6609	4222	1899	9651	7490	5427	3
58	873	338	1824	9342	6902	4516	2194	9948	7788	5427	2
59	017164	630	2116	9635	7196	4810	2489	140244	8086	6027	1
	89°	88°	87°	86°	85°	84°	83°	82°	81°	80°	Min.
	Natural Co-tangents.										
P. P. to 1″.	4·84	4·85	4·86	4·87	4·88	4·89	4·91	4·93	4·96	4·98	

Min.	10°	11°	12°	13°	14°	15°	16°	17°	18°	19°	
0	176327	194380	212557	230868	249328	267949	286745	305731	324920	344328	60
1	6627	4682	2861	1175	9637	8261	7060	6049	5241	4653	59
2	6927	4984	3165	1481	9946	8573	7375	6367	5563	4978	58
3	7227	5286	3469	1788	250255	8885	7690	6685	5885	5304	57
4	7527	5588	3773	2094	0564	9197	8005	7003	6207	5630	56
5	7827	5890	4077	2401	0873	9509	8320	7322	6528	5955	55
6	8127	6192	4381	2707	1183	9821	8635	7640	6850	6281	54
7	8427	6494	4686	3014	1492	270133	8950	7959	7172	6607	53
8	8727	6796	4990	3321	1801	0445	9266	8277	7494	6933	52
9	9028	7099	5294	3627	2111	0757	9581	8596	7817	7259	51
10	179328	197401	215599	233934	252420	271069	289896	308914	328139	347585	50
11	9628	7703	5903	4241	2729	1382	290211	9233	8461	7911	49
12	9928	8005	6208	4548	3039	1694	0527	9552	8783	8237	48
13	180229	8308	6512	4855	3348	2006	0842	9871	9106	8563	47
14	0529	8610	6817	4162	3658	2319	1158	310189	9428	8889	46
15	0829	8912	7121	5469	3968	2631	1473	0508	9751	9216	45
16	1130	9215	7426	5776	4277	2944	1789	0827	330073	9542	44
17	1430	9517	7731	6083	4587	3256	2105	1146	0396	9868	43
18	1731	9820	8035	6390	4897	3569	2420	1465	0718	350195	42
19	2031	200122	8340	6697	5207	3882	2736	1784	1041	0522	41
20	182332	200425	218645	237004	255516	274194	293052	312104	331364	350848	40
21	2632	0727	8950	7312	5826	4507	3368	2423	1687	1175	39
22	2933	1030	9254	7619	6136	4820	3684	2742	2010	1502	38
23	3234	1333	9559	7926	6446	5133	4000	3062	2333	1829	37
24	3534	1635	9864	8234	6756	5446	4316	3381	2656	2156	36
25	3835	1938	220169	8541	7066	5759	4632	3700	2679	2483	35
26	4136	2241	0474	8848	7377	6072	4948	4020	3302	2810	34
27	4437	2544	0779	9156	7687	6385	5265	4340	3625	3137	33
28	4737	2847	1084	9464	7997	6698	5581	4659	3949	3464	32
29	5038	3149	1389	9771	8307	7011	5897	4979	4272	3791	31
30	185339	203452	221695	240079	258618	277325	296213	315299	334595	354119	30
31	5640	3755	2000	0386	8928	7638	6530	5619	4919	4446	29
32	5941	4058	2305	0694	9238	7951	6846	5939	5242	4773	28
33	6242	4361	2610	1002	9549	8265	7163	6258	5566	5101	27
34	6543	4664	2916	1310	9859	8578	7480	6578	5890	5429	26
35	6844	4967	3221	1618	260170	8891	7796	6899	6213	5756	25
36	7145	5271	3526	1925	0480	9205	8113	7219	6537	6084	24
37	7446	5574	3832	2233	0791	9519	8430	7539	6861	6412	23
38	7747	5877	4137	2541	1102	9832	8747	7859	7185	6740	22
39	8048	6180	4443	2849	1413	280146	9063	8179	7509	7068	21
40	188349	206483	224748	243157	261723	280460	299380	318500	337833	357396	20
41	8651	6787	5054	3466	2034	0773	9697	8820	8157	7724	19
42	8952	7090	5360	3774	2345	1087	300014	9141	8481	8052	18
43	9253	7393	5665	4082	2656	1401	0331	9461	8806	8380	17
44	9555	7697	5971	4390	2967	1715	0649	9782	9130	8708	16
45	9856	8000	6277	4698	3278	2029	0966	320103	9454	9037	15
46	190157	8304	6583	5007	3589	2343	1283	0423	9779	9365	14
47	0459	8607	6889	5315	3900	2657	1600	0744	340103	9694	13
48	0760	8911	7194	5624	4211	2971	1918	1065	0428	360022	12
49	1062	9214	7500	5932	4523	3286	2235	1386	0752	0351	11
50	191363	209518	227806	246241	264834	283600	302553	321707	341077	360679	10
51	1665	9822	8112	6549	5145	3914	2870	2028	1402	1008	9
52	1966	210126	8418	6858	5457	4229	3188	2349	1727	1337	8
53	2268	0429	8724	7166	5768	4543	3506	2670	2052	1666	7
54	2570	0733	9031	7475	6079	4857	3823	2991	2377	1995	6
55	2871	1037	9337	7784	6391	5172	4141	3312	2702	2324	5
56	3173	1341	9643	8092	6702	5487	4459	3634	3027	2653	4
57	3475	1645	9949	8401	7014	5801	4777	3955	3352	2982	3
58	3777	1949	230255	8710	7326	6116	5095	4277	3677	3312	2
59	4078	2253	0562	9019	7637	6431	5413	4598	4002	3641	1
	79°	78°	77°	76°	75°	74°	73°	72°	71°	70°	

Natural Co-tangents. Min.

P. P. to 1′.	5·01	5·05	5·09	5·13	5·17	5·22	5·27	5·33	5·39	5·46	

Min.	20°	21°	22°	23°	24°	25°	26°	27°	28°	29°	
0	363970	383864	404026	424475	445229	466308	487733	509525	531709	554309	60
1	4300	4198	4365	4818	5577	6662	8093	9892	2083	4689	59
2	4629	4532	4703	5162	5926	7016	8453	510258	2456	5070	58
3	4959	4866	5042	5505	6275	7371	8813	0625	2829	5450	57
4	5288	5200	5380	5849	6624	7725	9174	0992	3203	5831	56
5	5618	5534	5719	6192	6973	8080	9534	1359	3577	6212	55
6	5948	5868	6058	6536	7322	8434	9895	1726	3950	6593	54
7	6278	6202	6397	6880	7671	8789	490256	2093	4324	6974	53
8	6608	6536	6736	7224	8020	9144	0617	2460	4698	7355	52
9	6938	6871	7075	7568	8369	9499	0978	2828	5072	7736	51
10	367268	387205	407414	427912	448719	469854	491339	513195	535446	558118	50
11	7598	7540	7753	8256	9068	470209	1700	3563	5821	8499	49
12	7928	7874	8092	8601	9418	0564	2061	3930	6195	8881	48
13	8259	8209	8432	8945	9768	0920	2422	4298	6570	9263	47
14	8589	8544	8771	9289	450117	1275	2784	4666	6945	9645	46
15	8919	8879	9111	9634	0467	1631	3145	5034	7319	560027	45
16	9250	9214	9450	9979	0817	1986	3507	5402	7694	0409	44
17	9581	9549	9790	430323	1167	2342	3869	5770	8069	0791	43
18	9911	9884	410130	0668	1517	2698	4231	6138	8445	1174	42
19	370242	390219	0470	1013	1868	3054	4593	6507	8820	1556	41
20	370573	390554	410810	431358	452218	473410	494955	516875	539195	561939	40
21	0804	0889	1150	1703	2568	3766	5317	7244	9571	2322	39
22	1235	1225	1490	2048	2919	4122	5679	7613	9946	2705	38
23	1566	1560	1830	2393	3269	4478	6042	7982	540322	3088	37
24	1897	1896	2170	2739	3620	4835	6404	8351	0698	3471	36
25	2228	2231	2511	3084	3971	5191	6767	8720	1074	3854	35
26	2559	2567	2851	3430	4322	5548	7130	9089	1450	4238	34
27	2890	2903	3192	3775	4673	5905	7492	9458	1826	4621	33
28	3222	3239	3532	4121	5024	6262	7855	9828	2203	5005	32
29	3553	3574	3873	4467	5375	6619	8218	520197	2579	5389	31
30	373885	393910	414214	434812	455726	476976	498582	520567	542956	565773	30
31	4216	4247	4554	5158	6078	7333	8945	0937	3332	6157	29
32	4548	4583	4895	5504	6429	7690	9308	1307	3709	6541	28
33	4880	4919	5236	5850	6781	8047	9672	1677	4086	6925	27
34	5211	5255	5577	6197	7132	8405	500035	2047	4463	7310	26
35	5543	5592	5919	6543	7484	8762	0399	2417	4840	7694	25
36	5875	5928	6260	6889	7836	9120	0763	2787	5218	8079	24
37	6207	6265	6601	7236	8188	9477	1127	3158	5595	8464	23
38	6539	6601	6943	7582	8540	9835	1491	3528	5973	8849	22
39	6872	6938	7284	7929	8892	480193	1855	3899	6350	9234	21
40	377204	397275	417626	438276	459244	480551	502219	524270	546728	569619	20
41	7536	7611	7967	8622	9596	0909	2583	4641	7106	570004	19
42	7869	7948	8309	8969	9949	1267	2948	5012	7484	0390	18
43	8201	8285	8651	9316	460301	1626	3312	5383	7862	0776	17
44	8534	8622	8993	9663	0654	1984	3677	5754	8240	1161	16
45	8866	8960	9335	440011	1006	2343	4041	6125	8619	1547	15
46	9199	9297	9677	0358	1359	2701	4406	6497	8997	1933	14
47	9532	9634	420019	0705	1712	3060	4771	6868	9376	2319	13
48	9864	9971	0361	1053	2065	3419	5136	7240	9755	2705	12
49	380197	400309	0704	1400	2418	3778	5502	7612	550134	3092	11
50	380530	400646	421046	441748	462771	484137	505867	527984	550513	573478	10
51	0863	0984	1389	2095	3124	4496	6232	8356	0892	3865	9
52	1196	1322	1731	2443	3478	4855	6598	8728	1271	4252	8
53	1530	1660	2074	2791	3831	5214	6963	9100	1650	4638	7
54	1863	1997	2417	3139	4185	5574	7329	9473	2030	5026	6
55	2196	2335	2759	3487	4538	5933	7695	9845	2409	5413	5
56	2530	2673	3102	3835	4892	6293	8061	530218	2789	5800	4
57	2863	3011	3445	4183	5246	6653	8427	0591	3169	6187	3
58	3197	3350	3788	4532	5600	7013	8793	0963	3549	6575	2
59	3530	3688	4132	4880	5954	7373	9159	1336	3929	6962	1
	69°	68°	67°	66°	65°	64°	63°	62°	61°	60°	Min.
	Natural Co-tangents.										
P. P. to 1'.	5·53	5 60	5 68	5 76	5 85	5 95	6 05	6 16	6.28	6 40	

Min.	30°	31°	32°	33°	34°	35°	36°	37°	38°	39°	
0	577350	600861	624869	649408	674509	700208	726543	753554	781286	809784	60
1	7738	1257	5274	9821	4932	0641	6987	4010	1754	810266	59
2	8126	1653	5679	650235	5355	1075	7432	4467	2223	0748	58
3	8514	2049	6083	0649	5779	1509	7877	4923	2692	1230	57
4	8903	2445	6488	1063	6203	1943	8322	5380	3161	1712	56
5	9291	2842	6894	1477	6627	2377	8767	5837	3631	2195	55
6	9680	3239	7299	1892	7051	2812	9213	6294	4100	2678	54
7	580068	3635	7704	2306	7475	3246	9658	6751	4570	3161	53
8	0457	4032	8110	2721	7900	3681	730104	7209	5040	3644	52
9	0846	4429	8516	3136	8324	4116	0550	7667	5510	4128	51
10	581235	604827	628921	653551	678749	704551	730996	758125	785981	814612	50
11	1625	5224	9327	3966	9174	4987	1443	8583	6451	5096	49
12	2014	5622	9734	4382	9599	5422	1889	9041	6922	5580	48
13	2403	6019	630140	4797	680025	5858	2336	9500	7394	6065	47
14	2793	6417	0546	5213	0450	6294	2783	9959	7865	6549	46
15	3183	6815	0953	5629	0876	6730	3230	760418	8336	7034	45
16	3573	7213	1360	6045	1302	7166	3678	0877	8808	7519	44
17	3963	7611	1767	6461	1728	7603	4125	1336	9280	8005	43
18	4353	8010	2174	6877	2154	8039	4573	1796	9752	8491	42
19	4743	8408	2581	7294	2580	8476	5021	2256	790225	8976	41
20	585134	608807	632988	657710	683007	708913	735469	762716	790697	819463	40
21	5524	9205	3396	8127	3433	9350	5917	3176	1170	9949	39
22	5915	9604	3804	8544	3860	9788	6366	3636	1643	820435	38
23	6306	610003	4211	8961	4287	710225	6815	4097	2117	0922	37
24	6697	0403	4619	9379	4714	0663	7264	4558	2590	1409	36
25	7088	0802	5027	9796	5142	1101	7713	5019	3064	1897	35
26	7479	1201	5436	660214	5569	1539	8162	5480	3538	2384	34
27	7870	1601	5844	0631	5997	1977	8611	5941	4012	2872	33
28	8262	2001	6253	1049	6425	2416	9061	6403	4486	3360	32
29	8653	2401	6661	1467	6853	2854	9511	6865	4961	3848	31
30	589045	612801	637070	661886	687281	713293	739961	767327	795436	824336	30
31	9437	3201	7479	2304	7709	3732	740411	7789	5911	4825	29
32	9829	3601	7888	2723	8138	4171	0862	8252	6386	5314	28
33	590221	4002	8298	3141	8567	4611	1312	8714	6862	5803	27
34	0613	4402	8707	3560	8995	5050	1763	9177	7337	6292	26
35	1006	4803	9117	3979	9425	5490	2214	9640	7813	6782	25
36	1398	5204	9527	4398	9854	5930	2666	770104	8290	7272	24
37	1791	5605	9937	4818	690283	6370	3117	0567	8766	7762	23
38	2184	6006	640347	5237	0713	6810	3569	1031	9242	8252	22
39	2577	6408	0757	5657	1143	7250	4020	1495	9719	8743	21
40	592970	616809	641167	666077	691572	717691	744472	771959	800196	829234	20
41	3363	7211	1578	6497	2003	8132	4925	2423	0674	9725	19
42	3757	7613	1989	6917	2433	8573	5377	2888	1151	830216	18
43	4150	8015	2399	7337	2863	9014	5830	3353	1629	0707	17
44	4544	8417	2810	7758	3294	9455	6282	3818	2107	1199	16
45	4937	8819	3222	8179	3725	9897	6735	4283	2585	1691	15
46	5331	9221	3633	8599	4156	720339	7189	4748	3063	2183	14
47	5725	9624	4044	9020	4587	0781	7642	5214	3542	2676	13
48	6120	620026	4456	9442	5018	1223	8096	5680	4021	3169	12
49	6514	0429	4868	9863	5450	1665	8549	6146	4500	3662	11
50	596908	620832	645280	670284	695881	722108	749003	776612	804979	834155	10
51	7303	1235	5692	0706	6313	2550	9458	7078	5458	4648	9
52	7698	1638	6104	1128	6745	2993	9912	7545	5938	5142	8
53	8093	2042	6516	1550	7177	3436	760366	8012	6418	5636	7
54	8488	2445	6929	1972	7610	3879	0821	8479	6898	6130	6
55	8883	2849	7342	2394	8042	4323	1276	8946	7379	6624	5
56	9278	3253	7755	2817	8475	4766	1731	9414	7859	7119	4
57	9674	3657	8168	3240	8908	5210	2187	9881	8340	7614	3
58	600069	4061	8581	3662	9341	5654	2642	780349	8821	8109	2
59	0465	4465	8994	4085	9774	6098	3098	0817	9303	8604	1
	59°	58°	57°	56°	55°	54°	53°	52°	51°	50°	Min.

Natural Co-tangents.

P. P. to 1″.	6·53	6·67	6·82	6·97	7·14	7·31	7·50	7·70	7·92	8·24	

Min.	40°	41°	42°	43°	44°	45°	46°	47°	48°	49°	
0	839100	869287	900404	932515	965689	1·00000	1·03553	1·07237	1·11061	1·15037	60
1	9595	9798	0931	3059	6251	0058	3613	7299	1126	5104	59
2	840092	870309	1458	3603	6814	0116	3674	7362	1191	5172	58
3	0588	0820	1985	4148	7377	0175	3734	7425	1256	5240	57
4	1084	1332	2513	4693	7940	0233	3794	7487	1321	5308	56
5	1581	1843	3041	5238	8504	0291	3855	7550	1387	5375	55
6	2078	2356	3569	5783	9067	0350	3915	7613	1452	5443	54
7	2575	2868	4098	6329	9632	0408	3976	7676	1517	5511	53
8	3073	3381	4627	6875	970196	0467	4036	7738	1582	5579	52
9	3571	3894	5156	7422	0761	0525	4097	7801	1648	5647	51
10	844069	874407	905685	937968	971326	1·00583	1·04158	1·07864	1·11713	1·15715	50
11	4567	4920	6215	8515	1892	0642	4218	7927	1778	5783	49
12	5066	5434	6745	9063	2458	0701	4279	7990	1844	5851	48
13	5564	5948	7275	9610	3024	0759	4340	8053	1909	5919	47
14	6063	6462	7805	940158	3590	0818	4401	8116	1975	5987	46
15	6562	6976	8336	0706	4157	0876	4461	8179	2041	6056	45
16	7062	7491	8867	1255	4724	0935	4522	8243	2106	6124	44
17	7562	8006	9398	1803	5291	0994	4583	8306	2172	6192	43
18	8062	8521	9930	2352	5859	1053	4644	8369	2238	6261	42
19	8562	9037	910462	2902	6427	1112	4705	8432	2303	6329	41
20	849062	879553	910994	943451	976996	1·01170	1·04766	1·08496	1·12369	1·16398	40
21	9563	880069	1526	4001	7564	1229	4827	8559	2435	6466	39
22	850064	0585	2059	4552	8133	1288	4888	8622	2501	6535	38
23	0565	1102	2592	5102	8703	1347	4949	8686	2567	6603	37
24	1067	1619	3125	5653	9272	1406	5010	8749	2633	6672	36
25	1568	2136	3659	5204	9842	1465	5072	8813	2699	6741	35
26	2070	2653	4193	6756	980413	1524	5133	8876	2765	6809	34
27	2573	3171	4727	7307	0983	1583	5194	8940	2831	6878	33
28	3075	3689	5261	7859	1554	1642	5255	9003	2897	6947	32
29	3578	4207	5796	8412	2126	1702	5317	9067	2963	7016	31
30	854081	884725	916331	948965	982697	1·01761	1·05378	1·09131	1·13029	1·17085	30
31	4584	5244	6866	9518	3269	1820	5439	9195	3096	7154	29
32	5087	5763	7402	950071	3842	1879	5501	9258	3162	7223	28
33	5591	6282	7938	0624	4414	1939	5562	9322	3228	7292	27
34	6095	6802	8474	1178	4987	1998	5624	9386	3295	7361	26
35	6599	7321	9010	1733	5560	2057	5685	9450	3361	7430	25
36	7104	7842	9547	2287	6134	2117	5747	9514	3428	7500	24
37	7608	8362	920084	2842	6708	2176	5809	9578	3494	7569	23
38	8113	8882	0621	3397	7282	2236	5870	9642	3561	7638	22
39	8619	9403	1159	3953	7857	2295	5932	9706	3627	7708	21
40	859124	889924	921697	954508	988432	1·02355	1·05994	1·09770	1·13694	1·17777	20
41	9630	890446	2235	5064	9007	2414	6056	9834	3761	7846	19
42	860136	0967	2773	5621	9582	2474	6117	9899	3828	7916	18
43	0642	1489	3312	6177	990158	2533	6179	9963	3894	7986	17
44	1148	2012	3851	6734	0735	2593	6241	1·00027	3961	8055	16
45	1655	2534	4390	7292	1311	2653	6303	0091	4028	8125	15
46	2162	3057	4930	7849	1888	2713	6365	0156	4095	8194	14
47	2669	3580	5470	8407	2465	2772	6427	0220	4162	8264	13
48	3177	4103	6010	8966	3043	2832	6489	0285	4229	8334	12
49	3685	4627	6551	9524	3621	2892	6551	0349	4296	8404	11
50	864193	895151	927091	960083	994199	1·02952	1·06613	1·10414	1·14363	1·18474	10
51	4701	5675	7632	0642	4778	3012	6676	0478	4430	8544	9
52	5209	6199	8174	1202	5357	3072	6738	0543	4498	8614	8
53	5718	6724	8715	1761	5936	3132	6800	0607	4565	8684	7
54	6227	7249	9257	2322	6515	3192	6862	0672	4632	8754	6
55	6736	7774	9800	2882	7095	3252	6925	0737	4699	8824	5
56	7246	8299	930342	3443	7676	3312	6987	0802	4767	8894	4
57	7756	8825	0885	4004	8256	3372	7049	0867	4834	8964	3
58	8266	9351	1428	4565	8837	3433	7112	0931	4902	9035	2
59	8776	9877	1971	5127	9418	3493	7174	0996	4969	9105	1
	49°	48°	47°	46°	45°	44°	43°	42°	41°	40°	Min.
	Natural Co-tangents.										
P. P. to 1″.	8·38	8·64	8·92	9·21	9·53	0·09	1·02	1·06	1·10	1·15	

Min.	50°	51°	52°	53°	54°	55°	56°	57°	58°	59°	
0	1·19175	1·23490	1·27994	1·32704	1·37638	1·42815	1·48256	1·53986	1·60033	1·66428	60
1	9246	3563	8071	2785	7722	2903	8349	4085	0137	6538	59
2	9316	3637	8148	2865	7807	2992	8442	4183	0241	6647	58
3	9387	3710	8225	2946	7891	3080	8536	4281	0345	6757	57
4	9457	3784	8302	3026	7976	3169	8629	4379	0449	6867	56
5	9528	3858	8379	3107	8060	3258	8722	4478	0553	6978	55
6	9599	3931	8456	3187	8145	3347	8816	4576	0657	7088	54
7	9669	4005	8533	3268	8229	3436	8909	4675	0761	7198	53
8	9740	4079	8610	3349	8314	3525	9003	4774	0865	7309	52
9	9811	4153	8687	3430	8399	3614	9097	4873	0970	7419	51
10	1·19882	1·24227	1·28764	1·33511	1·38484	1·43703	1·49190	1·54972	1·61074	1·67530	50
11	9953	4301	8842	3592	8658	3792	9284	5071	1179	7641	49
12	1·20024	4375	8919	3673	8653	3881	9378	5170	1283	7752	48
13	0095	4449	8997	3754	8738	3970	9472	5269	1388	7863	47
14	0166	4523	9074	3835	8824	4060	9566	5368	1493	7974	46
15	0237	4597	9152	3916	8909	4149	9661	5467	1598	8085	45
16	0308	4672	9229	3998	8994	5239	9755	5567	1703	8196	44
17	0379	4746	9307	4079	9079	4329	9849	5666	1808	8308	43
18	0451	4820	9385	4160	9165	4418	9944	5766	1914	8419	42
19	0522	4895	9463	4242	9250	4508	1·50038	5866	2019	8531	41
20	1·20593	1·24969	1·29541	1·34323	1·39336	1·44598	1·50133	1·55966	1·62125	1·68643	40
21	0665	5044	9618	4405	9421	4688	0228	6065	2230	8754	39
22	0736	5118	9696	4487	9507	4778	0322	6165	2336	8866	38
23	0808	5193	9775	4568	9593	4868	0417	6265	2442	8979	37
24	0879	5268	9853	4650	9679	4958	0512	6366	2548	9091	36
25	0951	5343	9931	4732	9764	5049	0607	6466	2654	9203	35
26	1023	5417	1·30009	4814	9850	5139	0702	6566	2760	9316	34
27	1094	5492	0087	4896	9936	5229	0797	6667	2866	9428	33
28	1166	5567	0166	4978	1·40022	5320	0893	6767	2972	9541	32
29	1238	5642	0244	5060	0109	5410	0988	6868	3079	9653	31
30	1·21310	1·25717	1·30323	1·35142	1·40195	1·45501	1·51084	1·56969	1·63185	1·69766	30
31	1382	5792	0401	5224	0281	5592	1179	7069	3292	9879	29
32	1454	5867	0480	5307	0367	5682	1275	7170	3398	9992	28
33	1526	5943	0558	5389	0454	5773	1370	7271	3505	1·70106	27
34	1598	6018	0637	5472	0540	5864	1466	7372	3612	0219	26
35	1670	6093	0716	5554	0627	5955	1562	7474	3719	0332	25
36	1742	6169	0695	5637	0714	6046	1658	7575	3826	0446	24
37	1814	6244	0873	5719	0800	6137	1754	7676	3934	0560	23
38	1886	6319	0952	5802	0887	6229	1850	7778	4041	0673	22
39	1959	6395	1031	5885	0974	6320	1946	7879	4148	0787	21
40	1·22031	1·26471	1·31110	1·35968	1·41061	1·46411	1·52043	1·57981	1·64256	1·70901	20
41	2104	6546	1190	6051	1148	6503	2139	8083	4363	1015	19
42	2176	6622	1269	6134	1235	6595	2235	8184	4471	1129	18
43	2249	6698	1348	6217	1322	6686	2332	8286	4579	1244	17
44	2321	6774	1427	6300	1409	6778	2429	8388	4687	1358	16
45	2394	6849	1507	6383	1497	6870	2525	8490	4795	1473	15
46	2467	6925	1586	6466	1584	6962	2622	8593	4903	1588	14
47	2539	7001	1666	6549	1672	7053	2719	8695	5011	1702	13
48	2612	7077	1745	6633	1759	7146	2816	8797	5120	1817	12
49	2685	7153	1825	6716	1847	7238	2913	8900	5228	1932	11
50	1·22758	1·27230	1·31904	1·36800	1·41934	1·47330	1·53010	1·59002	1·65337	1·72047	10
51	2831	7306	1984	6883	2022	7422	3107	9105	5445	2163	9
52	2904	7382	2064	6967	2110	7514	3205	9208	5554	2278	8
53	2977	7458	2144	7050	2198	7607	3302	9311	5663	2393	7
54	3050	7535	2224	7134	2286	7699	3400	9414	5772	2509	6
55	3123	7611	2304	7218	2374	7792	3497	9517	5881	2625	5
56	3196	7688	2384	7302	2462	7885	3595	9620	5990	2741	4
57	3270	7764	2464	7386	2550	7977	3693	9723	6099	2857	3
58	3343	7841	2544	7470	2638	8070	3791	9826	6209	2973	2
59	3416	7917	2624	7554	2726	8163	3888	9930	6318	3089	1
	39°	38°	37°	36°	35°	34°	33°	32°	31°	30°	Min.
	Natural Co-tangents.										
P. P. to 1″.	1·20	1·25	1·31	1·37	1·44	1·51	1·59	1·68	1·78	1·88	

Min.	60°	61°	62°	63°	64°	65°	66°	67°	68°	69°	
0	1·73205	1·80405	1·88073	1·96261	2·05030	2·14451	2·24604	2·35585	2·47509	2·60509	60
1	3321	0529	8205	6402	5182	4614	4780	5776	7716	0736	59
2	3438	0653	8337	6544	5333	4777	4956	5967	7924	0963	58
3	3555	0777	8469	6685	5485	4940	5132	6158	8132	1190	57
4	3671	0901	8602	6827	5637	5104	5309	5349	8340	1418	56
5	3788	1025	8734	6969	5790	5268	5486	6541	8549	1646	55
6	3905	1150	8867	7111	5942	5432	5663	6733	8758	1874	54
7	4022	1274	9000	7253	6094	5596	5840	6925	8967	2103	53
8	4140	1399	9133	7395	6247	5760	6018	7118	9177	2332	52
9	4257	1524	9266	7538	6400	5925	6196	7311	9386	2561	51
10	1·74375	1·81649	1·89400	1·97681	2·06553	2·16090	2·26374	2·37504	2·49597	2·62791	50
11	4492	1774	9533	7823	6706	6255	6552	7697	9807	3021	49
12	4610	1899	9667	7966	6860	6420	6730	7891	2·50018	3252	48
13	4728	2025	9801	8110	7014	6585	6909	8084	0229	3483	47
14	4846	2150	9935	8253	7167	6751	7088	8279	0440	3714	46
15	4964	2276	1·90069	8396	7321	6917	7267	8473	0652	3945	45
16	5082	2402	0203	8540	7476	7083	7447	8668	0864	4177	44
17	5200	2528	0337	8684	7630	7249	7626	8863	1076	4410	43
18	5319	2654	0472	8828	7785	7416	7806	9058	1289	4642	42
19	5437	2780	0607	8972	7939	7582	7987	9253	1502	4875	41
20	1·75556	1·82906	1·90741	1·99116	2·08094	2·17749	2·28167	2·39449	2·51715	2·65109	40
21	5675	3033	0876	9261	8250	7916	8348	9645	1929	5342	39
22	5794	3159	1012	9406	8405	8084	8528	9841	2142	5576	38
23	5913	3286	1147	9550	8560	8251	8710	2·40038	2357	5811	37
24	6032	3413	1282	9695	8716	8419	8891	0235	2571	6046	36
25	6151	3540	1418	9841	8872	8587	9073	0432	2786	6281	35
26	6271	3667	1554	9986	9028	8755	9254	0629	3001	6516	34
27	6390	3794	1690	2·00131	9184	8923	9437	0827	3217	6752	33
28	6510	3922	1826	0277	9341	9092	9619	1025	3432	6989	32
29	6630	4049	1962	0423	9498	9261	9801	1223	3648	7225	31
30	1·76749	1·84177	1·92098	2·00569	2·09654	2·19430	2·29984	2·41421	2·53865	2·67462	30
31	6869	4305	2235	0715	9811	9599	2·30167	1620	4082	7700	29
32	6990	4433	2371	0862	9969	9769	0351	1819	4299	7937	28
33	7110	4561	2508	1008	2·10126	9938	0534	2019	4516	8175	27
34	7230	4689	2645	1155	0284	2·20108	0718	2218	4734	8414	26
35	7351	4818	2782	1302	0442	0278	0902	2418	4952	8653	25
36	7471	4946	2920	1449	0600	0449	1086	2618	5170	8892	24
37	7592	5075	3057	1596	0758	0619	1271	2819	5389	9131	23
38	7713	5204	3195	1743	0916	0790	1456	3019	5608	9371	22
39	7834	5333	3332	1891	1075	0961	1641	3220	5827	9612	21
40	1·77955	1·85462	1·93470	2·02039	2·11233	2·21132	2·31826	2·43422	2·56046	2·69853	20
41	8077	5591	3608	2187	1392	1304	2012	3623	6266	2·70094	19
42	8198	5720	3746	2335	1552	1475	2197	3825	6487	0335	18
43	8319	5850	3885	2483	1711	1647	2383	4027	6707	0577	17
44	8441	5979	4023	2631	1871	1819	2570	4230	6928	0819	16
45	8563	6109	4162	2780	2030	1992	2756	4433	7150	1062	15
46	8685	6239	4301	2929	2190	2164	2943	4636	7371	1305	14
47	8807	6369	4440	3078	2350	2337	3130	4839	7593	1548	13
48	8929	6499	4579	3227	2511	2510	3317	5043	7815	1792	12
49	9051	6630	4718	3376	2671	2683	3505	5246	8038	2036	11
50	1·79174	1·86760	1·94858	2·03526	2·12832	2·22857	2·33693	2·45451	2·58261	2·72281	10
51	9296	6891	4997	3675	2993	3030	3881	5655	8484	2526	9
52	9419	7021	5137	3825	3154	3204	4069	5860	8708	2771	8
53	9542	7152	5277	3975	3316	3378	4258	6065	8932	3017	7
54	9665	7283	5417	4125	3477	3553	4447	6270	9156	3263	6
55	9788	7415	5557	4276	3639	3727	4636	6476	9381	3509	5
56	9911	7546	5698	4426	3801	3902	4825	6682	9606	3756	4
57	1·80034	7677	5838	4577	3963	4077	5015	6888	9831	4004	3
58	0158	7809	5979	4728	4125	4252	5205	7095	2·60057	4251	2
59	0281	7941	6120	4879	4288	4428	5395	7302	0283	4499	1
	29°	28°	27°	26°	25°	24°	23°	22°	21°	20°	Min.
	Natural Co-tangents.										
P. P. to 1″.	2·00	2·13	2·27	2·44	2·62	2·82	3·05	3·31	3·61	3·95	

Min.	70°	71°	72°	73°	74°	75°	76°	77°	78°	79°	
0	2·74748	2·90421	3·07768	3·27085	3·48741	3·73205	4·01078	4·33148	4·70463	5·14455	60
1	4997	0696	8073	7426	9125	3640	1576	3723	1137	5256	59
2	5246	0971	8379	7767	9509	4075	2074	4300	1813	6058	58
3	5496	1246	8685	8109	9894	4512	2574	4879	2490	6863	57
4	5746	1523	8991	8452	3·50279	4950	3076	5459	3170	7671	56
5	5996	1799	9298	8795	0666	5388	3578	6040	3851	8480	55
6	6247	2076	9606	9139	1053	5828	4081	6623	4534	9293	54
7	6498	2354	9914	9483	1441	6268	4586	7207	5219	5·20107	53
8	6750	2632	3·10223	9829	1829	6709	5092	7793	5906	0925	52
9	7002	2910	0532	3·30174	2219	7152	5599	8381	6595	1744	51
10	2·77254	2·93189	3·10842	3·30521	3·52609	3·77595	4·06107	4·38969	4·77286	5·22566	50
11	7507	3468	1153	0868	3001	8040	6616	9560	7978	3391	49
12	7761	3748	1464	1216	3393	8485	7127	4·40152	8673	4218	48
13	8014	4028	1775	1565	3785	8931	7639	0745	9370	5048	47
14	8269	4309	2087	1914	4179	9378	8152	1340	4·80068	5880	46
15	8523	4591	2400	2264	4573	9827	8666	1936	0769	6715	45
16	8778	4872	2713	2614	4968	3·80276	9182	2534	1471	7553	44
17	9033	5155	3027	2965	5364	0726	9699	3134	2175	8393	43
18	9289	5437	3341	3317	5761	1177	4·10216	3735	2882	9235	42
19	9545	5721	3656	3670	6159	1630	0736	4338	3590	5·30080	41
20	2·79802	2·96004	3·13972	3·34023	3·56557	3·82083	4·11256	4·44942	4·84300	5·30928	40
21	2·80059	6288	4288	4377	6957	2537	1778	5548	5013	1778	39
22	0316	6573	4605	4732	7357	2992	2301	6155	5727	2631	38
23	0574	6858	4922	5087	7758	3449	2825	6764	6444	3487	37
24	0833	7144	5240	5443	8160	3906	3350	7374	7162	4345	36
25	1091	7430	5558	5800	8562	4364	3877	7986	7882	5206	35
26	1350	7717	5877	6158	8966	4824	4405	8600	8605	6070	34
27	1610	8004	6197	6516	9370	5284	4934	9215	9330	6936	33
28	1870	8292	6517	6875	9775	5745	5465	9832	4·90056	7805	32
29	2130	8580	6838	7234	3·60181	6208	5997	4·50451	0785	8677	31
30	2·82391	2·98868	3·17159	3·37594	3·60588	3·86671	4·16530	4·51071	4·91516	5·39552	30
31	2653	9158	7481	7955	0996	7136	7064	1693	2249	5·40429	29
32	2914	9447	7804	8317	1405	7601	7600	2316	2984	1309	28
33	3176	9738	8127	8679	1814	8068	8137	2941	3721	2192	27
34	3439	3·00028	8451	9042	2224	8536	8675	3568	4460	3077	26
35	3702	0319	8775	9406	2636	9004	9215	4196	5201	3966	25
36	3965	0611	9100	9771	3048	9474	9756	4826	5945	4857	24
37	4229	0903	9426	3·40136	3461	9945	4·20298	5458	6690	5751	23
38	4494	1196	9752	0502	3874	3·90417	0842	6091	7438	6648	22
39	4758	1489	3·20079	0869	4289	0890	1387	6726	8188	7548	21
40	2·85023	3·01783	3·20406	3·41236	3·64705	3·91364	4·21933	4·57363	4·98940	5·48451	20
41	5289	2077	0734	1604	5121	1839	2481	8001	9695	9356	19
42	5555	2372	1063	1973	5538	2316	3030	8641	5·00451	5·50264	18
43	5822	2667	1392	2343	5957	2793	3580	9283	1210	1176	17
44	6089	2963	1722	2713	6376	3271	4132	9927	1971	2090	16
45	6356	3260	2053	3084	6796	3751	4685	4·60572	2734	3007	15
46	6624	3556	2384	3456	7217	4232	5239	1219	3499	3927	14
47	6892	3854	2715	3829	7638	4713	5795	1868	4267	4851	13
48	7161	4152	3048	4202	8061	5196	6352	2518	5037	5777	12
49	7430	4450	3381	4576	8485	5680	6911	3171	5809	6706	11
50	2·87700	3·04749	3·23714	3·44951	3·68909	3·96165	4·27471	4·63825	5·06584	5· 7638	10
51	7970	5049	4049	5327	9335	6651	8032	4480	7360	8573	9
52	8240	5349	4383	5703	9761	7139	8595	5138	8139	9511	8
53	8511	5649	4719	6080	3·70188	7627	9159	5797	8921	5·60452	7
54	8783	5950	5055	6458	0616	8117	9724	6458	9704	1397	6
55	9055	6252	5392	6837	1046	8607	4·30291	7121	5·10490	2344	5
56	9327	6554	5729	7216	1476	9099	0860	7786	1279	3293	4
57	9600	6857	6067	7596	1907	9592	1430	8452	2069	4248	3
58	9873	7160	6406	7977	2338	4·00086	2001	9121	2862	5205	2
59	2·90147	7464	6745	8359	2771	0582	2573	9791	3658	6165	1
	19°	18°	17°	16°	15°	14°	13°	12°	11°	10°	

Natural Co-tangents.

Min.

P. P. to 1′.	4·35	4 82	5 36	6 01	6 79	7 73	8 90	10 35	12 20	14·60

Min.	80°	81°	82°	83°	84°	85°	86°	87°	88°	89°	
0	5·67128	6·31375	7·11537	8·14435	9·51436	11·4301	14·3007	19·0811	28·6363	57·2900	60
1	8094	2566	3042	6398	4106	4685	3607	1879	8771	58·2612	59
2	9064	3761	4553	8370	6791	5072	4212	2939	29·1220	59·2659	58
3	5·70037	4961	6071	8·20352	9490	5461	4823	4051	3711	60·3058	57
4	1013	6165	7594	2344	9·62265	5853	5438	5156	6245	61·3829	56
5	1992	7374	9125	4345	4935	6248	6059	6273	8823	62·4992	55
6	2974	8587	7·20661	6355	7680	6645	6685	7403	30·1446	63·6567	54
7	3960	9804	2204	8376	9·70441	7045	7317	8546	4116	64·8580	53
8	4949	6·41026	3754	8·30406	3217	7448	7954	9702	6833	66·1055	52
9	5941	2253	5310	2446	6009	7853	8596	20·0872	9599	67·4019	51
10	5·76937	6·43484	7·26873	8·34496	9·78817	11·8262	14·9244	20·2056	31·2416	68·7501	50
11	7936	4720	8442	6555	9·81641	8673	9898	3253	5284	70·1533	49
12	8938	5961	7·30018	8625	4482	9087	15·0557	4465	8205	71·6151	48
13	9944	7206	1600	8·40705	7338	9504	1222	5691	32·1181	73·1390	47
14	5·80953	8456	3190	2795	9·90211	9923	1893	6932	4213	74·7292	46
15	1966	9710	4786	4896	3101	12·0346	2571	8188	7303	76·3900	45
16	2982	6·50970	6389	7007	6007	0772	3254	9460	33·0452	78·1263	44
17	4001	2234	7999	9128	8931	1201	3943	21·0747	3662	79·9434	43
18	5024	3503	9616	8·51259	10·0187	1632	4638	2049	6935	81·8470	42
19	6051	4777	7·41240	3402	0483	2067	5340	3369	34·0273	83·8435	41
20	5·87080	6·56055	7·42871	8·55555	10·0780	12·2505	15·6048	21·4704	34·3678	85·9398	40
21	8114	7339	4509	7718	1080	2946	6762	6056	7151	88·1436	39
22	9151	8627	6154	9893	1381	3390	7483	7426	35·0965	90·4633	38
23	5·90191	9921	7806	8·62078	1683	3838	8211	8813	4313	92·9085	37
24	1236	6·61219	9465	4275	1988	4288	8945	22·0217	8006	95·4895	36
25	2283	2523	7·51132	6482	2294	4742	9687	1640	36·1776	98·2179	35
26	3335	3831	2806	8701	2602	5199	16·0435	3081	5627	101·107	34
27	4390	5144	4487	8·70931	2913	5660	1190	4541	9560	104·171	33
28	5448	6463	6176	3172	3224	6124	1952	6020	37·3579	107·426	32
29	6510	7787	7872	5425	3538	6591	2722	7519	7686	110·892	31
30	5·97576	6·69116	7·59575	8·77689	10·3854	12·7062	16·3499	22·9032	38·1885	114·589	30
31	8646	6·70450	7·61287	9964	4172	7536	4283	23·0577	6177	118·540	29
32	9720	1789	3005	8·82252	4491	8014	5075	2137	39·0568	122·774	28
33	6·00797	3133	4732	4551	4813	8496	5874	3718	5059	127·321	27
34	1878	4483	6466	6862	5136	8981	6681	5321	9655	132·219	26
35	2962	5838	8208	9185	5462	9469	7496	6945	40·4358	137·507	25
36	4051	7199	9957	8·91520	5789	9962	8319	8593	9174	143·237	24
37	5143	8564	7·71715	3867	6118	13·0458	9150	24·0263	41·4106	149·465	23
38	6240	9936	3480	6227	6450	0958	9990	1957	9158	156·259	22
39	7340	6·81312	5254	8598	6783	1461	17·0837	3675	42·4335	163·700	21
40	6·08444	6·82694	7·77035	9·00983	10·7119	13·1969	17·1693	24·5418	42·9641	171·885	20
41	9552	4082	8825	3379	7457	2480	2558	7185	43·5081	180·932	19
42	6·10664	5475	7·80622	5789	7797	2996	3432	8978	44·0661	190·984	18
43	1779	6874	2428	8211	8139	3515	4314	25·0798	6386	202·219	17
44	2899	8278	4242	9·10646	8483	4039	5205	2644	45·2261	214·858	16
45	4023	9688	6064	3093	8829	4566	6106	4517	8294	229·182	15
46	5151	6·91104	7895	5554	9178	5098	7015	6418	46·4489	245·552	14
47	6283	2525	9734	8028	9529	5634	7934	8348	47·0853	264·441	13
48	7419	3952	7·91582	9·20516	9882	6174	8863	26·0307	7395	286·478	12
49	8559	5385	3438	3016	11·0237	6719	9802	2296	48·4121	312·521	11
50	6·19703	6·96823	7·95302	9·25530	11·0594	13·7267	18·0750	26·4316	49·1039	343·774	10
51	6·20851	8268	7176	8058	0954	7821	1708	6367	8157	381·971	9
52	2003	9718	9058	9·30599	1316	8378	2677	8450	50·5485	429·718	8
53	3160	7·01174	8·00948	3155	1681	8940	3655	27·0566	51·3032	491·106	7
54	4321	2637	2848	5724	2048	9507	4645	2715	52·0807	572·957	6
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